Invariance Property of a Conservation Law without Convexity

MIJOUNG KIM & YONG-JUNG KIM

ABSTRACT. The main goal of this paper is to investigate the mechanism of a conservation law that gives the N-wave like asymptotics. It turns out that the positivity of the flux function provides a certain invariance of solution which singles out the right asymptotics among two parameter family of N-waves. Two kinds of long time asymptotic convergence orders in L^1-norm to this N-wave are proved using a potential comparison technique. The first one is of the magnitude of the N-wave itself and the second one is of order $1/t$. We observe that these asymptotic convergence orders are related to space and time translations of potentials.

1. INTRODUCTION

This paper is devoted to the study of the long time asymptotics of bounded L^1 solutions to a general scalar conservation law,

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where the initial value $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and is compactly supported. We assume that the flux is continuously differentiable and that

$$f(0) = f'(0) = 0. \tag{1.2}$$

One may get this normalization assumption without loss of generality after a suitable change of variables. In this paper we consider a non-convex flux that satisfies

Indiana University Mathematics Journal ©, Vol. 58, No. 2 (2009)

the following three hypotheses:

$$\text{(H)} \qquad \begin{cases} f(u) \geq 0 \quad \text{for all } u \in \mathbb{R}, \\ f(u) \quad \text{has a finite number of inflection points}, \\ f(u)/|u| \to \infty \quad \text{as } |u| \to \infty. \end{cases}$$

Notice that the first one implies that $u = 0$ is a global minimum of the flux. This restriction to the flux is essential to obtain the invariance to be studied in this paper. The other two are rather technical. The second one has been used to construct fundamental solutions in [13, 14]. If one considers a bounded solution, the value of the flux at $|u|$ near infinity does not make any difference and therefore one may assume the third one without any loss of generality.

It is well known that N-waves are long time asymptotics of sign-changing solutions to a conservation law with a convex flux. In this paper we will see that the first hypothesis in (H) is the feature that makes a conservation law produce this long time behavior. On the other hand the asymptotic convergence order depends on the structure of initial value. One can not expect any L^1 convergence order with the generality of L^1 initial value. In this paper we consider compactly supported initial value such that

$$(1.3) \qquad -\infty < \int u_0(x)\,\mathrm{d}x = M < \infty, \quad \operatorname{spt}(u_0) \subset [-L, L],\ L \in \mathbb{R}.$$

(We reserve the letter $L > 0$ to denote the support of the initial value.) Under the positivity hypothesis in (H) the well-posedness of fundamental or source-type solutions has been shown by Liu and Pierre [19]. One can easily find an explicit formula of an N-wave for a convex case. For a non-convex case the structure of a fundamental solution is more complicate and shows interesting dynamics. Fundamental solutions for a non-convex case have been recently studied in [13, 14] under the assumptions in (H). N-waves are two-parameter families of special solutions and we denote them by $n_{p,q}(x,t)$. The N-waves satisfy $n_{p,q}(x,t) \geq 0$ for $x \geq 0$ and $n_{p,q}(x,t) \leq 0$ for $x \leq 0$. The two parameters p and q are given by the relations

$$(1.4) \qquad p = -\lim_{t\to 0} \int_{-\infty}^{0} n_{p,q}(y,t)\,\mathrm{d}y, \quad q = \lim_{t\to 0} \int_{0}^{\infty} n_{p,q}(y,t)\,\mathrm{d}y,$$

where the integrals are constant for all $t > 0$ due to the invariance property studied in this paper.

In this paper we show two kinds of convergence orders of a general solution $u(x,t)$ to the N-wave $n_{p,q}(x,t)$. First we show that if the initial value satisfies

$$(1.5) \qquad p = -\inf_{x} \int_{-\infty}^{x} u_0(y)\,\mathrm{d}y > 0, \quad q = M + p > 0,$$

the solution $u(x,t)$ converges to the N-wave $n_{p,q}(x,t)$ as $t \to \infty$ with the convergence order

$$\|u(t) - n_{p,q}(t)\|_1 = O(\|n_{p,q}(t)\|_\infty) \quad \text{as } t \to \infty. \tag{1.6}$$

