Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 244 (2008) 40–51
Journal of Differential Equations
www.elsevier.com/locate/jde

Potential comparison and asymptotics in scalar conservation laws without convexity ☆

Yong-Jung Kim

Division of Applied Mathematics, KAIST (Korea Advanced Institute of Science and Technology), Guseong-dong 373-1, Yuseong-gu, Daejeon 305-701, South Korea

Received 9 June 2006; revised 6 August 2006

Available online 4 October 2006

Abstract

Two kinds of optimal convergence orders in L^1-norm to a self-similar solution are proved or conjectured for various evolutionary problems so far. The first convergence order is of the magnitude of the similarity solution itself and the second one is of order $1/t$. Employing a potential comparison technique to scalar conservation laws we may easily see that these asymptotic convergence orders are related to space and time translation of potentials. We present the technique clearly in the simple setting of scalar conservation laws in one space dimension.

Keywords: Potential comparison principle; Primitive; Conservation laws; Optimal convergence order; Asymptotics; Convection

1. Introduction

Recently the optimal convergence orders have been actively studied for evolutionary problems with nonlinear convection or diffusion. In the literature one may find two kinds of optimal convergence orders in L^1-norm. The first one is the magnitude of the source-type solution. For example the Barenblatt solution to the nonlinear diffusion equation $u_t = \Delta u^m$ has the order $O(t^{-n/\lambda})$ for t large, where n is the space dimension and $\lambda := 2 - n(1-m)$. The L^1 convergence of exactly this order can be found in various cases [2,3,8,18,24].

☆ This work was supported in part by the Korea Research Foundation Grant funded by the Korean Government (KRF-2006-312-C00035).

E-mail address: ykim@amath.kaist.ac.kr.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.08.013

On the other hand, a different kind of convergence order $O(t^{-1})$ for t large has been observed for solutions to nonlinear diffusion equations [4,13], for solutions to its linearized problems [7,25], and for solutions to scalar conservation laws [12]. Similar convergence order $O(t^{-(1-\varepsilon)})$ for any $\varepsilon > 0$ has been obtained for convection [9] and fast diffusion equations [21]. Notice that this convergence order is independent of the space dimension and the similarity structure of the problem.

In this article we apply the potential comparison technique, which has been developed for nonlinear diffusion [13]. In the simplified setting of scalar conservation laws of this paper, one can easily see that those convergence orders are related to a space and a time shift of potentials. We hope this approach gives readers an insight on the role of potentials and on the asymptotics of evolutionary equations.

The study of the solutions to the Cauchy problem of scalar conservation laws in one-space dimension,

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \quad x \in \mathbf{R},\ t > 0, \tag{1}$$

serves as a prototype of hyperbolic conservation laws. Here the flux f is assumed to be smooth without the convexity assumption. In this paper we consider the primitive of the solution,

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy, \tag{2}$$

as its potential and show the optimal convergence orders to source-type solutions in L^1-norm as the time variable t tends to infinity.

Liu [19] proved that, if the flux is convex ($f''(u) \geqslant 0$), two quantities

$$p = -\inf_x U(x,t) \quad \text{and} \quad q = p + \lim_{x\uparrow\infty} U(x,t)$$

are constant and that the asymptotic structure of the solution is decided by these invariant constants. One can also find the primitive U explicitly from the Hopf–Lax formula [10, Section 3.3]. These clearly indicate that the primitive of the solution should play the key role in the asymptotics of the problem. The potential comparison technique presented in this note shows how the structure of the convection equation is decoupled by employing the primitive as its potential. Note that the Newtonian potential was taken as a potential for the fast diffusion equations [13] to decouple the Laplace operator in the problem.

