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# Potential theory and optimal convergence rates in fast nonlinear diffusion ☆

Yong Jung Kim [a,*], Robert J. McCann [b]

[a] *Division of Applied Mathematics, KAIST (Korea Advanced Institute of Science and Technology), Gusong-dong 373-1, Yusong-gu, Taejon 305-701, South Korea*

[b] *Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada*

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**Abstract**

A potential theoretic comparison technique is developed, which yields the conjectured optimal rate of convergence as $t \to \infty$ for solutions of the fast diffusion equation

$$u_t = \Delta(u^m), \quad (n-2)_+/n < m \leqslant n/(n+2),\ u, t \geqslant 0,\ \mathbf{x} \in \mathbf{R}^n,\ n \geqslant 1,$$

to a spreading self-similar profile, starting from integrable initial data with sufficiently small tails. This $1/t$ rate is achieved uniformly in relative error, and in weaker norms such as $L^1(\mathbf{R}^n)$. The range of permissible nonlinearities extends upwards towards $m = 1$ if the initial data shares enough of its moments with a specific self-similar profile. For example, in one space dimension, $n = 1$, the $1/t$ rate extends to the full range $m \in \,]0, 1[$ of nonlinearities provided the data is correctly centered.


**Résumé**

Dans les milieux dissipatifs, les perturbations initiales disparaissent progressivement, et seuls sont preservés leurs traits les plus grossiers, comme leur taille et leur position. Estimer précisément la vitesse de cette « disparition » est parfois une question d'un interêt primordial. Ici, nous donnons cette vitesse pour les diffusions non linéaires les plus rapides qui préservent la masse, pour le modèle qui gouverne la diffusion d'une densité initiale, intégrable et à support compact, vers un profil autosimilaire. Pour cela, nous établissons une théorie de comparaison des potentiels, ce qui permet de montrer que la vitesse précise de décroissance est en $1/t$ pour la norme $L^1(\mathbf{R}^n)$, et en fait uniforme pour l'erreur relative.


*Keywords:* Fast nonlinear diffusion; Large time convergence rate; Newtonian potential; Moments

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* Corresponding author.
*E-mail addresses:* ykim@amath.kaist.ac.kr (Y.J. Kim), mccann@math.toronto.edu (R.J. McCann).
*URL:* http://www.math.toronto.edu/~mccann (R.J. McCann).

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## 1. Introduction

In many diffusive settings, initial disturbances will gradually disappear and all but their crudest features—such as size and location—will eventually be forgotten. Quantifying the rate at which this information is lost is often a question of central interest. The present paper is devoted to resolving this issue for a range of nonlinearities in a model problem known the fast-diffusion equation (1.1). For other choices of the parameter $m$, this equation has been used to represent such diverse phenomena as heat transport, population spreading, fluid seepage, curvature flow, and avalanches in sandpiles. Although most of these applications lie outside the range of nonlinearities considered below, the evolution forms a paradigmatic example in nonlinear parabolic theory, and a complete understanding of its asymptotic behaviour is therefore desired. After much recent attention, the $1/t$ rate derived below for $m \in ]\frac{(n-2)_+}{n}, \frac{n}{n+2}]$ finally establishes the sharp, conjectured [11,24,25] power law rate of decay in this range, corresponding to the fastest conservative nonlinearities.

Fix $p > (2-n)_+ := \max\{2-n, 0\}$. We consider the asymptotic behavior as $t \to \infty$ of solutions $u(\mathbf{x}, t)$ to the nonlinear diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta(u^m), \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), \end{aligned} \tag{1.1}$$

on the whole space $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$, where $m := 1 - 2/(n+p)$. The initial value $u_0(\mathbf{x}) \geqslant 0$ is presumed non-negative and integrable. Notice $p > 0$ corresponds to the conservative range of fast diffusion $m \in ](n-2)/n, 1[$, in which the total mass of $u$ is preserved but the diffusivity $mu^{m-1}(\mathbf{x}, t)$ diverges at low densities. Our assumption $p > 2-n$ ensures $m > 0$, so the equation is forward- (not backward-) parabolic. It is well known that this problem is well-posed [33] and the solution $u(\mathbf{x}, t) > 0$ is smooth and strictly positive [2] for any $t > 0$ and $\mathbf{x} \in \mathbf{R}^n$. Such regularity has been demonstrated by Aronson and Bénilan. Following Herrero and Pierre, we suppose the initial condition $u_0 \in L^1(\mathbf{R}^n)$ is attained in the sense that $u \in C([0, \infty[; L^1_{\mathrm{loc}}(\mathbf{R}^n))$. Data $u_0 \geqslant 0$ which are Radon [16,52] or merely Borel [12] measures have been discussed by Dahlberg & Kenig, Pierre, and Chasseigne & Vázquez.

We impose a stronger localization on the initial data, by assuming the limit

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{n+p} u_0(\mathbf{x}) =: L_0 < \infty \tag{1.2}$$

exists and is finite, which is almost enough to ensure $p$ moments converge initially:

$$\int_{\mathbf{R}^n} |\mathbf{x}|^p u_0(\mathbf{x})\, \mathrm{d}\mathbf{x} < \infty. \tag{1.3}$$

Both conditions are satisfied by compactly supported initial data. Furthermore, (1.2) is natural in the sense that Carrillo & Vázquez and Lee & Vázquez have shown this tail condition to be propagated by the evolution: if $L_0$ is positive [11] or vanishes [44], the corresponding limit $\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{n+p} u(\mathbf{x}, t)$ of the solution at time $t > 0$ takes the value $L_t = (L_0^{2/(n+p)} + Bt)^{(n+p)/2}$ with $B = \frac{1}{2}(1 + \frac{n}{p})\frac{1}{n+p-2}$.

Let $\rho(\mathbf{x}, t)$ be a canonical (Barenblatt) spreading solution [5,50] that solves

$$\frac{\partial \rho}{\partial t} = \Delta(\rho^m), \quad \rho(\mathbf{x}, 0) = \delta(\mathbf{x}),\ \mathbf{x} \in \mathbf{R}^n,\ t > 0. \tag{1.4}$$

Since the work of Friedman and Kamin [30], the $L^1(\mathbf{R}^n)$ contractivity of the flow has been known to imply that the orbit $\rho(\mathbf{x}, t)$ attracts all non-negative solutions that share its mass. From the explicit formula (3.1) we see that the Barenblatt solution has tails $\rho(\mathbf{x}, t) = \mathrm{O}(1/|\mathbf{x}|^{n+p})$ as $|\mathbf{x}| \to \infty$, precisely consistent with hypothesis (1.2); the fast diffusion produces this exact algebraic rate of spatial decay. For initial data $u_0$ with tails thicker than $\mathrm{O}(1/|\mathbf{x}|^{n+p})$ however, the asymptotic rate of convergence to $\rho(\mathbf{x}, t)$ will reflect a competition between the initial structure and the fast diffusion, which is not our present concern. In fact, Vázquez [58] has shown that no $L^1$-contraction rate can hold uniformly among all $L^1(\mathbf{R}^n)$ initial data, building on work of Vázquez and Zuazua [59]. Extra restrictions such as finiteness of moments, entropy, or relative entropy must be imposed, and are employed throughout the literature to quantify decay.

The sharp convergence rate $\|u(t)-\rho(t)\|_{L^1(\mathbf{R}^n)} = \mathrm{O}(t^{-\alpha})$ of $\alpha = \frac{1}{2}(1+\frac{n}{p})$ for the large time limit was derived for integrable initial data with finite second moments

$$\int_{\mathbf{R}^n} |\mathbf{x}|^2 u_0(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty$$

and nonlinearity $p > n$ by Dolbeault and del Pino [26] and Otto [48]. For the faster range of diffusions, $p \in\, ](2-n)_+, n]$, a bound $\mathrm{O}(t^{-1/2})$ on the convergence order was found by Carrillo and Vázquez [11]. Note $\alpha = 1$ in the borderline case $p = n$, so these two bounds on the decay rate do not match. In the left end $p \in\, ](2-n)_+, 2]$ of this range of nonlinearities, we establish below a rate of convergence $\mathrm{O}(t^{-1})$ which is sharp in the sense that the exponent cannot be improved. In the complementary range $p \geqslant n$, a companion paper by McCann and Slepčev establishes the rate $\mathrm{O}(t^{-1+\delta})$ for any $\delta > 0$, assuming the center of mass (2.5) vanishes [46]. While the same $\mathrm{O}(t^{-1})$ rate is expected, also in the gap $p \in\, ]2, n[$, this remains a conjecture.

The results with Slepčev rely crucially on the spectrum of the linearized evolution found by Denzler and McCann [24,25]. However, the spectral calculation provides little information about the very fast diffusion regime $p \leqslant 2$. Thus the present challenge requires that a quite different approach be developed below. It is based on comparison of the Newtonian potential $U(\mathbf{x}, t) = \Delta^{-1} u$ of a solution with the Newtonian potential of the evolving Barenblatt profile $R(\mathbf{x}, t) = \Delta^{-1}\rho$; see (4.2) for a definition of $\Delta^{-1}$. Such potential comparisons were already used by Pierre [51] to show well-posedness of the porous medium flow starting from measures as initial data, and in a series of works by Dahlberg and Kenig [14–19] and Daskalopoulos and del Pino [20,21] to explore (among other things) which initial/boundary values yield bounded, non-vanishing solutions either globally or locally in time. Note the evolution

$$\frac{\partial U}{\partial t} = (\Delta U)^m > 0$$

of the Newtonian potential is pointwise monotone and enjoys a maximum principle (Section 5, Proposition 13). The strategy executed below is to use the convergence known by other methods [30,58] to deduce the existence of large enough times $S, T \geqslant 0$ so that the growing potential becomes trapped

$$R(\mathbf{x}, t-T) \leqslant U(\mathbf{x}, t) \leqslant R(\mathbf{x}, t+S) \tag{1.5}$$

between the potentials of two Barenblatt profiles when $t = T$ (Section 8, Theorem 17), and hence for all subsequent times $t \geqslant T$. Once (1.5) is established, the smoothing properties [42] of the evolution imply convergence of the original solution $u(\mathbf{x}, t) \to \rho(\mathbf{x}, t)$ (and not merely its Newtonian potential, Section 6, Theorem 14) at the same rate $\|\rho(t-T) - \rho(t+S)\|_{L^1(\mathbf{R}^n)} = \mathrm{O}(t^{-1})$ as the two delayed Barenblatts. Paradoxically, the thick tails of the Barenblatt profile which confound other analysis when $p \leqslant 2$ enable the present method, by providing a large enough gap between the barriers $R(\mathbf{x}, 0)$ and $R(\mathbf{x}, S+T)$ to squeeze the tails of $U(\mathbf{x}, T)$ in between them. When $p > 2$, this cannot be achieved unless $u_0(\mathbf{x})$ shares higher moments (2.6) with a particular Barenblatt $\rho(\mathbf{x}, \tau)$, but for $p \leqslant 2$ it is enough that their total mass and centers of mass coincide.

Our approach is akin to the one-dimensional argument used by Carrillo and Vázquez to establish $\mathrm{O}(t^{-1})$ convergence for all $p > 0$ and radial initial data $u_0(\mathbf{x}) = u_0(|\mathbf{x}|)$. However, their technique does not adapt to non-radial data, because it is based on comparing primitives $\tilde{u}(r, t) := \int_{|\mathbf{x}|<r} u(t, \mathbf{x})\,\mathrm{d}\mathbf{x}$ of the radial densities [55,56] instead of Newtonian potentials. Like Carrillo and Vázquez [11], we establish $\mathrm{O}(t^{-1})$ convergence not only in $L^1(\mathbf{R}^n)$, but uniformly in relative error (2.7). Convergence in this weighted $L^\infty$ norm

$$\lim_{t\to 0} \left\| \frac{u(\cdot, t) - \rho(\cdot, t)}{\rho(\cdot, t)} \right\|_{L^\infty(\mathbf{R}^n)} = 0 \tag{1.6}$$

was recently established by Vázquez' Theorem 21.1 [58] without any rate, and plays a key role in our reasoning.

Large time asymptotics for the porous medium regime $p < -n$ have been discussed by a number of authors in one [61,34,56,3,1,6] or several dimensions [2,30,4,17,35,36,60,41,10,7,23,48,26,44,58,9,47,49,54]. Some of these articles address fast-diffusion $p > 0$ as well, and a more modest literature is devoted exclusively to that regime [28,43,8,11,24, 25]. The long-time behaviour of non-conservative diffusion $p \in\, ]-n, 0[$ and the borderline case $p = 0$ [32] have also been examined [13,31,40,53,22]. Contributions by Alikakos, Angenent, Aronson, Bakry, Barenblatt, Bénilan, Bernoff, Carrillo, Chayes, Dahlberg, Daskalopoulos, Denzler, DiFrancesco, Dolbeault, Emery, Esteban, Friedman, Galaktionov, Hamilton, Jüngel, Kamin, Kenig, King, Koch, Lederman, Lee, Markowich, Newman, Osher, Otto, Peletier, del Pino, Ralston, Rodríguez, Rostamanian, Saez, Toscani, Unterreiter, Vázquez, Villani, Witelski, and Zel'dovich among others are reviewed in Carrillo and Vázquez [11], Vázquez [58], and the references there. The results of the current investigation were announced in [38].

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## 2. Statement of results

To permit $L_0 > 0$ in hypothesis (1.2), we relax (1.3) by assuming there exists $\tau \in [0, \infty[$ such that

$$\int_{\mathbf{R}^n} |\mathbf{x}|^p \big|u_0(\mathbf{x}) - \rho(\mathbf{x}, \tau)\big| \,\mathrm{d}\mathbf{x} < \infty. \tag{2.1}$$

In fact, $\tau = L_0^{2/(n+p)}/B$ without loss of generality. The initial value problem for fast diffusion can be formulated as

$$\frac{\partial u}{\partial t} = \Delta\big(u^{\frac{n+p-2}{n+p}}\big), \qquad 0 \leqslant u_0(\cdot) = \lim_{t\downarrow 0} u(\cdot\,, t) \quad \text{in } L^1_{\mathrm{loc}}\big(\mathbf{R}^n\big), \qquad (2-n)_+ < p < \infty, \tag{2.2}$$

where the limit

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{n+p} u_0(\mathbf{x}) =: L_0 < \infty \tag{2.3}$$

is assumed to converge. We may also assume both the initial value $u_0$ and the Barenblatt $\rho$ have total mass 1 and center of mass at the origin without losing generality, i.e.,

$$1 = \int_{\mathbf{R}^n} u_0(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \text{when } p > 0, \quad \text{and} \tag{2.4}$$

$$0 = \int_{\mathbf{R}^n} x_i u_0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad i = 1, \ldots, n, \ \text{assuming } p > 1. \tag{2.5}$$

If $\tau > 0$ in (2.1), the range of nonlinearities $p$ can be expanded in the rare case that the initial data happens to share further moments with the Barenblatt $\rho(\mathbf{x}, \tau)$; that is, if for each multi-index $\beta \in \mathbf{N}^n$ of order $|\beta| = \sum_{i=1}^n \beta_i$,

$$0 = \int_{\mathbf{R}^n} x_1^{\beta_1} \cdots x_n^{\beta_n} \big[u_0(\mathbf{x}) - \rho(\mathbf{x}, \tau)\big]\,\mathrm{d}\mathbf{x}, \quad \text{whenever } |\beta| < p. \tag{2.6}$$

The goal of this paper is to show that there exists $C = C(u_0)$ such that

$$\left\| \frac{u(\cdot\,, t) - \rho(\cdot\,, t)}{\rho(\cdot\,, t)} \right\|_{L^\infty(\mathbf{R}^n)} \leqslant \frac{C}{t} \quad \text{for } t \gg 1, \tag{2.7}$$

as conjectured by Carrillo and Vázquez [11] and Denzler and McCann [25]. It was already known that the relative uniform norm tends to zero (1.6) from the work of Vázquez. For radially symmetric solutions the convergence order (2.7) was established by Carrillo and Vázquez, but was not known to be better than $\mathrm{O}(t^{-1/2})$ in the nonradial case.

