Dynamics of Localized Solutions for Reaction-Diffusion Systems on Curved Surface

Shin-Ichiro Ei
Institute of Math. for Industry, Kyushu University
Hiroki Yagisita
Faculty of Science, Kyoto Sangyo University

1 Introduction

In nature, many kinds of spatial and/or temporal patterns are observed, some of them are simple and the others are complicated. To understand theoretically the dynamics of such patterns, many model equations have been proposed and analyzed. Among them, some sort of reaction-diffusion systems are one of the most familiar classes.

In this report, we consider general type of reaction-diffusion systems in $\boldsymbol{R}^2$ which possess stable spot solutions. Spot patterns are one of the most typical and important patterns from the phenomenological point of view. In fact, spot patterns correspond to localized patterns of materials which are observed in many phenomena such as biology and chemistry. Also in mathematical models, many model equations with spot solutions have been proposed as in the Gierer-Meinhardt model, Gray-Scott model (e.g. [5], [4], [7], [8]).

In this report, we assume the existence of a stable spot solution on $\boldsymbol{R}^2$ and the dynamics of it on a curved surface in $\boldsymbol{R}^3$. A curved surface is imaged as a cell membrane and a spot solution is a localized structure with high density of materials on the surface. In fact, [1] reported the localization of the phosphatidylinositol lipids on the cell membrane of Dictyostelium cells. As the consequence, we show that the motion of spot solution along the surface is essentially described as the gradient flow of the Gaussian curvature.

2 Setting and results

In this section, we will give the several assumptions in the general framework.

Let us consider general type of reaction-diffusion systems written by

$$\boldsymbol{u}_t = D\Delta\boldsymbol{u} + F(\boldsymbol{u}),\ t > 0,\ \boldsymbol{x} \in \boldsymbol{R}^2,\ \boldsymbol{u} \in \boldsymbol{R}^N, \tag{2.1}$$

where D is a diagonal matrix given by $D := diag\{d_1, d_2, \cdots, d_N\}$ with $d_j > 0$ and F is a smooth map from $\boldsymbol{R}^N$ to $\boldsymbol{R}^N$.

Let $\mathcal{L}(\boldsymbol{u}) := D\Delta\boldsymbol{u} + F(\boldsymbol{u})$ and $X := \{L^2(\boldsymbol{R}^2)\}^N$. The assumptions are as follows:

1) There exist a radially symmetric stationary pulse solution $S(r)$ of (2.1) such that $\mathcal{L}(S(r)) \equiv 0$, where $r := |\boldsymbol{x}|$.

Let L be the linearized operator $\mathcal{L}'(S(r))$ of (2.1) with respect to $S(r)$, that is, $L := D\Delta + {}^tF'(S(r))$.

2) The spectral set of L consists of two sets $\sigma_1 = \{0\}$ and $\sigma_2 \subset \{\lambda \in \boldsymbol{C};\ Re(\lambda) < -\gamma\}$ for a positive constant γ.

Remark 2.1 *0 is necessarily a spectrum of L because (2.1) has a translation invariance and $LS_{x_j} = 0$ $(j = 1, 2)$ hold, where $\boldsymbol{x} = (x_1, x_2)$.*

Let Q and R be projections corresponding to the spectral sets σ_1 and σ_2, respectively. That is, $Q := \frac{1}{2\pi i}\int_C (\lambda - L)^{-1} d\lambda$ and $R := Id - Q$, where C is a circle around 0 inside $\{\lambda \in \boldsymbol{C};\ Re(\lambda) > -\gamma\}$ and Id denotes the identity. Define $E := QX$, $E^\perp := RX$.

3) $E = span\{S_{x_1}, S_{x_2}\}$ holds.

Remark 2.2 *The assumption 3) means the stability of $S(x)$ except translation.*

Let L^* be the adjoint operator of L. Then L^* also has eigenfunctions $\phi_j^*(\boldsymbol{x})$ $(j = 1, 2)$ such that $L^*\phi_j^* = 0$ and $\langle\ S_{x_j}, \phi_j^*\ \rangle_X = \pi$. Then we note that ϕ_j^* is represented as $\phi^* = \partial_{x_j}\Phi^*$ by a radially symmetric function $\Phi^* = \Phi^*(r)$.

Under these three assumptions, we consider the reaction-diffusion systems (2.1) on a smoothly curved surface $\mathcal{M} \subset \boldsymbol{R}^3$:

$$\boldsymbol{u}_t = \delta^2 \Delta_{\mathcal{M}} \boldsymbol{u} + F(\boldsymbol{u}),\ t > 0,\ \boldsymbol{x} \in \mathcal{M}, \tag{2.2}$$

where $\delta > 0$ is a sufficiently small constant and $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$.