One of the main goals in the asymptotic study is to find the contraction order between two solutions. In this note we consider a positive solution with initial value $u_0(x)$ satisfying

$$u_0 \geqslant 0, \qquad \operatorname{spt}(u_0) \subset [0, L], \qquad \int_0^L u_0(x)\,dx = M > 0, \quad L > 0,$$

where constants $M, L > 0$ depend on the initial value u_0. Let $\rho(x,t)$ be the positive solution of the source-type sharing the same mass and $R(x,t)$ be its potential, i.e., $\rho(x,t)$ satisfies

$$\rho_t + f(\rho)_x = 0, \quad \rho(x,0) = M\delta(x), \quad x \in \mathbf{R},\ t > 0, \tag{3}$$

and

$$R(x,t) = \int_{-\infty}^{x} \rho(y,t)\,dy,$$

where $\delta(x)$ is the Dirac-delta measure.

The first convergence order we are going to show is that

$$\left\| u(t) - \rho(t) \right\|_1 \leqslant 2L \max_x \rho(x,t) = O\Big(\max_x \big|\rho(x,t)\big|\Big) \quad \text{as } t \to \infty. \tag{4}$$

(Here, we denote $u(t)$ for a function on $\mathbf{R}$ given by $u(t)|_x = u(x,t)$.) This convergence order is obtained in [12] under the convexity assumption employing a comparison technique between a solution and a rarefaction wave. The convergence order without a convexity assumption is a new result. However, the main contribution of this note is on the simplicity and the generality of the method.

One may expect a higher convergence order by placing the source-type solution at the correct spacial location. In fact we will see that there exists $c \in \mathbf{R}$ such that

$$\left\| u(t) - \rho_c(t) \right\|_1 = O\Big(f\Big(\max_x \big|\rho(x,t)\big|\Big)\Big) \quad \text{as } t \to \infty, \tag{5}$$

where ρ_c is the space translation given by $\rho_c(x,t) = \rho(x-c,t)$.

One of the goals of this article is to introduce a potential comparison technique in the simple setting of scalar conservation laws. In this article, the maximum potential of results is not pursued to keep the presentation simple. Extension of this method to more general cases including nonlinear diffusions is in progress.

This note consists as followings. In Section 2 several preliminary steps are constructed including the potential comparison principle. The asymptotic convergence orders in (4) and (5) are achieved in Sections 3 and 4, respectively. To show the convergence orders explicitly we apply the theory to the power law, $f(u) = u^q/q,\ q > 1$, in Section 5. In this case the convergence order in (4) corresponds to $O(t^{-1/q})$ which shows the dependence on the flux. However, the convergence order corresponding to (5) is $O(t^{-1})$ which is independent of the flux.

2. Preliminaries

The flux $f(u)$ is assumed to be smooth but not necessarily convex. Moreover, we may assume

$$f(0) = f'(0) = 0 \tag{6}$$

without loss of generality. For the simplicity we take a compactly supported positive initial value, i.e.,

$$u_0 \geqslant 0, \quad \operatorname{spt}(u_0) \subset [0,L], \quad \int_0^L u_0(x)\,dx = M > 0, \quad L > 0. \tag{7}$$

Due to the singularity property of the problem the solution is defined in a weak sense that satisfies

$$\iint \bigl(u\phi_t + f(u)\phi_x\bigr)\,dx\,dt = -\int u_0(x)\phi(x,0)\,dx \tag{8}$$

for any test function $\phi \in C_0^\infty(\mathbf{R}\times[0,\infty))$. However the weak solution is not unique and, to single out the physically meaningful one, we consider the solution that satisfies the Oleinik's entropy condition, i.e., for any point $x_0 \in \mathbf{R}$, $t > 0$,

$$l(u) \leqslant f(u) \quad \text{for all } u_l < u < u_r, \qquad \text{and} \qquad l(u) \geqslant f(u) \quad \text{for all } u_r < u < u_l, \tag{9}$$

where $l(u)$ is the linear function connecting two states u_r and u_l, i.e.,

$$l(u) = \frac{f(u_l) - f(u_r)}{u_l - u_r}(u - u_l) + f(u_l), \qquad u_l = \lim_{x\uparrow x_0} u(x,t),\ u_r = \lim_{x\downarrow x_0} u(x,t).$$