We immediately obtain an $L^1$ convergence rate from (2.7), namely

$$\big\|u(\cdot\,, t) - \rho(\cdot\,, t)\big\|_{L^1(\mathbf{R}^n)} = \mathrm{O}\big(t^{-1}\big) \quad \text{as } t \to \infty. \tag{2.8}$$

Since this convergence order is attained by the two Barenblatt solutions $\rho(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t + t_0)$ in Lemma 3, these rates are optimal. Because these two solutions are dilations of each other at each instant in time, Denzler and McCann referred to (2.8) as the *dilation-persistence conjecture*.

In this paper we develop a technique based on the Newtonian potential. We apply the technique to the fast diffusion using simple comparison and obtain following results:

**Theorem 1** *(Relative $L^\infty$ convergence rate). Let $u(\mathbf{x},t)$ solve* (2.2)–(2.4) *for $\mathbf{x} \in \mathbf{R}^n$, $t > 0$, while $\rho$ denotes the Barenblatt solution* (1.4).

(i) *If $0 < p \leqslant 2 \leqslant n$ and* (2.1) *holds then*

$$C(p,u_0) := \limsup_{t\to\infty} t \left\| \frac{u(\cdot\,,t) - \rho(\cdot\,,t)}{\rho(\cdot\,,t)} \right\|_{L^\infty(\mathbf{R}^n)} < +\infty. \tag{2.9}$$

(ii) *If $p > 2$, $n \geqslant 2$, and* (2.1)–(2.6) *hold, then again* (2.9) *is finite.*

(iii) *If the initial value $u_0(\mathbf{x}) = u_0(|\mathbf{x}|)$ is radially symmetric and $n \geqslant 3$, then* (2.9) *holds for all $p > 0$* (*or equivalently* $(n-2)/n < m < 1$).

(iv) *If $n = 1$ but* (2.5) *holds, then* (2.9) *is true for all $p > 1$* (*or* $0 < m < 1$).

Here (iii) is primarily a new proof of Carrillo and Vázquez' rate [11], whereas results (i), (ii) and (iv) were unknown. The sharp rate of convergence in $L^1(\mathbf{R}^n)$ norm is an immediate corollary:

**Corollary 2** *($L^1$ convergence rate). With the hypotheses and notation of Theorem* 1,

$$C_1(p,u_0) := \limsup_{t\to\infty} t \left\| u(\cdot\,,t) - \rho(\cdot\,,t) \right\|_{L^1(\mathbf{R}^n)} \tag{2.10}$$

*satisfies $C_1(p,u_0) \leqslant C(p,u_0)$, hence is finite in cases* (i)–(iv).

**Proof.** Given $\varepsilon > 0$, taking $t$ sufficiently large yields

$$t\left|u(\mathbf{x},t) - \rho(\mathbf{x},t)\right| \leqslant \bigl(C(p,u_0) + \varepsilon\bigr)\rho(\mathbf{x},t)$$

from (2.9). Since the Barenblatt solution $\rho(\mathbf{x},t)$ is normalized to have unit mass at each time, integrating this bound over $\mathbf{x} \in \mathbf{R}^n$ yields $C_1(p,u_0) \leqslant C(p,u_0) + \varepsilon$. Arbitrariness of $\varepsilon > 0$ concludes the corollary. □

The proof of the main result (Theorem 1) consists of several steps. Section 6, Theorem 14 exploits the smoothing properties of the equation to show the conjectured rate of convergence in relative error follows from the ordering (1.5). Since the Newtonian potentials satisfy a comparison principle (Section 5, Proposition 13), it suffices to establish this ordering at a single instant in time. This is accomplished in Section 8, Theorem 17, but requires a decay estimate $|U(\mathbf{x},0) - R(\mathbf{x},\tau)| = \mathrm{O}(1/|\mathbf{x}|^{n+p-2})$ as $|\mathbf{x}| \to \infty$ relating the Newtonian potential of our solution to that of a Barenblatt profile. For each separate case (i)–(iv) of Theorem 1, the desired decay is established in Section 4, Theorem 6, or Section 7, Propositions 15–16, using the appropriate moment conditions (2.4)–(2.6). The crucial ordering (1.5) amounts to showing that the Newtonian potential of an evolving solution will eventually be sandwiched between the Newtonian potentials of a concentrated and a diffuse Barenblatt. The rate of convergence of the evolving solution is therefore the same as the rate of convergence of the two Barenblatts. It is here that the small values $p \leqslant 2$ difficult to handle by other methods facilitate use of this comparison argument, because long tailed Barenblatts have sufficiently separated Newtonian potentials to fit the tails of an evolving solution's potential between them. In the porous medium case $p < -n$ such an approach would be doomed by Newton's theorem, which allows no room between the potentials of concentric equal mass Barenblatts outside of their compact support.

It is perhaps surprising that the optimal convergence order is independent of the nonlinearity $p > 0$. The same convergence order is also optimal in several different problems such as the inviscid conservation laws studied by Dolbeault and Escobedo [27] and Kim [37], and the Burgers equation studied by Kim and Ni [39]. It seems there may be a common contraction and scaling structure at work which produces this convergence rate.

## 3. Translated versus delayed Barenblatt asymptotics

The Barenblatt solution $\rho(\mathbf{x},t)$ is given explicitly by:

$$\rho(\mathbf{x},t) = \left(\frac{t}{At^{2\alpha} + B|\mathbf{x}|^2}\right)^{(n+p)/2} = \bigl(At^{n/p} + B|\mathbf{x}|^2 t^{-1}\bigr)^{-p\alpha}, \tag{3.1}$$

where $\alpha = 1/(n(m-1)+2) = (n+p)/(2p) > 0$ and $B = (1-m)\alpha/2m = \alpha/(n+p-2)$. The other constant $A > 0$ is decided by the total mass of the Barenblatt solution and we normalize it so that $\int \rho(\mathbf{x},t)\,d\mathbf{x} = 1$. For a fixed $t > 0$ or $\mathbf{x} \in \mathbf{R}^n$, we can easily check that

$$\rho(\mathbf{x},t) \sim \left(B|\mathbf{x}|^2/t\right)^{-(n+p)/2} = \mathrm{O}\left(1/|\mathbf{x}|^{n+p}\right) \quad \text{as } |\mathbf{x}| \to \infty, \tag{3.2}$$

$$\rho(\mathbf{x},t) \sim \left(A^p t^n\right)^{-\alpha} = \mathrm{O}\left(t^{-n\alpha}\right) \quad \text{as } t \to \infty. \tag{3.3}$$

The pressure $q(u)$ of a density $u(\mathbf{x},t)$ is defined by $q(u) = mu^{m-1}/(m-1)$, so

$$q(\rho) = -\left[(p+n-1)At^{n/p} + \left(1 + \frac{n}{p}\right)\frac{|\mathbf{x}|^2}{4t}\right]. \tag{3.4}$$

The case $A \leqslant 0$ also leads to a family of infinite mass Barenblatt profiles $\varrho(\mathbf{x},t)$ which will eventually prove to be convenient comparison solutions:

$$\varrho(\mathbf{x},t) = \begin{cases} \left(\dfrac{t}{B|\mathbf{x}|^2 - |A|t^{(n+p)/p}}\right)^{(n+p)/2} & \text{if } |\mathbf{x}|^2 > t^{(n+p)/p}|A|/B, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

*Barenblatt versus Barenblatt:*

It is sometimes useful to change to the so-called similarity variables,

$$t^{n\alpha}\rho(\mathbf{x},t) = \hat{\rho}(\mathbf{y}), \quad \mathbf{y} = \frac{\mathbf{x}}{t^\alpha}, \tag{3.6}$$

where $\hat{\rho}$ is given by

$$\hat{\rho}(\mathbf{y})^{m-1} = A + B|\mathbf{y}|^2, \quad \int \hat{\rho}(\mathbf{y})\,d\mathbf{y} = 1. \tag{3.7}$$

Using these variables we may easily compare two Barenblatt solutions.

The next lemma shows the $1/t$ convergence order of $\|u(t) - \rho(t)\|_{L^1(\mathbf{R}^n)}$ asserted by Theorem 1 and its corollary cannot be improved—neither in $L^1(\mathbf{R}^n)$ nor uniformly in relative error—without restrictions beyond (2.1)–(2.5). Indeed, (3.8) gives the precise coefficient of $1/t$ bounding the relative error between two time-delayed Barenblatts. This bound is achieved in the near- or the far-field limit depending on the sign of $n - p$.

**Lemma 3** *(Ratio of delayed Barenblatts converges like* $1/t$*). For* $t_0 > 0$*,*

$$\lim_{t\to\infty} t\left\|\frac{\rho(\mathbf{x},t+t_0)}{\rho(\mathbf{x},t)} - 1\right\|_{L^\infty(\mathbf{R}^n)} = (t_0/2)(n+p)\max\{1, n/p\} \tag{3.8}$$

*and*

$$0 < \lim_{t\to\infty} t\int \left|\rho(\mathbf{x},t) - \rho(\mathbf{x},t+t_0)\right| d\mathbf{x} < +\infty. \tag{3.9}$$

**Proof.** Treating $t_0 > 0$ as fixed, the binomial expansion of

$$\frac{\rho(\mathbf{x},t+t_0)}{\rho(\mathbf{x},t)} = \left(1 + \frac{t_0}{t}\right)^{p\alpha}\left(1 + \frac{(1+t_0/t)^{2\alpha} - 1}{1 + \frac{B}{A}|\mathbf{x}t^{-\alpha}|^2}\right)^{-p\alpha} \tag{3.10}$$

from (3.1) in the small parameter $t_0/t$ yields

$$\frac{\rho(\mathbf{x},t+t_0)}{\rho(\mathbf{x},t)} = 1 + \frac{p\alpha t_0}{t}\left(1 - \frac{2\alpha}{1 + \frac{B}{A}|\mathbf{x}t^{-\alpha}|^2}\right) + \mathrm{O}(t_0/t)^2 \tag{3.11}$$

as $t \to \infty$. The error bound $|\mathrm{O}(t_0/t)^2| \leqslant [C(\alpha,p)t_0/t]^2$ depends solely on $n$ and $p$ (by Taylor's remainder theorem or since the binomial series eventually alternates). Thus the limit (3.8) is attained for $\mathbf{x} = 0$ if $2\alpha - 1 = n/p > 1$, and for $\mathbf{x}/t^\alpha \to \infty$ otherwise.

Having established (3.8), the upper bound (3.9) follows immediately. However, since $\lim_{t\to\infty}\rho(\mathbf{x},t)=0$ vanishes pointwise but not in $L^1(\mathbf{R}^n)$, the lower bound is better obtained by Taylor expanding in similarity variables. There $E(t)=\|\rho(t)-\rho(t+t_0)\|_{L^1(\mathbf{R}^n)}$ can be reexpressed as

$$E(t)=\int\left|\hat\rho(\mathbf{y})-\left(\frac{t}{t+t_0}\right)^{n\alpha}\hat\rho\left(\mathbf{y}\left(\frac{t}{t+t_0}\right)^{\alpha}\right)\right|\mathrm{d}\mathbf{y}. \tag{3.12}$$

From the Taylor expansion

$$\hat\rho\big(\mathbf{y}(1+t_0/t)^{-\alpha}\big)=\hat\rho(\mathbf{y})+\sum_{i=1}^{n}\big((1+t_0/t)^{-\alpha}-1\big)\mathbf{y}_i\hat\rho_{\mathbf{y}_i}(\bar{\mathbf{y}}),$$

with $\bar{\mathbf{y}}=(1-s)\mathbf{y}+s\mathbf{y}(1+t_0/t)^{-\alpha}$ for $0<s<1$, we find

$$\begin{aligned}E(t)&=\big(1-(1+t_0/t)^{-n\alpha}\big)\int\left|\hat\rho(\mathbf{y})+\frac{1-(1+t_0/t)^{-\alpha}}{1-(1+t_0/t)^{-n\alpha}}\mathbf{y}\cdot\nabla\hat\rho(\bar{\mathbf{y}})\right|\mathrm{d}\mathbf{y}\\&=\mathrm{o}(1/t)+\frac{\alpha t_0}{t}\int\big|n\hat\rho+\mathbf{y}\cdot\nabla\hat\rho(\mathbf{y})\big|\,\mathrm{d}\mathbf{y}\quad\text{as } t\to\infty.\end{aligned}$$

The last identity follows from $0<-\mathbf{y}\cdot\nabla\hat\rho(\bar{\mathbf{y}})\leqslant(n+p)\hat\rho(\mathbf{y})$ by Lebesgue's dominated convergence theorem. Thus $tE(t)\to C\in\,]0,\infty[$ as desired. □

**Lemma 4** *(Convergence rate of displaced Barenblatts). For $\mathbf{0}\neq\mathbf{z}\in\mathbf{R}^n$,*

$$0<\lim_{t\to\infty}t^{\alpha}\int\big|\rho(\mathbf{x},t)-\rho(\mathbf{x}-\mathbf{z},t)\big|\,\mathrm{d}\mathbf{x}<+\infty$$

*and*

$$\lim_{t\to\infty}2t^{\alpha}\left\|\frac{\rho(\mathbf{x}-\mathbf{z},t)}{\rho(\mathbf{x},t)}-1\right\|_{L^\infty(\mathbf{R}^n)}=|\mathbf{z}|\sqrt{B/A}, \tag{3.13}$$

*where $\alpha=(1+p^{-1}n)/2$, $B=\alpha/(n+p-2)$, and $A$ is selected by* (3.7).

**Proof.** Similarly, $E(t):=\|\rho(\mathbf{x},t)-\rho(\mathbf{x}-\mathbf{z},t)\|_{L^1(\mathbf{R}^n)}$ can be rewritten as

$$E(t)=\int\big|\hat\rho(\mathbf{y})-\hat\rho\big(\mathbf{y}-\mathbf{z}t^{-\alpha}\big)\big|\,\mathrm{d}\mathbf{y}.$$

Consider the Taylor expansion:

$$\hat\rho\big(\mathbf{y}-\mathbf{z}t^{-\alpha}\big)=\hat\rho(\mathbf{y})+\sum_{i=1}^{n}z_it^{-\alpha}\hat\rho_{\mathbf{y}_i}(\bar{\mathbf{y}}),$$

where $|\mathbf{y}-\bar{\mathbf{y}}|<t^{-\alpha}|\mathbf{z}|$. By the dominated convergence theorem again,

$$t^{\alpha}E(t)=\int\big|\mathbf{z}\cdot\nabla\hat\rho(\bar{\mathbf{y}})\big|\,\mathrm{d}\mathbf{y}\to\int\big|\mathbf{z}\cdot\nabla\hat\rho(\mathbf{y})\big|\,\mathrm{d}\mathbf{y}\quad\text{as } t\to\infty.$$

Turning to (3.13), observe from (3.7) that

$$\frac{\rho(\mathbf{x}-\mathbf{z},t)}{\rho(\mathbf{x},t)}=\frac{\hat\rho(\mathbf{y}-\mathbf{z}/t^{\alpha})}{\hat\rho(\mathbf{y})}=(1-2\varepsilon)^{-(n+p)/2}=1+(n+p)\varepsilon+\mathrm{O}\big(\varepsilon^2\big) \tag{3.14}$$

as

$$\varepsilon=\frac{\mathbf{y}\cdot\mathbf{z}t^{-\alpha}-|\mathbf{z}t^{-\alpha}|^2/2}{|\mathbf{y}|^2+A/B}\to 0.$$

For fixed $(\mathbf{z},t)$, the extreme values of $\varepsilon$ are attained when $\mathbf{y}=\lambda\mathbf{z}t^{-\alpha}/2$, where $\lambda=1\pm\sqrt{\tau^2+1}$ and $\tau:=2(A/B)^{1/2}/|\mathbf{z}t^{-\alpha}|$. Thus the range $[\varepsilon_-,\varepsilon_+]$ of $\varepsilon$ is given by

$$\varepsilon_\pm:=\big(1\pm\sqrt{\tau^2+1}\big)^{-1}=\pm\frac{1}{\tau}+\mathrm{O}\big(1/\tau^2\big) \tag{3.15}$$

as $\tau\sim t^{\alpha}\to\infty$. Combining (3.14) with (3.15) yields (3.13) as desired. □

**Remark 5** *(Slow modes).* Note that $\alpha = 1$ for $p = n$ (or $m = \frac{n-1}{n}$). Roughly speaking, the preceding lemmas show both the $L^1$ and relative $L^\infty(\mathbf{R}^n)$ differences between two Barenblatt solutions dwindle faster after a time translation than after a spatial translation if $p > n$, while the reverse is true if $p < n$ [11,24]. Thus, for $p > n$, McCann and Slepčev [46] could improve on the rates found by Dolbeault, del Pino [26] and Otto [48] by centering the mass using condition (2.5), as Carrillo and Vázquez [11] had done in the radial case. For $p < n$, we may expect the convergence order $\mathrm{O}(1/t)$ without centering the mass. This explains why we are able to obtain convergence order $\mathrm{O}(1/t)$ for $p \leqslant 2 \leqslant n$ in Theorem 1(i) and its corollary without assuming (2.5).