One may expect a higher convergence order by placing the N-wave at the correct location. In fact under an extra condition on the initial value (2.8), we will show that there exists $c \in \mathbb{R}$ such that

$$\|u(t) - n^c_{p,q}(t)\|_1 = O(\|f(n_{p,q}(t))\|_\infty) \quad \text{as } t \to \infty, \tag{1.7}$$

where $n^c_{p,q}$ is a space translation $n^c_{p,q}(x,t) = n_{p,q}(x-c,t)$. This convergence order turns out to be order $O(1/t)$ under a general assumption

$$\liminf_{u\to 0} \frac{uf'(u)}{f(u)} = \gamma > 1. \tag{H1}$$

Similar convergence orders in L^1-norm can be found from the literature. For example the Barenblatt-type solution is a source solution of a nonlinear diffusion equation and decays with certain order depending on the dimension and the flux. The L^1 convergence of exactly this order can be found in various cases [3, 4, 9, 17, 22]. The convergence order $O(1/t)$ has been obtained for radial solutions, for solutions to fast diffusion equations [5, 15, 20, 23] and for solutions to its linearized problems [7, 24].

For solutions to scalar conservation laws the L^1 convergence order in (1.6) has been observed for convex cases [2, 8, 11, 16, 25]. Convergence order of (1.7) are found in [10, 11]. For the case with a non-convex flux one can find well-posedness and other estimates from [1, 6, 26]. However, the behavior of the solution is not well understood. Recently N-waves for the non-convex case has been suggested in [13] and the convergence orders in (1.6), (1.7) have been obtained for positive solutions [12].

The rest of the paper consists as follos. In Section 2 several preliminaries are given including the definition of entropy solutions, their potentials and the potential comparison principle. The main results are given in Theorem 2.1 which consists of three parts. In the succeeding three sections each of these three are proved. In Section 3 the invariance property of conservation laws is shown under hypotheses in (H). The convergence orders in (1.6) and (1.7) are obtained in Sections 4 and 5, respectively.

2. Preliminaries and Main Results

We consider a weak solution $u(x,t)$ of (1.1) that satisfies

$$\iint (u\varphi_t + f(u)\varphi_x)\,\mathrm{d}x\,\mathrm{d}t + \int u_0(x)\varphi(x,0)\,\mathrm{d}x = 0 \tag{2.1}$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}\times[0,\infty))$. If a weak solution has a discontinuity at $x = \xi(t)$, then its propagation speed is given by the Rankine-Hugoniot jump condition

$$\xi'(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}, \quad u_\ell = \lim_{y\uparrow x} u(y,t),\ u_r = \lim_{y\downarrow x} u(y,t). \tag{2.2}$$

Since a weak solution is not unique, one should consider a weak solution with a suitable admissibility condition to single out the physically right one. For a nonconvex flux the Oleinik entropy condition [21] is usually employed which accepts the discontinuities that satisfy

$$\begin{aligned} \ell(u) &\le f(u) \quad \text{for all } u_\ell < u < u_r,\\ \ell(u) &\ge f(u) \quad \text{for all } u_r < u < u_\ell, \end{aligned} \tag{2.3}$$

where $\ell(u)$ is the linear function connecting two states u_r and u_ℓ, i.e.,

$$\ell(u) = f(u_\ell) + \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}(u - u_\ell).$$

It is well known that the problem is well-posed under the entropy admissibility condition in the class of bounded and measurable solutions (see [1]) and we consider this unique solution only. It is also known that $u(x,t)$ is a solution if and only if it satisfies the conditions (2.2)–(2.3) at discontinuities and the conservation law in smooth regions.

Our approach for the asymptotic convergence is based on a potential comparison technique. We take the primitive of the solution,

$$U(x,t) = \int_{-\infty}^x u(y,t)\,\mathrm{d}y, \quad U_0(x) = \int_{-\infty}^x u_0(y)\,\mathrm{d}y, \tag{2.4}$$

as the potential of the solution $u(x,t)$. The potential of the N-wave $n_{p,q}(x,t)$ is similarly given by

$$N_{p,q}(x,t) = \int_{-\infty}^x n_{p,q}(y,t)\,\mathrm{d}y. \tag{2.5}$$

Notice that N-waves are usually denoted using the capital letter N. However, we denote an N-wave as $n_{p,q}(x,t)$ and reserve the capital letter to denote its potential. Now we are ready to state the main results of this paper:

Theorem 2.1. *Let $u(x,t)$ be the entropy solution of (1.1) with initial value $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that satisfies (1.3) and (1.5). Let $n_{p,q}(x,t)$ be the N-wave satisfying (1.4) and $-p = \int_{-\infty}^c u_0(y)\,\mathrm{d}y$ for certain $c \in \mathbb{R}$. If the smooth flux f satisfies (1.2) and (H), then the following hold.*