Such an entropy solution is well-posed [23] and one can easily check that, if the flux is convex, $f''(u) \geqslant 0$, the condition is equivalent to the condition

$$\lim_{x\uparrow x_0} u(x,t) \geqslant \lim_{x\downarrow x_0} u(x,t), \qquad x_0 \in \mathbf{R},\ t \geqslant 0. \tag{10}$$

Lemma 1. *For any given $t > t_0 \geqslant 0$ and $x \in \mathbf{R}$,*

$$U(x,t) = U(x,t_0) - \int_{t_0}^{t} f\bigl(u(x,s)\bigr)\,ds. \tag{11}$$

Proof. Since the wave speed is finite, for any fixed $t > 0$, there exists $x_0 \in \mathbf{R}$ such that $u(y,s) = 0$ for all $y < x_0$ and $s < t$. Let $\Omega := [x_0, x] \times [t_0, t]$ and consider the characteristic function $\phi(y,t) = \chi|_\Omega$. Since ϕ is not smooth, we may not directly apply ϕ to (8). However, using classical approximation arguments with smooth functions, $\phi_\varepsilon \to \phi$, $\mathrm{spt}(\phi_\varepsilon) \subset \Omega$, one may obtain

$$U(x,t) - U(x,t_0) = -\int_{t_0}^{t} \bigl[f\bigl(u(x,s)\bigr) - f\bigl(u(x_0,s)\bigr)\bigr]\,ds.$$

Since $u(x_0,s) = 0$ for all $t_0 \leqslant s \leqslant t$, one obtains (11). □

Remark 2. This lemma implies that the potential U is a weak solution of the following Hamilton–Jacobi equation,

$$U_t + f(U_x) = 0. \tag{12}$$

The *first* step of the potential comparison technique is to choose the potential function that may decouple the structure of the problem under consideration. The primitive in (2) will play the role of the potential to the conservation law. The *second* step is to obtain the potential comparison property.

Proposition 3 *(Potential comparison). Let $U(x,t)$ and $\tilde{U}(x,t)$ be the potentials of two integrable solutions $u,\tilde{u}$ to* (1). *If $U(x,0)\leqslant\tilde{U}(x,0)$ for all $x\in\mathbf{R}$, then $U(x,t)\leqslant\tilde{U}(x,t)$ for all $x\in\mathbf{R}$, $t>0$.*

Proof. Equation (11) implies that the potential U satisfies $U_t+f(u)=0$ in a weak sense and, hence, $E(x,t)=\tilde{U}(x,t)-U(x,t)$ is a weak solution of

$$E_t+a(x,t)E_x=0,\quad a(x,t)=\big(f(\tilde{u})-f(u)\big)/(\tilde{u}-u),$$

where $a(x,t)$ is understood as the derivative of the smooth flux if $u=\tilde{u}$. For any given point $(\xi_0,t_0)\in\mathbf{R}\times\mathbf{R}^+$, one may consider a backward characteristic $\xi(t)$ such that

$$\xi'(t)\in I(u_l,u_r),\quad \xi(t_0)=\xi_0,\quad 0<t<t_0 \text{ almost everywhere},$$

where $u_l=\lim_{x\uparrow\xi(t)}a(x,t)$, $u_r=\lim_{x\downarrow\xi(t)}a(x,t)$ and $I(u_l,u_r)$ is the closed interval having u_l,u_r as its end points. Since $E(x,0)\geqslant 0$, one clearly has $E(\xi(0)\pm,0)\geqslant 0$ for any characteristics that emanates from the point (ξ_0,t_0). Therefore, $E(\xi_0,t_0)\geqslant 0$ and hence $U(x,t)\leqslant\tilde{U}(x,t)$ for all $t>0$ and $x\in\mathbf{R}$. □

The theory of characteristics is employed in the proof of Proposition 3. For more detailed theory we refer readers to [5,6]. In the proof the convexity of the problem is not used. The only thing required is the finite speed of propagation which comes from the smoothness of the flux.