## 4. Newtonian potential and moments

The fundamental solution for the Laplace operator is given by

$$\phi(\mathbf{x}) := \begin{cases} -|\mathbf{x}|^{2-n}/c_n & \text{for } n \geqslant 3, \\ (2\pi)^{-1} \ln|\mathbf{x}| & \text{for } n = 2, \\ |\mathbf{x}|/2 & \text{for } n = 1, \end{cases} \tag{4.1}$$

where $c_n := (n-2)\omega_n$ and $\omega_n := 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbf{R}^n$. For any fixed $\mathbf{y} \in \mathbf{R}^n$, this means the equation

$$\Delta_{\mathbf{x}}\phi(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

is satisfied in the distributional sense.

The Newtonian potential $V(\mathbf{x})$ of a charge distribution $v(\mathbf{x})$ (i.e., of a signed Radon measure) is defined as a convolution with the fundamental solution:

$$V(\mathbf{x}) = \int_{\mathbf{R}^n} \phi(\mathbf{x}-\mathbf{y})v(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{4.2}$$

Since the fundamental solution $\phi(\mathbf{x})$ is locally integrable, the integration is well-defined as long as the density $v(\mathbf{x})$ of the Radon measure decays fast enough at infinity; for example,

$$v(\mathbf{x}) = \mathrm{O}\big(|\mathbf{x}|^{-(2+\varepsilon)}\big) \quad \text{as } |\mathbf{x}| \to \infty \tag{4.3}$$

for any $\varepsilon > 0$ will suffice, as in §9.7 of Lieb and Loss [45]. A priori, this decay rate (4.3) has nothing to do with fast diffusion. However, comparing this spatial decay to Corollary 9, we see a Newtonian potential can in fact be defined at each instant in time for any finite mass solution $u(\mathbf{x},t) = \mathrm{O}(|\mathbf{x}|^{-n-p})$ to the fast diffusion equation in the full range of nonlinearities $p > (2-n)_+$.

As was mentioned above, $L^1(\mathbf{R}^n)$ convergence rates can only be obtained by imposing additional restrictions on the initial data $0 \leqslant u_0 \in L^1(\mathbf{R}^n)$. Therefore, compact support or finiteness of certain moments have often been assumed in the literature. In this section we observe how the asymptotic behaviour of the potential $V(\mathbf{x})$ for large $|\mathbf{x}|$ is determined by the asymptotic structure plus certain moments of its density $v = \Delta V$. While this result is classical in flavor, and closely related to Hardy space theory, we were not successful at locating the precise statement we wanted elsewhere in the literature.

**Theorem 6** *(Spatial decay of Newtonian potential). Fix $\lambda, L, p > 0$ positive. Let $V(\mathbf{x})$ denote the Newtonian potential of a signed Radon measure $v(\mathbf{y})$ on $\mathbf{R}^n$, whose density satisfies*

$$|\mathbf{x}|^{n+p}\big|v(\mathbf{x})\big| < L \quad \textit{if } |\mathbf{x}| > \lambda, \quad \textit{and} \tag{4.4}$$

$$\int_{\mathbf{R}^n} |\mathbf{x}|^p\big|v(\mathbf{x})\big|\,\mathrm{d}\mathbf{x} =: M < \infty. \tag{4.5}$$

*Suppose*

$$\int \mathbf{x}^\beta v(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \tag{4.6}$$

*for each multi-index* $\beta \in \mathbf{N}^n$ *of degree* $0 \leqslant |\beta| := \beta_1 + \cdots + \beta_n < p$. *If* $p \geqslant (2-n)_+$, *there exists a constant* $C_p = C(n,p) < \infty$ *such that*

$$|\mathbf{x}|^{n+p-2}\big|V(\mathbf{x})\big| \leqslant \big(M + (n-1)L\big)C_p \quad \text{when } |\mathbf{x}| > 3\lambda. \tag{4.7}$$

**Proof.** To prove (4.7) for $n \neq 2$, decompose

$$-c_n|\mathbf{x}|^{n+p-2}V(\mathbf{x}) = |\mathbf{x}|^{n-2+p}\int_{\mathbf{R}^n}\frac{v(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}}\,d\mathbf{y} = |\mathbf{x}|^{n-2+p}\left(\int_{D_1} + \int_{D_2} + \int_{D_3}\right)\frac{v(\mathbf{y})\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{n-2}} = I_1 + I_2 + I_3,$$

into a sum of three integrals over disjoint regions $D_1 = B_r(\mathbf{0})$, $D_2 = B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbf{R}^n\colon\ |\mathbf{y}-\mathbf{x}| < r\}$ and $D_3 = \mathbf{R}^n - (B_r(\mathbf{0}) \cup B_r(\mathbf{x}))$, where $r := |\mathbf{x}|/3$. We estimate them separately.

On the main region $D_1$, we use Taylor's expansion for

$$f(\varepsilon) := \begin{cases} (1-\varepsilon)^{1-n/2} & \text{if } n \neq 2, \\ \frac{1}{2}\ln(1-\varepsilon) & \text{if } n = 2, \end{cases}$$

$$= \frac{f^{(q)}(\varepsilon_*)}{q!}\varepsilon^q + \sum_{k=0}^{q-1}\frac{f_k}{k!}\varepsilon^k, \tag{4.8}$$

where $f_k := f^{(k)}(0)$ and $\varepsilon_*/\varepsilon \in \,]0,1[$. Let $q$ be the smallest integer greater than or equal to $p$. Since

$$I_i = |\mathbf{x}|^p \int_{D_i} f\left(\frac{2\mathbf{x}\cdot\mathbf{y} - |\mathbf{y}|^2}{|\mathbf{x}|^2}\right)v(\mathbf{y})\,d\mathbf{y}, \quad i \in \{1,2,3\}, \tag{4.9}$$

the contribution from the first region is

$$I_1 = \int_{|\mathbf{y}|<r}\left(|\mathbf{x}|^p\varepsilon^q\frac{f^{(q)}(\varepsilon_*)}{q!} + \sum_{k=0}^{q-1}\sum_{j=0}^{k}\frac{f_k|\mathbf{x}|^{p-2k}}{j!(k-j)!}(2\mathbf{x}\cdot\mathbf{y})^{k-j}\big(-|\mathbf{y}|^2\big)^j\right)v(\mathbf{y})\,d\mathbf{y}, \tag{4.10}$$

with $\varepsilon = (2\mathbf{x}\cdot\mathbf{y} - |\mathbf{y}|^2)/|\mathbf{x}|^2$. In region $D_1$, we may easily check that $\varepsilon = \frac{\langle 2\mathbf{x}-\mathbf{y},\mathbf{y}\rangle}{|\mathbf{x}|^2} \in [-7/9, 2/9]$, where the maximum and minimum occur when $\mathbf{y} = \pm\mathbf{x}/3$. Monotonicity of $f^{(q)}(\varepsilon)$ on $\varepsilon < 1$ implies $|f^{(q)}(\varepsilon_*)| < |f^{(q)}(2/9)|$. Thus

$$\left|\int_{|\mathbf{y}|<r}|\mathbf{x}|^p\varepsilon^q f^{(q)}(\varepsilon_*)v(\mathbf{y})\,d\mathbf{y}\right| \leqslant 3^{p-q}\left(\frac{7}{3}\right)^q\left|f^{(q)}\left(\frac{2}{9}\right)\right|\int_{|\mathbf{y}|<r}|\mathbf{y}|^p\big|v(\mathbf{y})\big|\,d\mathbf{y}, \tag{4.11}$$

where we have used $p \leqslant q$ to estimate $|\mathbf{x}|^{p-q} \leqslant |3\mathbf{y}|^{p-q}$ and the triangle inequality to get $|\varepsilon\mathbf{x}| \leqslant |7\mathbf{y}/3|$ on $D_1$. This controls the first summand in (4.10). The remaining summands are estimated differently, depending on whether the degree $k+j$ in $\mathbf{y}$ exceeds $p$ or not.

If $k+j \geqslant p$, we use $|\mathbf{x}|^{p-k-j} \leqslant |3\mathbf{y}|^{p-k-j}$ on $D_1$ to deduce

$$\left|\int_{|\mathbf{y}|<r}|\mathbf{x}|^{p-2k}(2\mathbf{x}\cdot\mathbf{y})^{k-j}|\mathbf{y}|^{2j}v(\mathbf{y})\,d\mathbf{y}\right| \leqslant \frac{2^{k-j}}{3^{k+j-p}}\int_{|\mathbf{y}|<r}|\mathbf{y}|^p\big|v(\mathbf{y})\big|\,d\mathbf{y}. \tag{4.12}$$

On the other hand, if $k+j < p$ we observe that the vanishing moment condition (4.6) implies

$$\left|\int_{|\mathbf{y}|<r}(2\mathbf{x}\cdot\mathbf{y})^{k-j}|\mathbf{y}|^{2j}v(\mathbf{y})\,d\mathbf{y}\right| = \left|\int_{|\mathbf{y}|>r}(2\mathbf{x}\cdot\mathbf{y})^{k-j}|\mathbf{y}|^{2j}v(\mathbf{y})\,d\mathbf{y}\right| \leqslant \frac{2^{k-j}|\mathbf{x}|^{2k-p}}{3^{k+j-p}}\int_{|\mathbf{y}|>r}|\mathbf{y}|^p\big|v(\mathbf{y})\big|\,d\mathbf{y}. \tag{4.13}$$

Combining (4.10)–(4.13) yields $|I_1| \leqslant C_pM$ for $C_p$ large enough.

The remaining two integrals $I_2$ and $I_3$ take place on regions whose $p$th moment dwindles as $|\mathbf{x}| = 3r \to \infty$. We estimate them first in dimension $n \geqslant 3$. Since $|\mathbf{x}-\mathbf{y}|^{n-2} \geqslant r^{n-2}$ and $|\mathbf{y}| \geqslant r$ in region $D_3$,

$$|I_3| \leqslant \frac{|\mathbf{x}|^{n-2+p}}{r^{n-2+p}} r^p \int\limits_{|\mathbf{y}|\geqslant r} |v(\mathbf{y})| \,\mathrm{d}\mathbf{y} \leqslant 3^{n-2+p} \int\limits_{|\mathbf{y}|\geqslant r} |\mathbf{y}|^p |v(\mathbf{y})| \,\mathrm{d}\mathbf{y}.$$

In the region $D_2$, changing variables from $\mathbf{y}$ to $\mathbf{z} = \mathbf{x} - \mathbf{y}$ we find $2r \leqslant |\mathbf{x} - \mathbf{z}|$, so if $2r > \lambda$ the decay rate (4.4) yields

$$|I_2| = \left| |\mathbf{x}|^{n-2+p} \int\limits_{|\mathbf{z}|<r} \frac{v(\mathbf{x}-\mathbf{z})}{|\mathbf{z}|^{n-2}} \,\mathrm{d}\mathbf{z} \right| \leqslant (3r)^{n+p-2} \int\limits_{|\mathbf{z}|<r} \frac{L}{(2r)^{n+p} |\mathbf{z}|^{n-2}} \,\mathrm{d}\mathbf{z} = (3/2)^{n+p-2} \omega_n L/8.$$

Together, the estimates of $I_1 + I_2 + I_3$ yield (4.7) for $n \geqslant 3$.

Now consider one space dimension $n = 1$ but assume $p \geqslant 1$. The estimate for $I_1$ was carried out above, but we need to reconsider the estimates for $I_2$ and $I_3$ since $n - 2$ has changed signs. In the domains $D_2$ and $D_3$ we have $|x|^{p-1} \leqslant |3y|^{p-1}$ and $|x - y| < 4|y|$, with equality at the $y = -x/3$ boundary of $D_1$. Therefore,

$$|I_2 + I_3| = |x|^{p-1} \left| \int\limits_{|y|>r} v(y)|x-y| \,\mathrm{d}y \right| \leqslant 3^{p-1} 4 \int\limits_{|y|>r} |y|^p |v(y)| \,\mathrm{d}y \leqslant 3^{p-1} 4M.$$

Therefore, (4.7) is also valid for $n = 1$ with $p \geqslant 1$.

Finally, consider two space dimensions $n = 2$. Since $\int v(\mathbf{y}) \,\mathrm{d}\mathbf{y} = 0$, we may write,

$$2\pi |\mathbf{x}|^p V(\mathbf{x}) = \frac{|\mathbf{x}|^p}{2} \int\limits_{\mathbf{R}^2} v(\mathbf{y}) \big(\ln(|\mathbf{x} - \mathbf{y}|^2) - \ln(|\mathbf{x}|^2)\big) \,\mathrm{d}\mathbf{y} = I_1 + I_2 + I_3$$

with $I_i$ as in (4.8)–(4.9). Again $I_1$ was estimated above, but we need to examine $I_2$ and $I_3$. Changing the variable of integration from $\mathbf{y}$ to $\mathbf{z} = \mathbf{x} - \mathbf{y}$ yields

$$I_2 = |\mathbf{x}|^p \int\limits_{|\mathbf{z}|<r} v(\mathbf{x} - \mathbf{z}) \ln \frac{|\mathbf{z}|}{|\mathbf{x}|} \,\mathrm{d}\mathbf{z}.$$

Since $|\mathbf{x} - \mathbf{z}| \geqslant |\mathbf{x}| - |\mathbf{z}| \geqslant 2r$ in $D_2$, we may use (4.4) to obtain

$$|I_2| \leqslant |\mathbf{x}|^p \int\limits_{|\mathbf{z}|<r} \frac{L}{|\mathbf{x} - \mathbf{z}|^{2+p}} \ln \frac{|\mathbf{x}|}{|\mathbf{z}|} \,\mathrm{d}\mathbf{z} \leqslant \frac{3^p}{2^{2+p}} \frac{L}{r^2} \int\limits_0^r 2\pi s \ln \frac{3r}{s} \,\mathrm{d}s = \left(\frac{3}{2}\right)^{2+p} 2\pi L \int\limits_0^{1/3} t \ln \frac{1}{t} \,\mathrm{d}t \tag{4.14}$$

provided $2|\mathbf{x}|/3 = 2r > \lambda$. Turning now to $I_3$, changing variables from $\mathbf{z} = \mathbf{x} - \mathbf{y}$ to $\mathbf{w} = \mathbf{z}/(3r)$ yields

$$I_3 = |\mathbf{x}|^p \int\limits_{|\mathbf{z}|>r,\ |\mathbf{x}-\mathbf{z}|>r} v(\mathbf{x} - \mathbf{z}) \ln \frac{|\mathbf{z}|}{|\mathbf{x}|} \,\mathrm{d}\mathbf{z},$$

hence

$$|I_3| \leqslant (3r)^p L \int\limits_{|3\mathbf{w}|>1,\ |\hat{\mathbf{x}}-\mathbf{w}|>1/3} \frac{|\ln|\mathbf{w}||}{|3r(\hat{\mathbf{x}} - \mathbf{w})|^{2+p}} (3r)^2 \,\mathrm{d}\mathbf{w}, \tag{4.15}$$

where $\hat{\mathbf{x}} := \mathbf{x}/(3r)$, and $r > \lambda$ was used to invoke (4.4). Since both integrals (4.14) and (4.15) converge, $|I_2 + I_3| \leqslant C_p L$ provided $|\mathbf{x}| > 3\lambda$, establishing (4.7) for $n = 2$ and completing the proof of the theorem. □

For comparison, we exhibit the tail behaviour of the Newtonian potential for the Barenblatt profile $\hat{\rho}(\mathbf{x})$, which agrees with the Green's function $\phi(\mathbf{x})$ to leading order since its zeroth moment is normalized (2.4). Nevertheless, as in the theorem, the next asymptotic correction is a positive term of order $\mathrm{O}(1/|\mathbf{x}|^{n+p-2})$.