(i) *For all* $t > 0$,

$$-p = \inf_x \int_{-\infty}^{x} u(y,t)\,dy = \int_{-\infty}^{c} u(y,t)\,dy, \quad q = \int_{c}^{\infty} u(y,t)\,dy. \tag{2.6}$$

(ii) *If* $f(u) > 0$ *for all* $u \neq 0$, *then there exists* $T > 0$ *such that*

$$\|u(t) - n_{p,q}(t)\|_1 \leq 8L\|n_{p,q}(t)\|_\infty \quad \text{for } t > T. \tag{2.7}$$

(iii) *Furthermore, if the point* $c \in \mathbb{R}$ *satisfying* $-p = \int_{-\infty}^{c} u_0(y)\,dy$ *is unique,* p, $q > 0$ *and there exist constants* α, $\varepsilon > 0$ *satisfying*

$$\begin{aligned} u_0(x+c) &\geq n_{p,q}(x,\alpha), \qquad 0 \leq x \leq \varepsilon, \\ u_0(x+c) &\leq n_{p,q}(x,\alpha), \qquad -\varepsilon \leq x \leq 0, \end{aligned} \tag{2.8}$$

then there exist constants T, $C > 0$ *such that, for* $n^c_{p,q}(x,t) = n_{p,q}(x-c,t)$,

$$\|u(t) - n^c_{p,q}(t)\|_1 \leq C\|f(n_{p,q}(t))\|_\infty \quad \text{for } t > T. \tag{2.9}$$

The proof of the theorem is based on a potential comparison technique, which has been developed for nonlinear diffusion [15] and then applied to positive solution of conservation laws. The proof of the following comparison principle is given in [12] for positive solutions and it can be directly employed for sign changing cases. In the following we present the proof briefly.

Proposition 2.2 (Potential comparison)**.** *Let* $U_i(x,t)$, $i = 1,2$, *be the potentials of two integrable solutions* u_i, $i = 1, 2$, *respectively. If* $U_1(x,0) \leq U_2(x,0)$ *for all* $x \in \mathbb{R}$, *then* $U_1(x,t) \leq U_2(x,t)$ *for all* $x \in \mathbb{R}$, $t > 0$.

Proof. Roughly speaking, after an integration of (1.1) on interval $-\infty < y < x$, one obtains $U_t + f(u) = 0$ in a weak sense and, hence, $E(x,t) = U_1(x,t) - U_2(x,t)$ is a weak solution of

$$E_t + a(x,t)E_x = 0, \quad a(x,t) = (f(u_1) - f(u_2))/(u_1 - u_2),$$

where $a(x,t)$ is understood as the derivative of the smooth flux if $u_1 = u_2$. Hence the characteristic for E is same as the ones of solutions u_1 and u_2 if $u_1 = u_2$ and, otherwise, it is between them. Since E is constant along the characteristics and $E(x,0) \geq 0$, we have $E(x,t) \geq 0$, i.e., $U_1(x,t) \leq U_2(x,t)$ for all $x \in \mathbb{R}$, $t > 0$. ◻

3. Invariance Property

Theorem 2.1(i) claims that the global minimum value p of the potential $U(\cdot\,,t)$ and the minimum point $x = c$ are two invariant quantities. Then the other quantity q is automatically invariant. For its proof we study how a local extremum of a potential evolves. For the convex flux case this invariance has been shown in [18] using the fact that admissible discontinuities are decreasing ones. The solution may have increasing discontinuities and more complicate structures if the flux is non-convex.

Proof of Theorem 2.1(i). Let $t_0 < t_1$ and $x \in \mathbb{R}$ be given. Then, since the wave speed is finite and the support of the initial value is bounded, there exists $x_0 < x$ such that $u(y,s) = 0$ for all $y < x_0$ and $s < t_1$. Let $\Omega := [x_0, x] \times [t_0, t_1]$ and consider the characteristic function $\varphi(y,t) = \chi|_\Omega$. Since φ is not smooth, we may not directly apply φ to (2.1). However, using classical approximation arguments with smooth functions, $\varphi_\varepsilon \to \varphi$, one may obtain

$$U(x,t_1) - U(x,t_0) = -\int_{t_0}^{t_1} f(u(x,s))\,ds. \tag{3.1}$$

If u is continuous at a point (x,t), then one may easily obtain that

$$U_x(x,t) = u(x,t), \quad U_t(x,t) = -f(u(x,t)), \tag{3.2}$$

where the first one is from the definition of a potential and the second one is from the Lebesgue differentiation theorem applied to (3.1). These two relations will be frequently referred in the following.