If the flux is convex, then, under the assumptions in (6), we may easily see that, for all $t>0$,

$$\min\big\{\mathrm{spt}\big(\tilde{u}(t)\big)\big\}\leqslant\min\big\{\mathrm{spt}\big(u(t)\big)\big\}\quad\text{if}\quad\min\big\{\mathrm{spt}\big(\tilde{u}(0)\big)\big\}\leqslant\min\big\{\mathrm{spt}\big(u(0)\big)\big\}$$

since the wave speed is positive, and hence the minimum of the support of a solution is not changed. For a nonconvex flux, this relation does not hold in general. However, Lemma 3 immediately provides some useful information regarding evolution of support of solutions.

Corollary 4 *(Evolution of supports). Let $u,\tilde{u}$ satisfy* (1). *If*

$$\int_{-\infty}^{x}\big(\tilde{u}(y,0)-u(y,0)\big)\,dy\geqslant 0\quad\textit{for all } x\in\mathbf{R},$$

then $\min\{\mathrm{spt}(\tilde{u}(t))\}\leqslant\min\{\mathrm{spt}(u(t))\}$ *for all* $t\geqslant 0$.

Proof. Let $c=\min\{\mathrm{spt}(u(t))\}$ and $\tilde{c}=\min\{\mathrm{spt}(\tilde{u}(t))\}$. If $c<\tilde{c}$, then $U(x,t)>\tilde{U}(x,t)$ for $x\in(c,\tilde{c})$ which contradicts to Lemma 3. Therefore, $\tilde{c}\leqslant c$ for all $t>0$. □

The convergence order between two primitives can be transferred to their derivatives. One may find such regularity property from Ladyženskaja et al. [16] (see Theorem 4.1 in Chapter VII). In the following lemma we obtain similar result using the diminishing property of the number of intersection points between two solutions. The proof depends on the fact that $\rho(t)-u(t)$ changes its sign only once. For the convex case the Oleinik inequality implies that ρ is the steepest one and hence one can easily show that there is only one sign-changing point. In the following proof we employ the diminishing property of lap numbers of uniformly parabolic problems to show the uniqueness of the sign-changing point for a general flux without the convexity.

Lemma 5. *Let $u(x,t)$ be the solution to the Cauchy problem* (1), (7) *and ρ be the source-type solution* (3)*. Then,*

$$\left\|\rho(t)-u(t)\right\|_1 = 2\left\|R(t)-U(t)\right\|_\infty.$$

Proof. It is well known that the solution u^ε that satisfies the viscous problem,

$$u_t^\varepsilon + f\left(u^\varepsilon\right)_x = \varepsilon u_{xx}^\varepsilon, \qquad \lim_{t\downarrow 0} u^\varepsilon(x,t) = u_0(x),$$

converges to the solution u of the inviscid problem (1) as $\varepsilon \to 0$. Let $e^\varepsilon := \rho^\varepsilon - u^\varepsilon$, where ρ^ε and u^ε are the solutions to the viscous problem that converges to ρ and u, respectively. Then, e^ε satisfies

$$e_t^\varepsilon = \varepsilon e_{xx}^\varepsilon + f\left(\rho^\varepsilon\right)_x - f\left(u^\varepsilon\right)_x \left(= \varepsilon e_{xx}^\varepsilon + \frac{f(\rho^\varepsilon)_x - f(u^\varepsilon)_x}{\rho_x^\varepsilon - u_x^\varepsilon} e_x \right). \tag{13}$$

Let U^ε and R^ε be the potentials (or primitives) of u^ε and ρ^ε, respectively, and $E^\varepsilon = R^\varepsilon - U^\varepsilon$. Then, integrating the above relation on $(-\infty, x)$ gives the relation for E^ε, which is

$$E_t^\varepsilon = \varepsilon E_{xx}^\varepsilon + \frac{f(\rho^\varepsilon) - f(u^\varepsilon)}{\rho^\varepsilon - u^\varepsilon} E_x. \tag{14}$$