**Example 7** *(Barenblatt Newtonian potential).* The Newtonian potential $\hat{R} = \phi * \hat{\rho}$ of the normalized Barenblatt profile (3.7) takes the form

$$0 \leqslant \hat{R}(\mathbf{x}) - \phi(\mathbf{x}) = \frac{1}{\sqrt{B^{n+p}}} \int_{|\mathbf{x}|}^{\infty} \frac{\mathrm{d}r_1}{r_1^{n-1}} \int_{r_1}^{\infty} \left(1 + \frac{A}{Br^2}\right)^{-(n+p)/2} \frac{\mathrm{d}r}{r^{p+1}} \tag{4.16}$$

$$\leqslant \frac{B^{-(n+p)/2}}{p(n+p-2)|\mathbf{x}|^{n+p-2}}. \tag{4.17}$$

**Proof.** Since the potential $\hat{R}$ shares the spherical symmetry of the Barenblatt profile $\hat{\rho}$, we abuse notation by expressing both as functions of $r = |\mathbf{x}|$ rather than $\mathbf{x}$. The result (4.16) is obtained by integrating $\Delta \hat{R} = \hat{\rho}$ directly in spherical coordinates

$$\frac{1}{r^{n-1}} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1} \frac{\mathrm{d}\hat{R}}{\mathrm{d}r}\right) = \left(Br^2 + A\right)^{-(n+p)/2}$$

to get

$$\hat{R}'(r_1) = \frac{1}{r_1^{n-1}} \int_0^{r_1} r^{n-1} \hat{\rho}(r)\,\mathrm{d}r = \frac{1}{\omega_n r_1^{n-1}} - \frac{1}{r_1^{n-1}} \int_{r_1}^{\infty} r^{n-1} \hat{\rho}(r)\,\mathrm{d}r,$$

and then integrating again using the boundary condition

$$0 = \lim_{|\mathbf{x}| \to \infty} \hat{R}(\mathbf{x}) - \phi(\mathbf{x}).$$

The inequalities (4.16), (4.17) are obvious since the integrand varies inversely with $A > 0$. □

## 5. Newtonian potentials evolving under fast diffusion

A priori, potential theory has no relation to diffusion equations. In the preceding section we have merely observed that the structure of the potential for $|\mathbf{x}|$ large is controlled by moments of the density function. Now we consider the evolution of the Newtonian potential,

$$U(\mathbf{x}, t) = \int \phi(\mathbf{x} - \mathbf{y}) u(\mathbf{y}, t)\,\mathrm{d}\mathbf{y}, \tag{5.1}$$

when the density function $u$ is a solution of the fast diffusion equation (1.1). The monotonicity in time for any fixed $\mathbf{x} \in \mathbf{R}^n$ is obtained formally as follows:

$$\frac{\partial U}{\partial t} = \int \phi(\mathbf{x} - \mathbf{y}) u_t(\mathbf{y}, t)\,\mathrm{d}\mathbf{y} = \int \phi(\mathbf{x} - \mathbf{y}) \Delta\big(u^m(\mathbf{y}, t)\big)\,\mathrm{d}\mathbf{y} = u^m > 0. \tag{5.2}$$

Our main goals for this section are to justify the preceding formula rigorously in Proposition 10, and establish a comparison property for such potentials in Proposition 13. As mentioned already, the corresponding results were discovered earlier and independently in the porous medium setting $p < -n$ by Pierre [51], and subsequently extended to more general contexts (not quite encompassing the present one) by Dahlberg and Kenig [14–19]. We shall need a technical lemma, proved by Lee and Vázquez [44, Lemma 6.2] when $L_0 = 0$ in the tail hypothesis (2.3), and extended to the case $L_0 > 0$ by Carrillo and Vázquez [11, Lemma 5.1].

**Lemma 8** *(Infinite mass scaling limit). Suppose $u(\mathbf{x}, t)$ satisfies* (2.2), (2.3)*, with $T = BL_0^{2/(n+p)}$ and $(2pB)^{-1} = 1 - 2/(n+p)$. Then $u_\lambda(\mathbf{x}, t) := \lambda^{n+p} u(\lambda \mathbf{x}, t)$ converges in $C^\infty_{\mathrm{loc}}(Q')$ as $\lambda \to \infty$ to $v(\mathbf{x}, T + t)$, where $v(\mathbf{x}, t) := (B|\mathbf{x}|^2/t)^{-(n+p)/2}$ is the infinite mass Barenblatt, and $Q' = \{(\mathbf{x}, t) \in \mathbf{R}^{n+1} \mid \mathbf{x} \neq \mathbf{0},\ t > 0\}$.*

Its proof was based on the observation that $u_\lambda(\mathbf{x}, t)$ satisfies the same fast-diffusion equation as $u(\mathbf{x}, t)$, with initial condition tending to $v(\mathbf{x}, T) = \lim_{\lambda \to \infty} u_\lambda(\mathbf{x}, 0)$. Although Carrillo and Vázquez went on to derive fine asymptotics for the derivatives of $u$ (e.g., (5.6) but with $\varepsilon = C/|\mathbf{x}|$) we shall here be content with a simpler corollary, which asserts spatial decay of all derivatives of $u(\mathbf{x}, t)$ at the same rate as the Barenblatt.

**Corollary 9** *(Spatial decay of derivatives). Let* $u(\mathbf{x},t)$ *satisfy* (2.2)–(2.3). *To each integer* $i_0$ *and time interval* $[t_1,t_2]\subset\,]0,\infty[$ *corresponds a constant* $K<\infty$ *(depending only on* $i_o, t_1, t_2$ *and on* $u$*) such that*

$$\left|D^\beta u(\mathbf{x},t)\right| \leqslant \frac{K}{1+|\mathbf{x}|^{n+p+|\beta|}} \tag{5.3}$$

*for all* $(\mathbf{x},t)\in\mathbf{R}^n\times[t_1,t_2]$ *and multi-indices* $\beta=(\beta_1,\dots,\beta_n)\in\mathbf{N}^n$ *of order* $|\beta|:=\sum_{i=1}^n\beta_i\leqslant i_0$. *As usual,* $D^\beta u:=\partial^{|\beta|}u/\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}$.

**Proof.** Fix $\varepsilon>0$, an integer $i_0\geqslant 0$, and time interval $[t_1,t_2]\subset\,]0,\infty[$. Setting $u_\lambda(\mathbf{x},t):=\lambda^{n+p}u(\lambda\mathbf{x},t)$ and $v(\mathbf{x},t):=(B|\mathbf{x}|^2/t)^{-(n+p)/2}$, the preceding lemma yields $\lambda_0=\lambda_0(i_0,t_1,t_2,u)$ such that $\lambda\geqslant\lambda_0$ implies

$$\left|D^\beta u_\lambda(\hat{\mathbf{x}},t)-D^\beta v(\hat{\mathbf{x}},T+t)\right|\leqslant\varepsilon \tag{5.4}$$

for all $t\in[t_1,t_2]$, unit vectors $\hat{\mathbf{x}}\in\mathbf{R}^n$, and multi-indices $\beta$ of order $|\beta|\leqslant i_0$. Direct computation shows

$$D^\beta v(\mathbf{x},t)=\frac{1}{|\mathbf{x}|^{n+p+|\beta|}}D^\beta v\left(\frac{\mathbf{x}}{|\mathbf{x}|},t\right). \tag{5.5}$$

On the other hand, choosing $|\mathbf{x}|\geqslant\lambda_0$ and $\lambda:=|\mathbf{x}|$ yields

$$D^\beta u_\lambda\left(\frac{\mathbf{x}}{|\mathbf{x}|},t\right)=|\mathbf{x}|^{n+p+|\beta|}D^\beta u(\mathbf{x},t).$$

Thus (5.4) becomes

$$\left|D^\beta u(\mathbf{x},t)-D^\beta v(\mathbf{x},T+t)\right|\leqslant\varepsilon/|\mathbf{x}|^{n+p+|\beta|}, \tag{5.6}$$

which holds for all $|\mathbf{x}|\geqslant\lambda_0$ with $t\in[t_1,t_2]$ and $|\beta|\leqslant i_0$. For $|\mathbf{x}|\geqslant\lambda_0$, the triangle inequality now yields the desired bound (5.3) from (5.5)–(5.6), with

$$K=\varepsilon+(T+t_2)^{(n+p)/2}\max_{|\beta|\leqslant i_0}\sup_{|\hat{\mathbf{x}}|=1}\left|D^\beta v(\hat{\mathbf{x}},1)\right|.$$

Since $u\in C^\infty(\mathbf{R}^n\times\,]0,\infty[)$ as in [2], taking $K$ larger if necessary extends (5.3) to all $\mathbf{x}\in\mathbf{R}^n$. $\square$

**Proposition 10** *(Monotone growth of Newtonian potential). Let* $U=\phi*u$ *be the Newtonian potential of a solution* $u(\mathbf{x},t)\geqslant 0$ *to* (2.2)–(2.3). *Then*

$$\frac{\partial U}{\partial t}(\mathbf{x},t)=u^m(\mathbf{x},t)>0\quad\textit{for each }\mathbf{x}\in\mathbf{R}^n,\ t>0, \tag{5.7}$$

*where* $m=1-2/(n+p)$, *and*

$$\lim_{t\to\infty}\inf_{\mathbf{x}\in\mathbf{R}^n}U(\mathbf{x},t)=\begin{cases}0 & \textit{if } n\geqslant 3,\\ +\infty & \textit{if } n\leqslant 2.\end{cases} \tag{5.8}$$

**Proof.** Notice (2.2)–(2.3) imply $u_0\in L^1(\mathbf{R}^n)$; we normalize its mass (2.4) without loss of generality. Recall $u\in C^\infty(\mathbf{R}^n\times\,]0,\infty[)$ is strictly positive [2]. At each instant $t>0$ in time, the Newtonian potential is defined by $U=\phi*u$. Thus

$$\begin{aligned}U_t(\mathbf{x},t)&:=\lim_{h\to 0}\int_{\mathbf{R}^n}\phi(\mathbf{x}-\mathbf{y})\frac{u(\mathbf{y},t+h)-u(\mathbf{y},t)}{h}\,\mathrm{d}\mathbf{y}\\&=\lim_{h\to 0}\int_{\mathbf{R}^n}\phi(\mathbf{x}-\mathbf{y})u_t\big(\mathbf{y},\tau(\mathbf{y},h)\big)\,\mathrm{d}\mathbf{y},\end{aligned}$$

where $|\tau(\mathbf{y},h)-t|<|h|$ is provided by the mean value theorem. From Corollary 9 we discover

$$|u_t|_{(\mathbf{y},\tau(\mathbf{y},h))}=\left|\Delta\big(u^{\frac{n+p-2}{n+p}}\big)\right|_{(\mathbf{y},\tau(\mathbf{y},h))}\leqslant\frac{K_2}{1+|\mathbf{y}|^{n+p}},$$

for all $|h| < t/2$, where $K_2$ depends only on $t$ and $u$. Recalling definition (4.1) of $\phi(\mathbf{x}) \sim 1/|\mathbf{x}|^{n-2}$ or $\ln|\mathbf{x}|$, Lebesgue's dominated convergence theorem yields $U_t = \phi * u_t$ in the full range $p > (2-n)_+$ of nonlinearities. Since $u_t = \Delta(u^m)$, it remains only to show the spatial derivatives of $u^m$ decay quickly enough to justify the standard argument that $\phi * \Delta(u^m) = u^m$. Integrating twice by parts, $\Delta\phi(\mathbf{y}) = 0$ for $\mathbf{y} \neq 0$ and the explicit form (4.1) of the Green's function $\phi(\mathbf{y})$ give:

$$\phi * \Delta(u^m)|_{(\mathbf{x},t)} = \lim_{r\to 0, R\to\infty} \int\limits_{r<|\mathbf{y}|<R} \phi(\mathbf{y}) \Delta u^m(\mathbf{x}-\mathbf{y},t)\,\mathrm{d}\mathbf{y}$$
$$= u^m(\mathbf{x},t) + \lim_{R\to\infty} \int\limits_{\partial B_R^n(\mathbf{0})} \left[\phi\nabla u^m - u^m\nabla\phi\right] \cdot \frac{\mathbf{y}}{|\mathbf{y}|}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) = u^m(\mathbf{x},t).$$

Since $p > 2-n$, the last limit vanishes by Corollary 9, which asserts

$$u^m|_{(\mathbf{x}-\mathbf{y},t)} \leqslant \frac{K_0}{1+|\mathbf{x}-\mathbf{y}|^{n+p-2}} \quad \text{and} \quad \left|\nabla u^m\right|_{(\mathbf{x}-\mathbf{y},t)} \leqslant \frac{K_1}{1+|\mathbf{x}-\mathbf{y}|^{n+p-1}}.$$

Thus $U_t = \phi * u_t = \phi * \Delta(u^m) = u^m > 0$ and (5.7) is established.

To address the value of the long time limit, abuse notation by setting $\phi(|\mathbf{x}|) := \phi(\mathbf{x})$, and $\Phi(r) := \int_{B_r^n(\mathbf{0})} |\phi(\mathbf{y})|\,\mathrm{d}\mathbf{y} = r^2/|2(n-2)|$ (unless $n=2$). First consider the case with $n \geqslant 3$. Since $u(\mathbf{x},t) \to 0$ uniformly as $t\to\infty$, given $r>0$ there exists $T(r)>0$ such that $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} < \frac{1}{r\Phi(r)}$ for all $t > T(r)$. Then,

$$\left|U(\mathbf{x},t)\right| = \int \left|\phi(\mathbf{x}-\mathbf{y})\right| u(\mathbf{y},t)\,\mathrm{d}\mathbf{y} \leqslant \frac{1}{r\Phi(r)} \int\limits_{B_r^n(\mathbf{x})} \left|\phi(\mathbf{x}-\mathbf{y})\right|\mathrm{d}\mathbf{y} + \left|\phi(r)\right| \int\limits_{\mathbf{R}^n - B_r^n(\mathbf{x})} u(\mathbf{y},t)\,\mathrm{d}\mathbf{y}$$
$$\leqslant \frac{1}{r} + \frac{1}{c_n r^{n-2}}.$$

Thus for each $r>0$

$$\lim_{t\to\infty}\left|U(\mathbf{x},t)\right| \leqslant \frac{1}{r} + \frac{1}{c_n r^{n-2}}$$

uniformly in $\mathbf{x}$, hence the first part of (5.8) is obtained.