For simplicity, we assume the surface $\mathcal{M}$ is expressed by the isothermal coodinate as $\mathcal{M} = \{\Gamma(u,v);\ (u,v) \in \boldsymbol{R}^2\}$ with $ds^2 = E(u,v)(du^2 + dv^2)$. Let $K(u,v)$ be the Gaussian curvature of $\mathcal{M}$ and $T := \delta^2 t$. Then we have:

Theorem 2.1 *For $\boldsymbol{H} \in \mathcal{M}$, there exists a map $\Psi = \Psi(\boldsymbol{w};\boldsymbol{H}) \in \boldsymbol{R}^2$ defined for $|\boldsymbol{w}| << 1$ such that $\Psi(0;\boldsymbol{H}) = 0$ and the solution $\boldsymbol{u}(t,\boldsymbol{x})$ of (2.2) is expressed as*

$$\boldsymbol{u}(t,\boldsymbol{x}) = S(|\Psi(\boldsymbol{x} - \boldsymbol{H}(T);\boldsymbol{H}(T))|/\delta) + O(\delta)$$

and $\boldsymbol{H}(T)$ satisfies

$$\frac{d\boldsymbol{H}}{dT} = \frac{M_0}{E(\boldsymbol{h}(T))}\{K_u(\boldsymbol{h}(T))\Gamma_u(\boldsymbol{h}(T)) + K_v(\boldsymbol{h}(T))\Gamma_v(\boldsymbol{h}(T))\} + O(\delta), \tag{2.3}$$

where $\boldsymbol{H} = \Gamma(\boldsymbol{h}) \in \mathcal{M}$, $\boldsymbol{h} = (h_1,h_2) \in \boldsymbol{R}^2$, $M_0 := \dfrac{1}{4}\displaystyle\int_0^\infty r^3 \langle\, DS_r, \Phi_r^* \,\rangle\, dr$.

(2.3) implies that the spot solution moves toward points with maximal Gaussian curvature if M_0 is positive and toward points with minimal Gaussian curvature if M_0 is negative while the sign is depending on each model. The proofs will be basically given by similar manner to [2].

Finally, we give the explicit value of the constant M_0 for the Gierer-Mienhardt model. We consider the following Gierer-Meinhardt model for sufficiently small $\varepsilon > 0$, $0 \le \tau << 1$ and $a > 0$:

$$\left\{\begin{array}{lcl} u_t &=& \varepsilon^2 \Delta u - u + \dfrac{u^{p_1}}{(v+a)^{p_2}}, \\ \tau v_t &=& d\Delta v - v + \dfrac{u^{p_3}}{(v+a)^{p_4}}, \end{array}\right. \tag{2.4}$$

where p_j are the positive constants satisfying $p_1 > 1$ and $0 < \dfrac{p_1 - 1}{p_2} < \dfrac{p_3}{p_4 + 1}$. (2.4) has a stable radially symmetric spiky solution in the form of $S(r) = (U(r/\varepsilon), V(r))$ for $r = |\boldsymbol{x}|$ and functions $U(r)$, $V(r)$ ([3], [5], [6]). In particular, when $p_3 = 2$ and $1 < p_1 < 3$ or

$p_3 = p_1 + 1$ and $1 < p_1 < \infty$, $\Phi^*(r)$ is given by $\Phi^*(r) = c(U(r/\varepsilon), 0) + o(1)$ as $\varepsilon \downarrow 0$ ([3]) for a constant c. Then we have the following theorem on the value of the constant M_0:

Theorem 2.2 *If $p_3 = 2$ and $1 < p_1 < 3$ or $p_3 = p_1 + 1$ and $1 < p_1 < \infty$ are satisfied for (2.4), then the constant M_0 is given by*

$$M_0 = \varepsilon^3 \frac{\displaystyle\int_0^\infty r^3 (U'(r))^2 dr}{4 \displaystyle\int_0^\infty r (U'(r))^2 dr} + o(\varepsilon^3) > 0.$$

The above theorem says that the spot solution moves toward points with maximal Gaussian curvature.

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Institute of Math. for Industry

Kyushu University

Fukuoka 819-0395

JAPAN

E-mail address ichiro@imi.kyushu-u.ac.jp ,

Faculty of Science

Kyoto Sangyo University

Kyoto 603-8555

JAPAN

E-mail address hrk0ygst@cc.kyoto-su.ac.jp .