Employing the theory of intersection comparison (see [11, Chapter 1]) or of lap number (see [22]), we may conclude that the number of sign changes is at most once since $\lim_{t\downarrow 0}\rho(x,t) = \delta(x)$ and $\operatorname{spt}(u_0) \subset [0,L]$. Let $x^\varepsilon(t)$ be the sign-changing point of $e^\varepsilon = \rho^\varepsilon - u^\varepsilon$. Then, clearly,

$$\begin{aligned}
\left\|u(t)-\rho(t)\right\|_1 &= \int \left|u(x,t)-\rho(x,t)\right| dx = \lim_{\varepsilon\downarrow 0} \int \left|u^\varepsilon(x,t) - \rho^\varepsilon(x,t)\right| dx \\
&= 2\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{x^\varepsilon} \left[\rho^\varepsilon(x,t) - u^\varepsilon(x,t)\right] dx = 2\lim_{\varepsilon\downarrow 0}\left\|U^\varepsilon(t) - R^\varepsilon(t)\right\|_\infty \\
&= 2\left\|U(t)-R(t)\right\|_\infty. \qquad \square
\end{aligned}$$

3. Convergence order of the similarity scale

The next step, which is the *third* one, is to trap the potential U between R and its translation. In this section we take a space translation of R and show the convergence order in (4).

Lemma 6 *(Trapped!). Let u be the entropy solution of* (1), (7) *and ρ be the canonical solution* (3)*. Then, for all $x \in \mathbf{R}$, $t > 0$,*

$$R(x-L,t) \leqslant U(x,t) \leqslant R(x,t), \tag{15}$$

$$\min\{\operatorname{spt}(\rho(t))\} \leqslant \min\{\operatorname{spt}(u(t))\}. \tag{16}$$

Proof. Since $\rho(x,0)=\delta(x)$, $R(x,0)$ is the Heaviside step function

$$R(x,0)=\begin{cases} 0, & \text{if } x<0, \\ 1, & \text{if } x \geqslant 0. \end{cases}$$

Therefore the restrictions in (7) imply that

$$R(x-L,0) \leqslant U(x,0) \leqslant R(x,0).$$

Proposition 3 implies that (15) holds for all $t>0$. The estimate (16) comes from Corollary 4. □

The *fourth* step is to compute the decay rate of the potential difference which comes directly from the estimate (15):

Lemma 7 *(Convergence rate of potentials). Under the same conditions as in Lemma* 6,

$$\|U(t)-R(t)\|_\infty \leqslant L \max_x \rho(x,t). \tag{17}$$

Proof. Using the comparison inequality (15), one obtains

$$|U(x,t)-R(x,t)| \leqslant |R(x-L,t)-R(x,t)| = \int_{x-L}^{x} \rho(y,t)\,dy \leqslant L \max_x \rho(x,t).$$

The right-hand side is independent of the point x and hence the estimate is uniform. □

The last step is to transfer the decay order of the potential difference to the convergence order of the general solution u to the source-type solution ρ in L^1 sense. This step is already obtained in Lemma 5 and, hence, the following theorem on the convergence rate immediately follows.

Theorem 8. *Let $u(x,t)$ be the solution to the Cauchy problem* (1), (7) *and ρ be the source-type solution* (3). *Then,*

$$\|u(t)-\rho(t)\|_1 \leqslant 2L \max_x \rho(x,t) \quad \text{as } t \to \infty. \tag{18}$$

Employing the potential comparison technique makes the proof simple without using the theory of characteristics which is rather complicate for the nonconvex case. In the proof of Proposition 3 only the finite speed of characteristics is used. One may also obtain the comparison property by employing the maximum principle to (14) and then taking the zero viscosity limit.

Remark 9. The asymptotic convergence in the Wasserstein metric has been shown for solutions to scalar conservation laws (1) in [1]. Their method is also based on the primitives of solutions and the corresponding convergence order in the L^1 norm is the one in (18). This result is under a general convexity assumption and the technique is based on the Hopf–Lax formula for the Hamilton–Jacobi equation (12).