Turning to $n \leqslant 2$, again $u(\mathbf{x},t) \to 0$ uniformly as $t\to\infty$. Given $r>0$ we may therefore find $T(r)>0$ so that $\int_{B_r(\mathbf{x})} u(\mathbf{y},t)\,\mathrm{d}\mathbf{y} < 1/2$ for all $t > T(r)$ and $\mathbf{x} \in \mathbf{R}^n$. If $n=1$, then $\phi(y) = |y|/2$, and

$$\left|U(x,t)\right| \geqslant \frac{1}{2} \int\limits_{\mathbf{R}-B_r^n(x)} |x-y| u(y,t)\,\mathrm{d}\mathbf{y} \geqslant \frac{r}{2} \int\limits_{\mathbf{R}-B_r^n(x)} u(y,t)\,\mathrm{d}\mathbf{y} = \frac{r}{4}.$$

On the other hand $\phi(\mathbf{y}) = (2\pi)^{-1}\ln|\mathbf{y}|$ changes its sign at $|\mathbf{y}|=1$ when $n=2$. Let $0<\varepsilon<1<r$ be given constants. Then there exists $T(r,\varepsilon)>0$ such that $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} < \varepsilon$ and $\int_{B_r(\mathbf{x})} u(\mathbf{y},t)\,\mathrm{d}\mathbf{y} < 1/2$ for all $t > T(r,\varepsilon)$ and $\mathbf{x}\in\mathbf{R}^2$. Thus,

$$2\pi U(\mathbf{x},t) = \left(\int\limits_{|\mathbf{y}-\mathbf{x}|>r} + \int\limits_{1<|\mathbf{y}-\mathbf{x}|<r} + \int\limits_{|\mathbf{y}-\mathbf{x}|<1}\right) u(\mathbf{y},t)\ln|\mathbf{x}-\mathbf{y}|\,\mathrm{d}\mathbf{y}$$
$$\geqslant \frac{1}{2}\ln r + 0 + \varepsilon \int\limits_{B_1^n(\mathbf{x})} \ln|\mathbf{x}-\mathbf{y}|\,\mathrm{d}\mathbf{y}.$$

Since $\int_{B_1^n(\mathbf{x})} \ln|\mathbf{x}-\mathbf{y}|\,\mathrm{d}\mathbf{y}$ is finite and independent of $\mathbf{x}$, by choosing $r$ large and $\varepsilon$ small, we see (5.8) holds for $n=1,2$. $\square$

Since $\int u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = 1$, we may view $U(\mathbf{x},t)$ as a weighted average of $\phi(\mathbf{x}-\mathbf{y})$. We can easily see that this average is dominated by the value of $\phi(\mathbf{x})$ for large $|\mathbf{x}|$ since the solution is diffusive. In this sense the limits (5.8) were expected. We will also require the limiting behaviour of the Newtonian potential as $t \to 0$, to understand in what sense the Newtonian evolution achieves its initial condition.

**Lemma 11** *(Initial Newtonian potential). Let $U = \phi * u$ denote the Newtonian potential at each instant in time of a solution $u(\mathbf{x}, t)$ to* (2.2)–(2.4)*. Then $T > 0$ large enough yields a uniform tail control $u(\mathbf{x}, t) \leqslant \varrho(\mathbf{x}, 2T)$ for $(\mathbf{x}, t) \in \mathbf{R}^n \times [0, T]$ given by the modified Barenblatt* (3.5)*. Furthermore*

$$U_0(\mathbf{x}) := (\phi * u_0)(\mathbf{x}) = \lim_{t \downarrow 0} U(\mathbf{x}, t) \quad \textit{a.e. } \mathbf{x} \in \mathbf{R}^n, \tag{5.9}$$

*and the limit converges both pointwise and in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$.*

**Proof.** The preceding proposition shows $U(\mathbf{x}, t)$ increases with $t > 0$ for fixed $\mathbf{x}$, so the limit (5.9) converges pointwise; the only question is whether or not it converges to $U_0(\mathbf{x})$.

Introduce the weight function

$$w(\mathbf{x}) := \begin{cases} (1 + |\mathbf{x}|)^{2-n} & \text{if } n \neq 2, \\ \ln(2 + |\mathbf{x}|) & \text{if } n = 2. \end{cases}$$

A classical estimate based on Fubini's theorem shows continuity of the operation $\Delta^{-1} : L^1(\mathbf{R}^n, w(\mathbf{x})\, d\mathbf{x}) \to L^1_{\mathrm{loc}}(\mathbf{R}^n)$ defined by $\Delta^{-1} u := \phi * u$; see the proof of Theorem 9.7 in Lieb and Loss [45]. Using the tail condition (2.3) we shall convert the $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ convergence in (2.2) of

$$\lim_{t \downarrow 0} u(\cdot, t) = u_0(\cdot) \tag{5.10}$$

to convergence in the weighted space $L^1(\mathbf{R}^n, w(\mathbf{x})\, d\mathbf{x})$; it will then follow that (5.9) holds in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. To do this, define the infinite mass modification

$$\varrho(\mathbf{x}, t) = \begin{cases} \left( \dfrac{t}{B|\mathbf{x}|^2 - At^{(n+p)/p}} \right)^{(n+p)/2} & \text{if } |\mathbf{x}|^2 > t^{(n+p)/p} A/B, \\ +\infty & \text{otherwise,} \end{cases}$$

of the Barenblatt solution as a comparison function. For $T > BL_0^{2/(n+p)}$ and $|\mathbf{x}|$ large enough, (2.3) implies $\varrho(\mathbf{x}, T) \geqslant u_0(\mathbf{x})$. Since $\lim_{t \to \infty} \varrho(\mathbf{x}, t) = +\infty$ monotonically, and reaches its limit in finite time on compact sets, taking $T$ larger still ensures $\varrho(\mathbf{x}, T) \geqslant u_0(\mathbf{x})$ globally. The maximum principle then implies $\varrho(\mathbf{x}, T + t) \geqslant u(\mathbf{x}, t)$ on $\mathbf{R}^n$, and $\varrho(\mathbf{x}, 2T) \geqslant u(\mathbf{x}, t)$ for all $t \leqslant T$. Choose $r > 0$ so $r^2 = (3T)^{(n+p)/p} A/B$. Now $u(\cdot, t) \to u_0(\cdot)$ in $L^1(B^n_r(\mathbf{0}), d\mathbf{x})$ according to (2.2), with or without the weight $w(\mathbf{x})$. Outside the ball $B^n_r(\mathbf{0})$, every subsequence admits a sub-subsequence $u(\cdot, t_k) \to u_0(\cdot)$ which converges pointwise almost everywhere as $t_k \downarrow 0$. Since this subsequence is dominated by $\varrho(\cdot, 2T) \in L^1(\mathbf{R}^n - B^n_r(\mathbf{0}), w(\mathbf{x})\, d\mathbf{x})$ we conclude the full sequence (5.10) converges in the weighted space $L^1(\mathbf{R}^n, w(\mathbf{x})\, d\mathbf{x})$ by the dominated convergence theorem. This implies $U(\cdot, t) \to U_0$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $t \downarrow 0$. Again, a subsequence converges pointwise almost everywhere to $U_0(\cdot)$. A priori, the full limit (5.9) converged pointwise, so its value has been identified, and the proof of the lemma concluded. In dimension $n \neq 2$, both the limit and the convolution $U_0(\mathbf{x})$ are upper semicontinuous, so (5.9) holds at every point $\mathbf{x} \in \mathbf{R}^n$—not just almost everywhere. □

**Lemma 12** *(Diffusion coefficient for potential difference). For $m \in \mathbf{R}$, the function $f_m : ]0, \infty[ \to \mathbf{R}$ defined by*

$$f_m(s) = \begin{cases} m & \textit{if } s = 1, \\ (s^m - 1)/(s - 1) & \textit{otherwise,} \end{cases} \tag{5.11}$$

*is $C^\infty$-smooth and non-vanishing unless $m = 0$. Thus when $0 < m < 1$, if $u(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ are positive continuous functions, so is*

$$a(\mathbf{x}, t) = \rho^{m-1} f_m(u/\rho) = \begin{cases} m\rho^{m-1}(\mathbf{x}, t) & \textit{where } u(\mathbf{x}, t) = \rho(\mathbf{x}, t), \\ (u^m - \rho^m)/(u - \rho) & \textit{elsewhere.} \end{cases} \tag{5.12}$$

*If $(1 - \varepsilon)\rho < u < (1 + \varepsilon)\rho$ holds for some constant $\varepsilon > 0$ sufficiently small then $0 < m\rho^{m-1}(\mathbf{x}, t)/2 < a(\mathbf{x}) < 2m\rho^{m-1}(\mathbf{x}, t)$. Moreover,*

$$\rho^{2-m} \nabla a(\mathbf{x}, t) = f'_m(u/\rho) \nabla(u - \rho) + m f_{m-1}(u/\rho) \nabla \rho. \tag{5.13}$$

**Proof.** The function $f_m(s)$ defined by (5.11) is smooth for all $s > 0$, except possibly at $s = 1$. L'Hopital's rule shows continuity of $f_m(s)$ at $s = 1$. Since $f_m(s)$ is a ratio of two holomorphic functions, the singularity is removable and $f_m(s)$ is holomorphic in a neighbourhood of $s = 1$. For $s > 0$, $f_m(s)$ takes the same sign as $m$, since the sign of the denominator determines the sign of the numerator. Continuity at $s = 1$ gives an $\varepsilon > 0$ for which $|s - 1| < \varepsilon$ forces $f_m(s) \in ]m/2, 2m[$. Since $a = \rho^{m-1} f_m(u/\rho)$ with $u(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ positive and continuous functions, the first four claims of the lemma have been proved. The remaining formula (5.13) follows by straightforward differentiation of $a = \rho^{m-1} f_m(u/\rho)$ using the identity $(m - 1) f_m(s) - (s - 1) f_m'(s) = m f_{m-1}(s)$. □

The main advantage of employing the Newtonian potential results from its monotonicity in time (5.2). Since $U(\mathbf{x}, t)$ is increasing in time for any fixed $\mathbf{x} \in \mathbf{R}^n$, $U(\mathbf{x}, t)$ and $U(\mathbf{x}, t')$ form disjoint layers when $t \neq t'$. The following lemma implies that if the Newtonian potential of a solution lies between two such layers, it stays trapped between them forever.

**Proposition 13** *(Potential comparison). Let $U(\mathbf{x}, t)$ and $\widetilde{U}(\mathbf{x}, t)$ be the Newtonian potentials of two bounded solutions $u, \tilde{u} \in L^\infty(\mathbf{R}_+^{n+1})$ to* (2.2)–(2.4) (*plus* (2.5) *if $n = 1$*). *Then $U(\mathbf{x}, t)$ is continuous on the closure of the halfspace $\mathbf{R}_+^{n+1} := \mathbf{R}^n \times ]0, \infty[$, and $U(\mathbf{x}, 0) \leqslant \widetilde{U}(\mathbf{x}, 0)$ for all $\mathbf{x} \in \mathbf{R}^n$ implies $U(\mathbf{x}, t) \leqslant \widetilde{U}(\mathbf{x}, t)$ for all $t > 0$.*

**Proof.** Let $V(\mathbf{x}, t) = U(\mathbf{x}, t) - \widetilde{U}(\mathbf{x}, t)$. Then $V$ is the potential function of the difference $v = u - \tilde{u}$ at each instant in time. Proposition 10 shows it satisfies

$$\frac{\partial V}{\partial t} = u^m(\mathbf{x}, t) - \tilde{u}^m(\mathbf{x}, t) = a \Delta V, \tag{5.14}$$

where $a(\mathbf{x}, t) = (u^m(\mathbf{x}, t) - \tilde{u}^m(\mathbf{x}, t))/(u(\mathbf{x}, t) - \tilde{u}(\mathbf{x}, t))$ is positive and continuous according to Lemma 12. Viewing $a(\mathbf{x}, t)$ as frozen (independent of $V$), we may apply the maximum principle for linear parabolic equations, e.g., Friedman's Lemma 2.5 [29], to conclude that $U(\mathbf{x}, t) \leqslant \widetilde{U}(\mathbf{x}, t)$ for all $\mathbf{x} \in \mathbf{R}^n$ and $t > 0$ since $V(\mathbf{x}, 0) = U(\mathbf{x}, 0) - \widetilde{U}(\mathbf{x}, 0) \geqslant 0$. The hypotheses which remain to be verified for the maximum principle to apply are: continuity of $V(\mathbf{x}, t)$ for $t \geqslant 0$, and the existence of $\tau > 0$ for which the limit

$$\liminf_{|\mathbf{x}| \to \infty} \min_{t \in [0, \tau]} V(\mathbf{x}, t) = 0 \tag{5.15}$$

vanishes.

Recall from Lemma 11 that taking $T > 0$ large enough implies $u(\mathbf{x}, t) \leqslant \varrho(\mathbf{x}, 2T)$ for all $\mathbf{x} \in \mathbf{R}^n$ and $t \leqslant T$, where $\varrho(\mathbf{x}, t)$ is the modified Barenblatt (3.5). Since $T$ can be arbitrarily large, it suffices to establish continuity of $U(\mathbf{x}, t)$ on $\mathbf{R}^n \times [0, T[$. In fact, continuous and monotone dependence on $t \geqslant 0$ is implied by Proposition 10 and Lemma 11; we need only show $U(\mathbf{x}, t)$ is a continuous function of $\mathbf{x} \in \mathbf{R}^n$ for each fixed $t \in [0, T[$, and then invoke semicontinuity to conclude the monotone limit (5.9) agrees with $U_0(\mathbf{x})$ everywhere. Notice that $\varrho(\mathbf{x} + \mathbf{z}, 2T) \leqslant \varrho(\mathbf{x}, 3T)$ as long as the translations $|\mathbf{z}|^2 \leqslant 4AT^{n/p}(3^{n/p} - 2^{n/p})/B$ are sufficiently small. Thus

$$\lim_{\mathbf{z} \to 0} U(\mathbf{x} + \mathbf{z}, t) = \lim_{\mathbf{z} \to 0} \int_{\mathbf{R}^n} u(\mathbf{x} + \mathbf{z} - \mathbf{y}, t) \phi(\mathbf{y}) \, d\mathbf{y} = U(\mathbf{x}, t)$$

by Lebesgue's dominated convergence theorem: $p > (2 - n)_+$ implies integrability of the dominating function $|\phi(\cdot)| \min\{\varrho(\mathbf{x} - \cdot, 3T), \|u_0\|_{L^\infty(\mathbf{R}^n)}\}$.