4. Convergence order beyond the similarity scale

In this section we show the convergence order in (5). The main idea is to estimate the potential difference $U - R_c$ using R and its time translation such as

$$\left\|U(t) - R_c(t)\right\|_\infty \leqslant \left\|R(t) - R(t+T)\right\|_\infty,$$

where $R_c(x,t) = R(x-c,t)$. Suppose that $u = 0$ is not a limit point of the inflection points of the flux $f(u)$. Then there exists $u_1 > 0$ such that there is no inflection point on the interval $(0, u_1)$. Since the solution decays to zero, there exists $S > 0$ such that $\rho(x,t), u(x,t) < u_1$ for all $t > S$. By taking $T := T + S$ if needed, the convexity assumption is acceptable for the estimate using a time translation in this section. Therefore, we assume that the flux is convex in this section, i.e.,

$$f''(u) \geqslant 0. \tag{19}$$

Under the convexity hypothesis, $f'(u)$ is an increasing function and one may consider the profile $u = g(x)$ defined uniquely by the relation

$$g(0) = 0, \quad f'(g(x)) = x, \quad x \in \mathbf{R}. \tag{20}$$

One may easily check that $g(x)$ is also an increasing function and rarefaction waves are given by $u(x,t) = g((x - x_0)/(t + t_0))$ for some constants $x_0 \in \mathbf{R}$, $t_0 \geqslant 0$.

It is well known that the positive source-type solution ρ is given explicitly by

$$\rho(x,t) = \begin{cases} g(x/t), & 0 \leqslant x \leqslant b(t), \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $b(t) > 0$ satisfy

$$M = \int_0^{b(t)} g(y/t)\,dy. \tag{22}$$

One can easily check that $\rho(x,t)$ satisfies Eq. (1) at a regularity point and the entropy condition (10) at the unique singularity point $x = b(t)$.

Lemma 10 *(Trapped!). Let u be the entropy solution of* (1), (7)*, ρ be the canonical solution* (21)*, and the flux $f(u)$ be convex. Let U and R be the potentials of u and ρ, respectively, and $c =$* $\min(\operatorname{spt}(u_0))$*. If there exist $\varepsilon, t_0 > 0$ that satisfy*

$$R(x - c, t_0) \leqslant U(x, 0), \quad c < x < c + \epsilon, \tag{23}$$

then there exists $T > 0$ such that

$$R(x - c, t + T) \leqslant U(x,t) \leqslant R(x - c, t) \quad \textit{for all } t > 0,\ x \in \mathbf{R}. \tag{24}$$

Proof. We may assume $c = 0$ after a translation. From the explicit formula (21) we have

$$R(L,t) = \int_0^L g(x/t)\,dx = t\int_0^{L/t} g(y)\,dy \leqslant g(L/t)L \to 0 \quad \text{as } t \to \infty.$$

Therefore, there exists $T > t_0$ such that $R(L,T) \leqslant U(\varepsilon,0)$. Furthermore, since $U(x,t)$ and $R(x,t)$ are increasing functions in x variable, $U(x,0) = 1$ for all $x \geqslant L$ and $R(x,T) \leqslant 1$, we obtain $R(x,T) \leqslant U(x,0)$ for all $x > 0$, which complete our initial comparison

$$R(x,T) \leqslant U(x,0) \leqslant R(x,0).$$

Therefore, Proposition 3 implies (24) for all $t > 0$. □

The *fourth* step is to compute the decay rate of the potential difference which comes directly from the estimate (24):

Lemma 11 *(Convergence rate of potentials). Under the same conditions as in Lemma* 10*, there exists* $T > 0$ *such that*

$$\left\|U(t) - R_c(t)\right\|_\infty \leqslant Tf\Big(\max_x \rho(x,t)\Big). \tag{25}$$

Proof. Using the comparison inequality (24) and the evolution equation for potentials (11), we obtain

$$\begin{aligned}\left|U(x,t) - R(x-c,t)\right| &\leqslant \left|R(x-c,t+T) - R(x-c,t)\right| \\ &= \int_t^{t+T} f\big(\rho(x-c,s)\big)\,ds \leqslant Tf\Big(\max_x \rho(x,t)\Big).\end{aligned}$$

Since the right-hand side is independent of $x \in \mathbf{R}$, the estimate is uniform. □

The last step is to transfer the decay order in (25) to the convergence order of the solution u, which is already done in Lemma 5. Therefore, the convergence order immediately follows.

Theorem 12. *Let u be the entropy solution of* (1), (7)*, ρ be the canonical solution* (21)*, and the flux $f(u)$ be convex. Let U and R be the potentials of u and ρ, respectively, and* $c = \min(\operatorname{spt}(u_0))$*. If there exist* $\epsilon, t_0 > 0$ *that satisfies* (23)*, then there exists* $T > 0$ *such that*

$$\left\|u(t) - \rho_c(t)\right\|_1 \leqslant 2Tf\Big(\max_x \rho(x,t)\Big). \tag{26}$$

Remark 13. The convergence order (18) is based on the fact that u and ρ share the same total mass which is preserved. On the other hand, the order (26) has been obtained after placing ρ at the correct spacial location. It seems that the center of mass is controlled asymptotically if ρ is located at $c = \min(\operatorname{spt}(u_0))$.

Remark 14. Condition (23) on the initial value u_0 is a necessary one which corresponds to condition (15) in [12]. Notice that rarefaction waves are given by $g(x/t)$ and become flatter as the they are getting older, i.e., as $t \to \infty$. Condition (23) implies that the initial value $u_0(x)$ is steeper than $g(x/t_0)$ on the interval $c < x < c + \varepsilon$ and one may roughly say that the initial value is younger than the age of t_0. It is natural to ask if one may improve the convergence order beyond $O(1/t)$ by considering space and time shifts together, i.e.,

$$\left\|u(t) - \rho_{c,k}(t)\right\|_1 = O\left(t^{-\alpha}\right) \quad \text{as } t \to \infty,$$

where $\rho_{c,k}(x,t) = \rho(x - c, t + k)$. This kind of approach has been made for linearized problems by setting the variance using the extra time shift (see [25]) or controlling higher moments (see [7,14]). However the variance and higher moments are not conserved for nonlinear problems and any higher convergence order is not known.

5. Explicit computations of convergence orders

As a simplified model the power law is commonly considered:

$$f(u) = u^q/q, \quad q > 1. \tag{27}$$

Then the rarefaction profile is simply given by $g(x) = (f')^{-1}(x) = \sqrt[q-1]{x}$. The convergence to source-type solution is well studied in [20]. In this case we can compute the convergence orders in previous sections explicitly. First the positive source-type solution (or a positive N-wave), is given by

$$\rho(x,t) = \begin{cases} \sqrt[q-1]{x/t}, & 0 \leqslant x \leqslant (\frac{qM}{q-1})^{\frac{q-1}{q}} \sqrt[q]{t}, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Substituting the end point of support we can easily check that

$$\max_x \left|\rho(x,t)\right| = \left((q-1)t/(qM)\right)^{-1/q}. \tag{29}$$

Therefore, the convergence order in Theorem 8 corresponds to

$$\left\|u(t) - \rho(t)\right\|_1 \leqslant 2L\left((q-1)t/(qM)\right)^{-1/q} = O\left(t^{-1/q}\right) \quad \text{as } t \to \infty.$$

For the well-known Burgers equation case, $f(u) = u^2/2$, this estimate gives the well-known result of convergence order $1/\sqrt{t}$, [17], i.e.,

$$\left\|u(t) - \rho(t)\right\|_1 \leqslant 2L\left(t/(2M)\right)^{-1/2} = O\left(t^{-1/2}\right) \quad \text{as } t \to \infty.$$

On the other hand, the convergence order in Theorem 12 corresponds to

$$\left\|u(t) - \rho_c(t)\right\|_1 \leqslant \frac{2TM}{q-1} t^{-1} = O\left(t^{-1}\right) \quad \text{as } t \to \infty,$$

where the convergence order $O(t^{-1})$ is independent from the power of the flux.

The convergence order in (18) seems natural since the order of the magnitude of the source-type solution depends on the flux. However, the convergence order in (26) is independent of the flux at least for the power law case. One may guess the convergence order should be $O(t^{-1})$ for a more general flux, but we could not show that.