Turning now to the uniform limit (5.15), fix $\varepsilon > 0$ such that $(2 - n)_+ < p - \varepsilon \leqslant 1 + (2 - n)_+$. Taking $T$ larger if necessary, the modified Barenblatt bound on $u$ and $\tilde{u}(\cdot, t) \leqslant \varrho(\cdot, 2T)$ for $t \in [0, T]$ implies

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{n+p-\varepsilon} \big|v(\mathbf{x}, t)\big| \leqslant \lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{n+p-\varepsilon} 2\varrho(\mathbf{x}, 2T) = 0 \quad \text{and} \tag{5.16}$$

$$M = \int_{\mathbf{R}^n} |\mathbf{x}|^{p-\varepsilon} \big|v(\mathbf{x}, t)\big| \, d\mathbf{x} \leqslant \int_{\mathbf{R}^n} |\mathbf{x}|^{p-\varepsilon} \min\big\{2\varrho(\mathbf{x}, 2T), \|v\|_{L^\infty(\mathbf{R}_+^{n+1})}\big\} \, d\mathbf{x} < +\infty. \tag{5.17}$$

Now the zeroth moment (2.4) of $v = u - \tilde{u}$ vanishes, and if $n = 1$ the first moment (or center of mass) vanishes also (2.5). Since the bounds (5.16) and (5.17) depend on $T$ but not $t$, Theorem 6 yields $\sup_{t \in [0, T]} |V(\mathbf{x}, t)| \leqslant (1 + M) C_{p-\varepsilon}/|\mathbf{x}|^{n+p-\varepsilon-2}$ for large $\mathbf{x} \in \mathbf{R}^n$; (5.15) follows, concluding the proof of the lemma. □

## 6. Relative uniform rate from potential convergence

Building on results of the preceding sections, we shall eventually prove there exist constants $S, T \geqslant 0$ such that $R(\mathbf{x}, t-T) \leqslant U(\mathbf{x}, t) \leqslant R(\mathbf{x}, t+S)$ at time $t = T$, and hence at all subsequent times. It then follows that

$$\left|U(\mathbf{x}, t) - R(\mathbf{x}, t)\right| \leqslant \left|R(\mathbf{x}, t+S) - R(\mathbf{x}, t-T)\right|, \quad \text{for all } t > T.$$

In this case the potential difference $|U - R|$ is bounded by the difference of a single potential $R$ with two different starting times. Since $\rho(\mathbf{x}, t)$ is known to converge to a Barenblatt profile, it is natural to expect $u(\mathbf{x}, t)$ to be contracted towards the same profile at the rate which two Barenblatts attract each other. This section is devoted to proving the following theorem:

**Theorem 14** *(Relative uniform rate from potential convergence). Let $U$, $R$ be the Newtonian potentials of a solution $u$ to* (2.2)–(2.4)*, and of the Barenblatt solution $\rho$, respectively. If there exist $T, S > 0$ such that*

$$R(\mathbf{x}, t-T) \leqslant U(\mathbf{x}, t) \leqslant R(\mathbf{x}, t+S) \quad \text{if } \mathbf{x} \in \mathbf{R}^n,\ t \geqslant T, \tag{6.1}$$

*then*

$$\limsup_{t\to\infty} t \left\| \frac{u(\mathbf{x}, t)}{\rho(\mathbf{x}, t)} - 1 \right\|_{L^\infty(\mathbf{R}^n)} < \infty. \tag{6.2}$$

**Proof.** The proof is subdivided into eight steps. That $1/t$ convergence of the densities corresponds to $t^{-\frac{n}{2p}(n+p-2)}$ convergence of the Newtonian potentials can be expected from the spatiotemporal scaling (6.3) of the evolution.

**Claim 1** *(Potentials converge like $t^{-\frac{n}{2p}(n+p-2)}$). Taking $A = A(n, p)$ from* (3.7)*, hypothesis* (6.1) *implies*

$$\left|U(\mathbf{x}, t) - R(\mathbf{x}, t)\right| \leqslant \frac{S+T}{((t-T)^{n/p} A)^{(n+p-2)/2}} \quad \text{if } \mathbf{x} \in \mathbf{R}^n,\ t > T.$$

**Proof of Claim 1.** From hypothesis (6.1), the monotonicity $\partial R/\partial t = \rho^m > 0$ of Proposition 10 yields

$$\begin{aligned}\left|U(\mathbf{x}, t) - R(\mathbf{x}, t)\right| &\leqslant \left|R(\mathbf{x}, t+S) - R(\mathbf{x}, t-T)\right| = (S+T)\frac{\partial R}{\partial t}\big(\mathbf{x}, \tau(t, \mathbf{x})\big) \\ &= (S+T)\rho^m\big(\mathbf{x}, \tau(t, \mathbf{x})\big) \leqslant (S+T)\rho^{\frac{n+p-2}{n+p}}(\mathbf{0}, t-T)\end{aligned}$$

for all $\mathbf{x} \in \mathbf{R}^n$, $t > T$ and some $\tau(\mathbf{x}, t) \in\, ]t-T, t+S[$. The explicit form (3.1) of $\rho$ completes the claim. Here $A = A(n, p)$ normalizes the mass of $\rho$. □

To derive a convergence rate for the density $u - \rho = \Delta(U - R)$ from the rate just established for its Newtonian potential, we need a result which allows us to take spatial derivatives. The parabolic regularity theory laid out in Ladyženskaja, Solonnikov and Ural'ceva [42] provides a key ingredient. Following the argument of Carrillo and Vázquez from the radially symmetric setting [11], we exploit the invariances of the equation by working with the family of rescaled solutions

$$u_\lambda(\mathbf{x}, t) = \lambda^{n\alpha} u\big(\lambda^\alpha \mathbf{x}, \lambda t\big), \qquad \rho_\lambda(\mathbf{x}, t) = \lambda^{n\alpha}\rho\big(\lambda^\alpha \mathbf{x}, \lambda t\big). \tag{6.3}$$

Note that $u_\lambda$ solves the same equation (1.1) as $u$ does, while the Barenblatt solution $\rho_\lambda(\mathbf{x}, t) = \rho(\mathbf{x}, t)$ is unchanged by the scaling. In fact, $\rho(\mathbf{x}, t) = \lim_{\lambda\to\infty} u_\lambda(\mathbf{x}, t)$, as can be guessed from the result at $t = 0$; see Vázquez [58] and Claim 3. This rescaling will allow us to derive asymptotic results while working on a compact subset of space-time, thereby avoiding the degeneracies at infinity which hinder the regularity theory. Ultimately, a separate argument will be supplied by Claim 7 to control the tail evolution of the solutions by comparison with retarded and advanced Barenblatts. The same tactics were implemented by Carrillo and Vázquez to separate bulk from tail behaviour in their radial argument [11].

**Claim 2** *(Rescaled potentials converge like $1/\lambda$). Denote by $U_\lambda = \phi * u_\lambda$ and $R_\lambda = R$ the Newtonian potentials of $u_\lambda$ and $\rho_\lambda$ from* (6.3). *For all $(\mathbf{x}, t)$ in the open halfspace $\mathbf{R}^{n+1}_+$, Claim* 1 *implies*

$$\lambda\big|U_\lambda(\mathbf{x},t) - R_\lambda(\mathbf{x},t)\big| \leqslant \frac{T+S}{((t-T/\lambda)^{n/p}A)^{(n+p-2)/2}} \quad \text{if } \lambda > T/t. \tag{6.4}$$

**Proof of Claim 2.** For $n \neq 2$, the fundamental solution $\phi$ of (4.1) yields

$$\begin{aligned} U_\lambda(\mathbf{x},t) &= -\lambda^{(n-2)\alpha} \int |\lambda^\alpha \mathbf{x} - \lambda^\alpha \mathbf{y}|^{2-n} u(\lambda^\alpha \mathbf{y}, \lambda t)\, \mathrm{d}(\lambda^\alpha \mathbf{y})/c_n \\ &= \lambda^{(n-2)\alpha} U(\lambda^\alpha \mathbf{x}, \lambda t). \end{aligned}$$

For two space dimensions, (2.4) implies the corresponding identity

$$\begin{aligned} U_\lambda(\mathbf{x},t) &= \frac{1}{\omega_2} \int \big[\ln|\lambda^\alpha \mathbf{x} - \lambda^\alpha \mathbf{y}| - \ln \lambda^\alpha\big] u(\lambda^\alpha \mathbf{y}, \lambda t)\, \mathrm{d}(\lambda^\alpha \mathbf{y}) \\ &= U(\lambda^\alpha \mathbf{x}, \lambda t) - (\ln \lambda)\alpha/\omega_2, \end{aligned}$$

with $\omega_2 = 2\pi$. Since the same expressions apply to $R_\lambda(\mathbf{x}, t)$, the scaling relation

$$\big|U_\lambda(\mathbf{x},t) - R_\lambda(\mathbf{x},t)\big| = \lambda^{(n-2)\alpha}\big|U(\lambda^\alpha \mathbf{x}, \lambda t) - R(\lambda^\alpha \mathbf{x}, \lambda t)\big|$$

holds for all $n \geqslant 1$. Recalling $\alpha = (n+p)/(2p)$, Claim 1 yields (6.4). □

We shall also need local gradient bounds which are uniform as $\lambda \to \infty$; these follow from the gradient limits given, for example, by the next claim (without a rate).

**Claim 3** *(Uniform scaling limit). For a solution $u(\mathbf{x}, t)$ to* (2.2)–(2.4)*, the limit* $\lim_{\lambda\to\infty} u_\lambda(\mathbf{x},t) = \rho(\mathbf{x},t)$ *of* (6.3) *converges in $C^1_{\mathrm{loc}}(\mathbf{R}^{n+1}_+)$, and uniformly in relative error*

$$\lim_{\lambda\to\infty} \left\| \frac{u_\lambda(\mathbf{x},1)}{\rho(\mathbf{x},1)} - 1 \right\|_{L^\infty(\mathbf{R}^n)} = 0. \tag{6.5}$$

**Proof of Claim 3.** From Lee and Vázquez [44], one expects $\lim_{\lambda\to\infty} u_\lambda$ to converge in $C^k_{\mathrm{loc}}(\mathbf{R}^{n+1}_+)$ for all $k \geqslant 0$. When $k = 0$, this follows from Vázquez' Theorem 21.1 [58], or Friedman and Kamin [30]. Indeed, for each $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon, u_0)$ such that $\lambda t \geqslant t_0$ yields

$$\big|u(\mathbf{x}, \lambda t) - \rho(\mathbf{x}, \lambda t)\big| \leqslant \varepsilon \rho(\mathbf{x}, \lambda t)$$

for all $\mathbf{x} \in \mathbf{R}^n$. Replacing $\mathbf{x}$ by $\lambda^\alpha \mathbf{x}$ and recalling $\rho_\lambda = \rho$ gives

$$\big|u_\lambda(\mathbf{x}, t) - \rho(\mathbf{x}, t)\big| \leqslant \varepsilon \rho(\mathbf{x}, t),$$

which is bounded uniformly by $\varepsilon\rho(\mathbf{0}, t_1)$ on $\mathbf{R}^n \times ]t_1, \infty[$ for all $\lambda > t_0/t_1$. In particular, (6.5) is established.

Since $\rho$ is bounded away from zero and infinity on the cylinder $Q = B^n_{\Upsilon+1} \times ]t_1, t_2[$ of radius $\Upsilon + 1$, the same will hold for $u_\lambda$ when $\lambda$ is large. Thus the pressure $q_\lambda(\mathbf{x}, t) := m u_\lambda(\mathbf{x}, t)^{m-1}/(m-1)$ converges uniformly on $Q$ to the smooth function $q_\infty(\mathbf{x}, t) = -\frac{m}{1-m}(At^{2\alpha} + B|\mathbf{x}|^2)/t$. Also, for $\lambda$ sufficiently large (depending on $u_0$ and $t_1$), Lee and Vázquez' Theorem 6.1 [44] assert concavity of $q_\lambda(\mathbf{x}, t) = \lambda^{1-2\alpha} q_1(\lambda^\alpha \mathbf{x}, \lambda t)$ on the ball $B^n_{\Upsilon+1} \subset \mathbf{R}^n$; in fact, they show

$$\lim_{\lambda\to\infty} \sup_{\mathbf{x}\in\mathbf{R}^n} \sup_{t\geqslant t_1} \left| \frac{\partial^2}{\partial x_i^2}(q_\lambda - q_\infty)\right|_{(\mathbf{x},t)} = 0, \quad i = 1, \ldots, n. \tag{6.6}$$

Uniform convergence of a sequence of concave functions to a smooth limit implies convergence of their gradients $\lim_{\lambda\to\infty} \nabla q_\lambda = \nabla \rho$. This convergence is uniform on the slightly smaller ball $B^n_\Upsilon$, and with a rate independent of $t$ on the interval $]t_1, t_2[$. Although it is not needed subsequently, $m u_\lambda^{m-2} \partial u_\lambda/\partial t = \partial q_\lambda/\partial t = (m-1) q_\lambda \Delta q_\lambda + |\nabla q_\lambda|^2$ converges uniformly on $B^n_\Upsilon \times ]t_1, t_2[$ by (6.6), hence the same is true of $\partial u_\lambda/\partial t$. □

From (5.7), the Newtonian potential $U_\lambda(\mathbf{x},t)$ satisfies an equation

$$\frac{\partial U_\lambda}{\partial t} = (u_\lambda)^{m-1}\Delta U_\lambda$$

which is uniformly parabolic on compact subdomains of the open halfspace $\mathbf{R}^{n+1}_+$. Thus, on the ball $B^n_\Upsilon := \{\mathbf{x}\in\mathbf{R}^n \mid |\mathbf{x}| < \Upsilon\}$ cross the interval $[t_1,t_2]\subset\,]0,\infty[$, both $u_\lambda$ and $U_\lambda$ satisfy uniformly parabolic equations and $\Delta U_\lambda = u_\lambda$. We employ the following a priori estimate from the classical regularity theory for uniformly parabolic systems:

**Claim 4.** (Ladyženskaja, Solonnikov and Ural'ceva [42].) *Let $Q = \Omega \times\, ]t_1,t_2[$ with $\Omega\subset\mathbf{R}^n$ a bounded domain, $0<t_1<t_2<\infty$, and $Q'\subset Q$ an open subset with $d = dist(Q',\partial Q) > 0$. Let $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t),\dots,u_N(\mathbf{x},t))$ be a smooth solution of a system of the form*

$$\frac{\partial \mathbf{u}}{\partial t} = a(\mathbf{x},t)\Delta\mathbf{u} + \sum_{j=1}^n \mathbf{B}_j(\mathbf{x},t)\frac{\partial\mathbf{u}}{\partial x_j}, \quad (\mathbf{x},t)\in Q,$$

*where the coefficient $a(\mathbf{x},t)$ is scalar valued and each $\mathbf{B}_j(\mathbf{x},t)$ is an $N\times N$ matrix. If*

$$0<\nu_1<a(\mathbf{x},t)<\nu_2<\infty \quad \text{when } (\mathbf{x},t)\in Q, \tag{6.7}$$

*and*

$$\bigl|\nabla a(\mathbf{x},t)\bigr| < \mu, \quad \bigl|\mathbf{B}_j(\mathbf{x},t)\bigr| < \mu \quad \text{when } (\mathbf{x},t)\in Q,\ j=1,\dots,n, \tag{6.8}$$

*then*

$$\max_{(\mathbf{x},t)\in Q'}\bigl|\nabla\mathbf{u}(\mathbf{x},t)\bigr| < C\bigl(d,\nu_1,\nu_2,\mu,\|\mathbf{u}\|_{L^\infty(Q;\mathbf{R}^N)}\bigr).$$

**Proof of Claim 4.** This claim is a special case of Theorem 4.1 in Chapter VII of Ladyženskaja, Solonnikov and Ural'ceva [42]. The original version of the theorem is written in divergence form, and is applicable to more general equations with weaker conditions on the coefficients. □

We now use this estimate to transfer the convergence order of the Newtonian potentials to their density functions.

**Claim 5** *(Relative error decays like $1/t$ locally). Let $u_\lambda(\mathbf{x},t)$ denote the rescaling* (6.3) *of a solution to* (2.2)–(2.4)*, and $\rho$ the Barenblatt of the same mass. For each $\Upsilon<\infty$,* (6.1) *implies there exists $C = C(p,\Upsilon,u_0)$ such that*

$$\left|\frac{u_\lambda(\mathbf{x},t)-\rho(\mathbf{x},t)}{\rho(\mathbf{x},t)}\right| \leqslant \frac{C}{\lambda t} \quad \text{if } |\mathbf{x}|\leqslant \Upsilon t^\alpha,\ \lambda t\geqslant 1. \tag{6.9}$$

**Proof of Claim 5.** Fix $Q = \{\mathbf{x} \mid |\mathbf{x}| < \Upsilon+2\}\times\,]0.1,2[$ and $Q' = \{\mathbf{x} \mid |\mathbf{x}| < \Upsilon+1\}\times\,]0.2,1.9[$. Set $V_\lambda(\mathbf{x},t) = \lambda(U_\lambda(\mathbf{x},t) - R(\mathbf{x},t))$, where $U_\lambda$ and $R = R_\lambda$ are the Newtonian potentials of the respective solutions $u_\lambda$ and $\rho_\lambda = \rho$. Then, (6.4) implies that $V_\lambda$ are uniformly bounded,

$$\|V_\lambda\|_{L^\infty(Q)} < (T+S)M(n,p) \quad \text{for all } \lambda\geqslant 20T,$$

and satisfy

$$\frac{\partial V_\lambda}{\partial t} = a_\lambda(\mathbf{x},t)\Delta V_\lambda \quad \text{with } a_\lambda(\mathbf{x},t) = (u_\lambda^m-\rho^m)/(u_\lambda-\rho) \tag{6.10}$$

from (5.7). Combining Lemma 12 with Claim 3, we see on the compact domain $Q$, that the coefficients $a_\lambda(\mathbf{x},t)$ are uniformly bounded away from zero and infinity for $\lambda$ large, i.e., (6.7) holds uniformly for all $a_\lambda$ with $\lambda$ large. Since $|\nabla\rho|$ and $|\nabla u_k|$ are uniformly bounded by Claim 3, the same lemma (5.13) shows $|\nabla a_\lambda|$ to be uniformly bounded hence (6.8) holds uniformly on $Q$ with some constant $\mu>0$ for $\lambda$ large.

Now we may apply Ladyženskaja et al. (Claim 4) for the scalar case with $\mathbf{B}_j = 0$ and obtain

$$\max_{(\mathbf{x},t)\in Q'}\bigl|\nabla V_\lambda(\mathbf{x},t)\bigr| < C_0$$

for some constant $C_0 = C_0(p, \Upsilon, u_0)$ depending implicitly on $T$, $S$ and $n$ through $u_0$. With respect to $\lambda$, this bound is uniform for $\lambda$ large.

Now we consider the second order derivatives. Let $\mathbf{w}_\lambda = \nabla V_\lambda$. Then, after differentiating (6.10), we obtain

$$\frac{\partial \mathbf{w}_\lambda}{\partial t} = a_\lambda(\mathbf{x}, t)\Delta \mathbf{w}_\lambda + \sum_{j=1}^{n} \mathbf{B}_j^\lambda(\mathbf{x}, t)\frac{\partial \mathbf{w}_\lambda}{\partial x_j},$$

where the $j$th column of $\mathbf{B}_j^\lambda$ is $\nabla a_\lambda$ and other elements are all zero. Therefore,

$$|\mathbf{B}_j^\lambda| = |\nabla a_\lambda| < \mu, \quad j = 1, \dots, n.$$

After applying Ladyženskaja et al. one more time, we obtain a uniform bound for the second order derivatives of $V_\lambda$ and, hence, there exists $C_1 = C_1(p, \Upsilon, \mathbf{u}_0) > 0$ such that

$$\left|\Delta V_\lambda(\mathbf{x}, \tau)\right| = \left|\lambda(u_\lambda - \rho)(\mathbf{x}, \tau)\right| < C_1$$

for all $|\mathbf{x}| \leqslant \Upsilon$, $0.3 < \tau < 1.8$ and $\lambda$ large. Taking $C_1$ larger if necessary, the inequality extends to all $\lambda > 1$. Fixing $\tau = 1$, and introducing new variables $\mathbf{y} \in \mathbf{R}^n$ and $t > 0$, the preceding formula reads

$$\begin{aligned} \frac{C_1}{\lambda t} &> \left|u_{\lambda t}(\mathbf{y}, 1) - \rho(\mathbf{y}, 1)\right| \qquad \forall\, |\mathbf{y}| \leqslant \Upsilon,\ \lambda t > 1, \\ &= t^{n\alpha}\left|u_\lambda(t^\alpha \mathbf{y}, t) - \rho(t^\alpha \mathbf{y}, t)\right| \end{aligned} \tag{6.11}$$

from the scaling relation (6.3). Now (3.1) shows

$$\rho(t^\alpha \mathbf{y}, t) \geqslant t^{-n\alpha}\left(A + B\Upsilon^2\right)^{-p\alpha} \quad \forall\, |\mathbf{y}| \leqslant \Upsilon,\ t > 0. \tag{6.12}$$

Combining (6.11)–(6.12) with the identifications $C = (A + B\Upsilon^2)^{p\alpha} C_1$ and $\mathbf{x} = t^\alpha \mathbf{y}$ yields the desired estimate (6.9). $\square$

Setting $\lambda = 1$ in Claim 5 yields the uniform bound (6.2) on growing balls $|\mathbf{x}| \leqslant \Upsilon t^\alpha$. It remains only to show $C = C(p, \Upsilon, u_0)$ can be chosen independent of $\Upsilon$ as $\Upsilon \to \infty$, by constructing a tail estimate which controls the complementary region $|\mathbf{x}| \geqslant \Upsilon t^\alpha$. This estimate relies on trapping the tails of $u_k$ between two time delayed Barenblatts, which requires the next claim, suggested by (3.8).

**Claim 6** *(Barenblatt tail separation). Let* $1 + B\Upsilon_0^2/A = 4\alpha(1 + p\alpha)$ *define* $\Upsilon_0$*, with* $p, \alpha, A, B$ *and* $\rho(\mathbf{x}, t)$ *from* (3.1). *Then*

$$\frac{\rho(\mathbf{x}, t+1)}{\rho(\mathbf{x}, t)} \geqslant 1 + \frac{p\alpha}{2t} \quad \text{if } |\mathbf{x}| \geqslant \Upsilon_0 t^\alpha,\ t \geqslant 1. \tag{6.13}$$

**Proof of Claim 6.** Fix $t \geqslant 1$. Applying the mean value theorem to (3.10) yields $t_* = t_*(\alpha, t) \geqslant t$ such that

$$\begin{aligned} \frac{\rho(\mathbf{x}, t+1)}{\rho(\mathbf{x}, t)} &= \left(1 + \frac{1}{t}\right)^{p\alpha}\left(1 + \frac{2\alpha/t_*}{1 + \frac{B}{A}|\mathbf{x}t^{-\alpha}|^2}\right)^{-p\alpha} \geqslant \left(1 + \frac{p\alpha}{t}\right)\left(1 - \frac{2p\alpha^2/t_*}{1 + \frac{B}{A}|\mathbf{x}t^{-\alpha}|^2}\right) \\ &= 1 + \frac{p\alpha}{t}\left(1 - \frac{2\alpha}{1 + \frac{B}{A}|\mathbf{x}t^{-\alpha}|^2}\left(\frac{t + p\alpha}{t_*}\right)\right) \geqslant 1 + \frac{p\alpha}{2t}. \end{aligned}$$

The first inequality follows from the convexity of $(1+s)^{\pm p\alpha} \geqslant 1 \pm p\alpha s$ on $s > -1$, and the second from $|\mathbf{x}t^{-\alpha}| \geqslant \Upsilon_0$, $t_* \geqslant t \geqslant 1$, and our choice of $\Upsilon_0$. $\square$

**Claim 7** *(Tails lie between two Barenblatts). Fix a solution* $u(\mathbf{x}, t)$ *to* (2.2)–(2.4) *which satisfies* (6.1) *and its rescalings* (6.3). *For* $\Upsilon_0$ *from* (6.13) *and* $\lambda$ *sufficiently large,*

$$\rho(\mathbf{x}, t-1) \leqslant u_\lambda(\mathbf{x}, t) \leqslant \rho(\mathbf{x}, t+1) \quad \text{if } |\mathbf{x}| \geqslant \Upsilon_0 t^\alpha,\ t \geqslant 2. \tag{6.14}$$

**Proof of Claim 7.** The upper and lower bounds (6.14) are proved similarly, using the maximum principle on a suitably chosen domain. We prove the upper bound first. Taking $t = 1$ in (6.13) yields

$$\rho(\mathbf{x},2) \geqslant (1+p\alpha/2)\rho(\mathbf{x},1) \quad \text{if } |\mathbf{x}| \geqslant \Upsilon_0 \tag{6.15}$$

$$\geqslant u_\lambda(\mathbf{x},1) \tag{6.16}$$

for $\lambda > 1$ sufficiently large, from (6.5). Furthermore, $\lambda \geqslant (1+2/(p\alpha))C$ with $C = C(p,\Upsilon_0,u_0)$ from (6.9), ensures

$$u_\lambda(\mathbf{x},t) \leqslant \begin{cases} \left(1+\dfrac{C}{\lambda t}\right)\rho(\mathbf{x},t) & \text{if } |\mathbf{x}| \leqslant \Upsilon_0 t^\alpha,\ t \geqslant 1, \\ \rho(\mathbf{x},t+1) & \text{if } |\mathbf{x}| = \Upsilon_0 t^\alpha,\ t \geqslant 1 \end{cases} \tag{6.17}$$

by (6.13). The maximum principle orders the solutions $u_\lambda(\mathbf{x},t) \leqslant \rho(\mathbf{x},t+1)$ of (2.2) on the entire outer region $|\mathbf{x}| \geqslant \Upsilon_0 t^\alpha$, $t \geqslant 1$, since this ordering holds on its boundary (6.16)–(6.17).

Turning now to the lower bound (6.14), taking $\lambda$ large enough in (6.5) yields

$$u_\lambda(\mathbf{x},2) \geqslant \rho(\mathbf{x},2)/(1+p\alpha/2)$$

$$\geqslant \rho(\mathbf{x},1) \quad \text{if } |\mathbf{x}| \geqslant \Upsilon_0 \tag{6.18}$$

from (6.15). Moreover, $\lambda \geqslant (1+2/(p\alpha))C$ in Claim 5 ensures

$$u_\lambda(\mathbf{x},t) \geqslant \begin{cases} \left(1-\dfrac{C}{\lambda t}\right)\rho(\mathbf{x},t) & \text{if } |\mathbf{x}| \leqslant \Upsilon_0 t^\alpha,\ t \geqslant 1, \\ \rho(\mathbf{x},t-1) & \text{if } |\mathbf{x}| \geqslant \Upsilon_0 (t-1)^\alpha,\ t \geqslant 2 \end{cases} \tag{6.19}$$

by (6.13). The maximum principle again orders the solutions $u_\lambda(\mathbf{x},t) \geqslant \rho(\mathbf{x},t-1)$ of (2.2) on the outer region $|\mathbf{x}| \geqslant \Upsilon_0 t^\alpha$, $t \geqslant 2$, since this ordering holds on its boundary (6.18)–(6.19). This establishes (6.14). □

**Claim 8** *(Relative error decays like* $1/t$ *uniformly). If* $u(\mathbf{x},t)$ *solves* (2.2)–(2.4)*, and satisfies* (6.1)*, then* (6.2) *holds.*

**Proof of Claim 8.** Let $u_\lambda(\mathbf{x},t)$ denote the family of rescaled solutions (6.3), and take $\Upsilon_0 = \Upsilon_0(p,n)$ from Claim 6. For $\lambda > 1$ sufficiently large, (6.14) combines with (3.8) to yield $C_1 = C_1(p,n)$ such that

$$\left|\frac{u_\lambda(\mathbf{x},t)}{\rho(\mathbf{x},t)} - 1\right| \leqslant \frac{C_1}{t} \quad \text{if } |\mathbf{x}| \geqslant \Upsilon_0 t^\alpha,\ t \geqslant 2. \tag{6.20}$$

Claim 5 extends (6.20) to all $\mathbf{x} \in \mathbf{R}^n$, but with a larger constant $C = C(p,u_0) \geqslant C_1$. Thus for all $t > 2$,

$$\frac{C}{t} \geqslant \left\|\frac{u_\lambda}{\rho}(\mathbf{x},t) - 1\right\|_{L^\infty(\mathbf{R}^n)} = \left\|\frac{u}{\rho}(\lambda^\alpha \mathbf{x},\lambda t) - 1\right\|_{L^\infty(\mathbf{R}^n)} = \left\|\frac{u}{\rho}(\mathbf{x},\lambda t) - 1\right\|_{L^\infty(\mathbf{R}^n)}$$

which completes the proof of Claim 8 and the theorem. □

## 7. Initial comparisons in special cases

Certain peculiarities make the Newtonian potential easier to handle in one space dimension (or with radial symmetry) than in the general case. These permit the desired bounds to be obtained for all $p > (2-n)_+$ without moment vanishing conditions—in particular, without appealing to Theorem 6. In such special cases, the Coulomb potential of a single point charge by itself forms a lower barrier for the Newtonian potential of any centered distribution with appropriate tails. The one-dimensional estimates are facilitated by convexity of the integral kernel $\phi(x)$, while in higher dimensions radial symmetry permits Newton's theorem to be invoked. The reader interested only in higher dimensions with non-radial data can omit the present section.

**Proposition 15** *(Initial comparison in one dimension). Fix* $n = 1$ *and* $p > 1$*, so* $\phi(x) = |x|/2$*. Let* $U_0 = \phi * u_0$ *be a potential whose density* $0 \leqslant u_0 \in L^1(\mathbf{R})$ *satisfies* (2.3)–(2.5)*. Then there exists* $C > 0$ *such that*

$$0 \leqslant U_0(x) - \phi(x) \leqslant C/|x|^{p-1} \quad \textit{for all } x \in \mathbf{R}. \tag{7.1}$$

**Proof.** Recall that

$$U_0(x) := \int \phi(y) u_0(x-y)\,\mathrm{d}y.$$

Let $\mathrm{d}\mu(y) = u_0(x-y)\,\mathrm{d}y$. Then, changing variables to $z = x - y$, the mass and center of mass normalizations (2.4)–(2.5) give

$$\int y\,\mathrm{d}\mu(y) = -\int (x-z)u_0(z)\,\mathrm{d}z = -x\int u_0(z)\,\mathrm{d}z + \int z u_0(z)\,\mathrm{d}z = -x.$$

Since $\phi(x) = |x|/2$ is convex and $\mu$ is a positive measure with unit total mass, for any $x \in \mathbf{R}$, Jensen's inequality yields

$$U_0(x) = \int \phi(y)\,\mathrm{d}\mu(y) \geqslant \phi\biggl(\int y\,\mathrm{d}\mu(y)\biggr) = \phi(x). \tag{7.2}$$

Now we consider the difference between $U_0(x)$ and $\phi(x)$ for large $|x|$. Let $V(x) = U_0(x) - \phi(x)$. Then $V(x)$ is the Newtonian potential of $v(x) := u_0(x) - \delta(x)$ and

$$V(x) = \frac{1}{2}\int |y-x| v(y)\,\mathrm{d}y = \frac{1}{2}\int (y-x)v(y)\,\mathrm{d}y + \int_{-\infty}^{x} (x-y)v(y)\,\mathrm{d}y = \int_{-\infty}^{x} (x-y)v(y)\,\mathrm{d}y \tag{7.3}$$

since $v_0$ has zero total mass and center of mass. Now $u_0(y) \leqslant \mathrm{O}(1/|y|^{p+1})$ from (2.3), so after integration (7.3) we obtain $|V(x)| = \mathrm{O}(1/|x|^{p-1})$ as $|x| \to \infty$. Thus there exist $C > 0$ and $r_0 > 0$ such that

$$V(x) < C/|x|^{p-1} \quad \text{for } |x| > r_0. \tag{7.4}$$

Since the continuous function $V(x)$ attains its maximum on the interval $|x| \leqslant r_0$, whereas $1/|x|^{p-1}$ is bounded below, taking $C$ larger if necessary extends the estimate (7.4) to all $x \in \mathbf{R}$, concluding the proof of both (7.1) and the proposition. □

Turning to spherically symmetric data in dimensions $n \geqslant 3$, we derive a similar estimate which allows us to recover the decay rate proved by Carrillo and Vázquez [11] under this symmetry hypothesis (in all dimensions). Let us begin by recalling Newton's theorem (§9.7 of Lieb and Loss [45]):