Acknowledgments

A.E. Tzavaras suggested the author to use the primitive of the solution in the study of metastability in the Burgers equation which was a part of his PhD thesis [15]. The suggestion is finally comes to light in this article. The author is also grateful to an anonymous referee for the suggestions that improved this article.

References

[1] J.A. Carrillo, M. Di Francesco, C. Lattanzio, Contractivity of Wasserstein metrics and asymptotic profiles for scalar conservation laws, J. Differential Equations, in press.
[2] J.A. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani, A. Unterreiter, Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities, Monatsh. Math. 133 (2001) 1–82.
[3] J.A. Carrillo, G. Toscani, Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity, Indiana Univ. Math. J. 49 (2000) 113–141.
[4] J.A. Carrillo, J.L. Vázquez, Fine asymptotics for fast diffusion equations, Comm. Partial Differential Equations 28 (2003) 1023–1056.
[5] C.M. Dafermos, Regularity and large time behaviour of solutions of a conservation law without convexity, Proc. Roy. Soc. Edinburgh Sect. A 99 (1985) 201–239.
[6] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Grundlehren Math. Wiss., vol. 325, Springer-Verlag, 2000, xvi+443 pp.
[7] J. Denzler, R.J. McCann, Fast diffusion to self-similarity: Complete spectrum, long time asymptotics, and numerology, Arch. Ration. Mech. Anal. 175 (2005) 301–342.
[8] J. Dolbeault, M. del Pino, Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions, J. Math. Pures Appl. 81 (2002) 847–875.
[9] J. Dolbeault, M. Escobedo, L^1 and L^∞ intermediate asymptotics for scalar conservation laws, Asymptot. Anal. 41 (2005) 189–213.
[10] L.C. Evans, Partial Differential Equations, Grad. Stud. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1998, xviii+662 pp.
[11] V.A. Galaktionov, Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications, Chapman and Hall/CRC, Boca Raton, FL, 2004.
[12] Y.-J. Kim, Asymptotic behavior of solutions to scalar conservation laws and optimal convergence orders to N-waves, J. Differential Equations 192 (1) (2003) 202–224.
[13] Y.-J. Kim, R.J. McCann, Potential theory and optimal convergence rates in fast nonlinear diffusion, J. Math. Pures Appl. 86 (1) (2006) 42–67.
[14] Y.-J. Kim, W.-M. Ni, Higher order approximations in the heat equation and the truncated moment problem, preprint.
[15] Y.-J. Kim, A.E. Tzavaras, Diffusive N-waves and metastability in the Burgers equation, SIAM J. Math. Anal. 33 (3) (2001) 607–633.
[16] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type (in Russian). Translated from the Russian by S. Smith, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1967.
[17] P.D. Lax, Hyperbolic systems of conservation laws. II, Comm. Pure Appl. Math. 10 (1957) 537–566.
[18] C. Lederman, P.A. Markowich, On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass, Comm. Partial Differential Equations 28 (2001) 301–332.
[19] T.-P. Liu, Invariants and asymptotic behavior of solutions of a conservation law, Proc. Amer. Math. Soc. 71 (1978) 227–231.
[20] T.-P. Liu, M. Pierre, Source-solutions and asymptotic behavior in conservation laws, J. Differential Equations 51 (1984) 419–441.

[21] R.J. McCann, D. Slepčev, Second-order asymptotics for the fast-diffusion equation, Int. Math. Res. Not. 24947 (2006) 1–22.
[22] H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (2) (1982) 401–441.
[23] O.A. Oleinik, Existence of solutions for a simple hyperbolic equation, Amer. Math. Soc. Transl. Ser. 2 285 (1965) 33.
[24] F. Otto, The geometry of dissipative evolution equations: The porous medium equation, Comm. Partial Differential Equations 26 (2001) 101–174.
[25] T.P. Witelski, A.J. Bernoff, Self-similar asymptotics for linear and nonlinear diffusion equations, Stud. Appl. Math. 100 (2) (1998) 153–193.