**Newton's theorem.** *Let $v \in L^1(\mathbf{R}^n)$ be a radially symmetric function with compact support* $\mathrm{spt}(v) \subset B_r(\mathbf{0})$. *Then, its Newtonian potential $V(\mathbf{x})$ satisfies*

$$V(\mathbf{x}) = \phi(\mathbf{x}) \int_{\mathbf{R}^n} v(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad |\mathbf{x}| \geqslant r,$$

$$V(\mathbf{x}) \geqslant -\bigl|\phi(\mathbf{x})\bigr| \int_{\mathbf{R}^n} \bigl|v(\mathbf{y})\bigr|\,\mathrm{d}\mathbf{y}, \quad |\mathbf{x}| \leqslant r.$$

**Proposition 16** *(Initial comparison assuming radial symmetry). Fix $n \geqslant 3$ and $p > 0$. Let $U_0 = \phi * u_0$ be the Newtonian potential of a density $0 \leqslant u_0 \in L^1(\mathbf{R}^n)$ which is radially symmetric and satisfies* (2.3)–(2.5). *Then there exists $C > 0$ such that*

$$0 \leqslant U_0(\mathbf{x}) - \phi(\mathbf{x}) \leqslant C/|\mathbf{x}|^{n+p-2} \quad \textit{for all } \mathbf{x} \in \mathbf{R}^n. \tag{7.5}$$

**Proof.** For $\lambda > 0$, let $U_\lambda(\mathbf{x}) \leqslant 0$ denote the Newtonian potential of the truncated density $\chi_{B^n_\lambda(\mathbf{0})} u_0$. Set

$$\varepsilon(\lambda) := \int_{\mathbf{R}^n - B^n_\lambda(\mathbf{0})} u_0(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Newton's theorem implies $U_\lambda(\mathbf{x}) \geqslant (1 - \varepsilon(\lambda))\phi(\mathbf{x})$ for each $\lambda > 0$ and $\mathbf{x} \in \mathbf{R}^n$. On the other hand, since

$$U_\lambda(\mathbf{x}) = \int\limits_{B^n_\lambda(\mathbf{0})} \phi(\mathbf{x}-\mathbf{y}) u_0(\mathbf{y})\,\mathrm{d}\mathbf{y} \geqslant \int\limits_{\mathbf{R}^n} \phi(\mathbf{x}-\mathbf{y}) u_0(\mathbf{y})\,\mathrm{d}\mathbf{y} = U_0(\mathbf{x}),$$

Lebesgue's monotone convergence theorem implies that $U_\lambda(\mathbf{x}) \downarrow U_0(\mathbf{x})$ as $\lambda \to \infty$. Since $\varepsilon(\lambda) \to 0$ in the same limit, we obtain the first inequality in (7.5)

$$U_0(\mathbf{x}) \geqslant \phi(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbf{R}^n.$$

Now consider the second inequality. Set $r = |\mathbf{x}|$. Then Newton's theorem implies that

$$U_r(\mathbf{x}) = \big(1 - \varepsilon(r)\big)\phi(\mathbf{x}).$$

Since $u_0(\mathbf{x}) = \mathrm{O}(1/r^{n+p})$ is assumed for $|\mathbf{x}|$ large (2.3),

$$\varepsilon(r) = \int\limits_r^\infty u_0(s) s^{n-1}\,\mathrm{d}s = \mathrm{O}\big(1/r^p\big) \quad \text{as } r \to \infty.$$

Since $\phi(\mathbf{x}) = \mathrm{O}(1/r^{n-2})$ for $n \geqslant 3$, we conclude $\varepsilon(r)\phi(\mathbf{x}) = \mathrm{O}(1/r^{n+p-2})$. Therefore, there exist $C$ and $r_0 > 0$ such that

$$U_0(\mathbf{x}) \leqslant U_r(\mathbf{x}) \leqslant \phi(\mathbf{x}) + C/r^{n+p-2} \quad \text{for } |\mathbf{x}| \geqslant r_0.$$

Furthermore, since $U_0 \leqslant 0$ from the definition $\phi(\mathbf{x}) = -|\mathbf{x}|^{2-n}/c_n$, taking $C$ larger if necessary yields

$$U_0(\mathbf{x}) - \phi(\mathbf{x}) \leqslant 1/|c_n r^{n-2}| \leqslant C/r^{n+p-2} \quad \text{for } |\mathbf{x}| \leqslant r_0,$$

to conclude the proof of (7.5). □

## 8. Convergence rates for general solutions

Elementary examples with discrete measures show the time zero comparisons of the preceding section cannot generally hold true in several dimensions. However, by allowing some time to elapse, we deduce below a leap-frog (or "tortoise and hare" type) theorem, which shows that a large enough headstart enables the Newtonian potential of any solution to overtake its competitors. This theorem requires a decay (8.1) for the initial potentials, which may be expected in view of Theorem 6. Its corollary will allow us to prove our main theorem by invoking the results of Section 6.

**Theorem 17** *(Newtonian potentials leap-frog). Fix $n \geqslant 1$ and $p > (2-n)_+$. Let $U = \phi * u$ and $\widetilde{U} = \phi * \tilde{u}$ be the Newtonian potentials at each instant in time, of two solutions $u(\mathbf{x}, t)$ and $\tilde{u}(\mathbf{x}, t)$ to* (2.2)–(2.4)*, whose initial difference satisfies the decay condition*

$$\limsup_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{n+p-2} \big|U(\mathbf{x}, 0) - \widetilde{U}(\mathbf{x}, 0)\big| < \infty. \tag{8.1}$$

*Given $T_0 > 0$, taking $T > 0$ large enough ensures*

$$U(\mathbf{x}, t) \geqslant \widetilde{U}(\mathbf{x}, T_0) \quad \textit{for all } t \geqslant T \textit{ and } \mathbf{x} \in \mathbf{R}^n. \tag{8.2}$$

**Proof.** Fix $T_0 > 0$. Hypothesis (8.1) provides $C > 0$ and $r_0 > 0$ such that

$$U(\mathbf{x}, 0) > \widetilde{U}(\mathbf{x}, 0) - C/|\mathbf{x}|^{n+p-2} \quad \text{for } |\mathbf{x}| > r_0.$$

Since $U_t = u^m$ in Proposition 10, using the continuity and bound $\varrho(\mathbf{x}, 2T_1)$ on $\tilde{u}(x, t)$ provided for $t \in [0, T_1]$ by taking $T_1 > T_0$ large enough in Lemma 11, one finds

$$\begin{aligned}\widetilde{U}(\mathbf{x}, T_0) - \widetilde{U}(\mathbf{x}, 0) &= \int_0^{T_0} \tilde{u}^m(\mathbf{x}, s)\,\mathrm{d}s \quad \text{for a.e. } \mathbf{x} \in \mathbf{R}^n,\\ &\leqslant \varrho^m(\mathbf{x}, 2T_1) \int_0^{T_0} \mathrm{d}s\\ &\leqslant T_0\left(\frac{4T_1}{B|\mathbf{x}|^2}\right)^{(n+p-2)/2} \quad \text{when } |\mathbf{x}| > r_1,\end{aligned}$$

with $r_1 = \sqrt{2(2T_1)^{(n+p)/p} A/B}$ as in (3.5). The same reasoning plus two inequalities preceding yield

$$U(\mathbf{x}, t) = U(\mathbf{x}, 0) + \int_0^t u^m(\mathbf{x}, s)\,\mathrm{d}s > \widetilde{U}(\mathbf{x}, T_0) - C_0/|\mathbf{x}|^{n+p-2} + \int_0^t u^m(\mathbf{x}, s)\,\mathrm{d}s \quad \text{a.e. } |\mathbf{x}| > r_2, \tag{8.3}$$

where $C_0 = C + T_0(4T_1/B)^{(n+p-2)/2}$ and $r_2 = \max\{r_0, r_1\}$. The remainder of the proof is devoted to showing that for $t$ large enough, the positive integral more than compensates for the negative corrector in (8.3).

Vázquez' Theorem 21.1 [58] provides a time $T_2$ such that

$$\frac{3}{2}\rho^m(\mathbf{x}, t) > u^m(\mathbf{x}, t) > \frac{1}{2}\rho^m(\mathbf{x}, t) \quad \text{for } t > T_2, \tag{8.4}$$

where

$$\rho^m(\mathbf{x}, t) = \left(At^{n/p} + B|\mathbf{x}|^2 t^{-1}\right)^{-(n+p-2)/2}.$$

Take $T > 0$ large enough that

$$\frac{1}{2}(2B)^{(2-n-p)/2} \int_{T_2}^{T} s^{(n+p-2)/2}\,\mathrm{d}s > C_0.$$

Then $|\mathbf{x}| > r := \sqrt{AT^{1+(n/p)}/B}$ implies

$$\frac{1}{2}\int_{T_2}^{T} \rho^m(\mathbf{x}, s)\,\mathrm{d}s > \frac{1}{2}\int_{T_2}^{T} \left(2B|\mathbf{x}|^2 s^{-1}\right)^{-(n+p-2)/2}\mathrm{d}s > C_0/|\mathbf{x}|^{n+p-2}. \tag{8.5}$$

Combining (8.3), (8.4) and (8.5) yields

$$U(\mathbf{x}, T) > \widetilde{U}(\mathbf{x}, T_0) \quad \text{for all } |\mathbf{x}| > \max\{r, r_2\}, \tag{8.6}$$

not just a.e. $\mathbf{x}$ since now both functions are continuous, as in Proposition 13. As $t \to \infty$, Proposition 10 asserts $U(\mathbf{x}, t) \uparrow 0$ or $U(\mathbf{x}, t) \uparrow +\infty$ strictly monotonically, the convergence being uniform on the compact set $|\mathbf{x}| \leqslant \max\{r, r_2\}$, and the value of the limit depending on dimension only. Taking $T$ larger if necessary therefore extends (8.6) to all $\mathbf{x} \in \mathbf{R}^n$, establishing (8.2) to conclude the proof of the proposition. □

**Remark 18.** Theorem 17 and its proof extend to the case where either $u$ or $\tilde{u}$ is replaced by the Barenblatt solution $\rho$, even though the initial condition (2.2) is violated in the sense that $\lim_{t\downarrow 0} \rho(\mathbf{x}, t)$ does not converge in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

**Corollary 19** *(Barenblatt sandwich). Let $U = \phi * u$ and $R = \phi * \rho$ denote the Newtonian potentials at each instant in time of a solution $u(\mathbf{x}, t)$ to* (2.2)–(2.5)*, and of the Barenblatt solution $\rho$ respectively. Under any of the additional hypotheses of Theorem* 1*, there exist constants $S, T \geqslant 0$ such that*

$$R(\mathbf{x}, t - T) \leqslant U(\mathbf{x}, t) \leqslant R(\mathbf{x}, t + S) \quad \textit{for } \mathbf{x} \in \mathbf{R}^n,\ t \geqslant T. \tag{8.7}$$

**Proof.** The densities $u(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ are bounded functions in space at each positive time $t>0$ due to the $L^1$–$L^\infty$ smoothing effect of Bénilan and Véron reviewed in Vázquez [57]; in fact $\|u(\cdot\,,t)\|_{L^\infty(\mathbf{R}^n)} \leqslant \rho(\mathbf{0},t)$ decreases to zero as $t\to\infty$. It suffices to exhibit positive constants $T_0, T, T_1 > 0$ such that

$$R(\mathbf{x},T_0) \leqslant U(\mathbf{x},T) \leqslant R(\mathbf{x},T_1) \quad \text{for } \mathbf{x}\in\mathbf{R}^n, \tag{8.8}$$

since

$$R(\mathbf{x},s) \leqslant R(\mathbf{x},s+T_0) \leqslant U(\mathbf{x},s+T) \leqslant R(\mathbf{x},s+T_1) \quad \text{for } (\mathbf{x},s)\in\mathbf{R}^{n+1}_+$$

then follows from the monotonicity $\partial R/\partial s > 0$ of Proposition 10 and the comparison principle of Proposition 13. The change of variables $s = t-T$ and $S = (T_1 - T)_+$ then yields (8.7).

To deduce (8.8), we must first check the hypotheses of Theorem 17 are satisfied. If $n \geqslant 2$ and $p > 0$, as in Theorem 1(i) and (ii), then by assumptions (2.3)–(2.5)—and (2.6) also if $p > 2$—all moments of $v_0(\mathbf{x}) := u(\mathbf{x},0) - \rho(\mathbf{x},\tau)$ up to but excluding $p$ vanish, where $\tau \geqslant 0$ is from (2.1). Thus (4.4)–(4.6) are satisfied, and the desired tail decay (8.1) of the potential $\phi * v_0$ is asserted by Theorem 6. If instead, as in Theorem 1(iii) or (iv), $n = 1 < p$, or else $v_0$ is radial but $n \geqslant 3$, the same tail decay estimate follows from Proposition 15 or 16.

Either way, fixing $T_0 > 0$ and taking $\tilde{u} = \rho$ in the remark following Theorem 17 yields a positive constant $T > 0$ such that

$$U(\mathbf{x},T) \geqslant R(\mathbf{x},T_0) \quad \text{if } \mathbf{x}\in\mathbf{R}^n.$$

Interchanging the roles of $u \leftrightarrow \tilde{u}$ in the preceding argument, yields $T_1 > 0$ such that

$$R(\mathbf{x},T_1) \geqslant U(\mathbf{x},T) \quad \text{if } \mathbf{x}\in\mathbf{R}^n.$$

This concludes the proof of (8.8) and the corollary. □

**Proof of Theorem 1.** First assume $u_0(\mathbf{x})$ has vanishing center of mass (2.5), in addition to the other requirements of Theorem 1. Combining Corollary 19 with Theorem 14 yields the desired rate of convergence (2.9). This concludes the proof of the theorem, except if $p \in\, ]1,2]$ and $n \geqslant 2$ in case (i), when we do not wish to assume (2.5). The tails (2.3) of $u_0(\mathbf{x})$ are sufficiently small to ensure its center of mass of $u_0(\mathbf{x})$ converges to some $\mathbf{z}\in\mathbf{R}^n$. Since the diffusion (2.2) commutes with translation, the preceding argument implies the ratio

$$\left\| \frac{u(\mathbf{x},t)}{\rho(\mathbf{x}-\mathbf{z},t)} - 1 \right\|_{L^\infty(\mathbf{R}^n)} \leqslant \frac{C}{t}, \quad t \gg 1,$$

converges uniformly at rate $\mathrm{O}(1/t)$ as $t\to\infty$. In the same limit, (3.13) shows the ratio $\rho(\mathbf{x}-\mathbf{z},t)/\rho(\mathbf{x},t) \to 1$ converges uniformly at rate $\mathrm{O}(1/t^\alpha)$. Since $\alpha = (n+p)/2p \geqslant 1$ in the range $1 < p \leqslant 2 \leqslant n$ we are dealing with, the triangle inequality yields

$$\begin{aligned}
\left| \frac{u(\mathbf{x},t)}{\rho(\mathbf{x},t)} - 1 \right| &\leqslant \left| \frac{u(\mathbf{x},t)}{\rho(\mathbf{x}-\mathbf{z},t)} - 1 \right| \left| \frac{\rho(\mathbf{x}-\mathbf{z},t)}{\rho(\mathbf{x},t)} \right| + \left| \frac{\rho(\mathbf{x}-\mathbf{z},t)}{\rho(\mathbf{x},t)} - 1 \right| \\
&\leqslant \frac{C}{t}\left(1 + \frac{C'}{t^\alpha}\right) + \frac{C'}{t^\alpha}, \quad t \gg 1, \\
&= \mathrm{O}(1/t) \qquad \text{as } t\to\infty.
\end{aligned}$$

The constants $C$ and $C'$ cannot depend on anything other than $p$ and $u_0$, since these determine $n$, $\mathbf{z}$ and $\rho$. □

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