Suppose that $x = \xi(t)$ is the (global) minimum point of $U(\cdot,t)$ and $u(\cdot,t)$ is continuous at that point. Then $U_x(\xi,t) = u(\xi,t) = 0$ and hence $x = \xi(t)$ is a characteristic line carrying the zero value, which implies that $\xi'(t) = f'(0) = 0$. Therefore, the minimum $-p(t) = U(\xi(t),t)$ satisfies

$$-p'(t) = \frac{d}{dt}U(\xi(t),t) = \xi'(t)u(\xi(t),t) - f(u(\xi(t),t)) = 0,$$

which implies that the minimum value p is constant. Notice that the invariance of p does not depend on the assumptions (H) if the solution is continuous at the minimum point of the potential. Furthermore, since $\xi'(t) = 0$ and p is constant, we have $U(c,t) = p$ for $c = \xi(0)$ as long as u is continuous at the point.

Now we consider the case when $u(\cdot,t)$ has a discontinuity at the minimum point $x = \xi(t)$. Let u_ℓ and u_r be the left and the right hand side limits, respectively, and $\ell(u)$ be their linear connection. Then, since $U(\cdot,t)$ has a minimum at the point $\xi(t)$, it is clear that $u_\ell \le 0 \le u_r$ and $u_\ell \ne u_r$. (Note that, if the flux is convex, this kind of discontinuities are not allowed.) Then since $u = 0$ is a global minimum point of the flux, one can easily see that the entropy condition (2.3) holds only if u_ℓ and u_r are global minimum points and hence $f(u_{\ell,r}) = f'(u_{\ell,r}) = 0$. Therefore, we still have $\xi'(t) = 0$ by the Rankine-Hugoniot condition (2.2) and hence $p'(t) = 0$.

Since the total mass M is preserved, the other quantity q in (1.5) is also constant and the proof of Theorem 2.1(i) is complete. □

The potential has local extrema at zero points of the solution and the previous proof shows how they evolve. Let $U(x,t)$ have a local extremum at $x=\xi(t)$ and $u(x,t)$ be continuous at that point. Then, $u(\xi(t),t)=0$, $\xi'(t)=f'(0)=0$, and

$$\frac{d}{dt}U(\xi(t),t)=\xi'(t)u(\xi(t),t)-f(u(\xi(t),t))=0.$$

Therefore, the local extremum point and the value are constant until it meets a shock discontinuity. If a minimum meets a decreasing shock, then it is not a local minimum anymore. In the proof it is basically shown that, if a minimum meets an increasing discontinuity, it should be a harmless shock in the sense that the onesided limits satisfy $f(u_{\ell,r})=f'(u_{\ell,r})=0$ and the minimum value and the point are not changed. This kind of shock has been called a contact shock of type II in [14]. If $f(u)>0$ for all $u\neq 0$ (i.e., $u=0$ is the only global minimum point of the flux), then any local minimum of the potential does not meet a shock discontinuity and hence one may say that the entropy condition basically prohibits discontinuities at a local minimum point of the potential.

Suppose that $u(x,t)$ has a shock at a local extremum point $x=\xi(t)$ and hence $U(x,t)$ has a local maximum at that point. Let

$$u_\ell(t)=\lim_{y\uparrow\xi(t)}u(y,t),\qquad u_r(t)=\lim_{y\downarrow\xi(t)}u(y,t).$$

Then, the limits satisfy $u_\ell\geq 0\geq u_r$ and $u_\ell\neq u_r$. The wave speed of the discontinuity is given by the Rankine-Hugoniot jump condition which is

$$\xi'(t)=\frac{f(u_\ell)-f(u_r)}{u_\ell-u_r}.$$

Clearly, there exists $\varepsilon>0$ small such that $u(\cdot\,,t)$ is continuous on $(\xi_\varepsilon(t),\xi(t))$, where $\xi_\varepsilon(t)=\xi(t)-\varepsilon$. Then, $\xi_\varepsilon'(t)=\xi'(t)$ and

$$\begin{aligned}\frac{d}{dt}U(\xi_\varepsilon(t),t)&=U_x(\xi_\varepsilon(t),t)\xi'(t)+U_t(\xi_\varepsilon(t),t)\\&=u(\xi_\varepsilon(t),t)\xi'(t)-f(u(\xi_\varepsilon(t),t)).\end{aligned}$$

Taking $\varepsilon\to 0$ gives

$$\frac{d}{dt}U(\xi(t),t)=u_\ell\frac{f(u_\ell)-f(u_r)}{u_\ell-u_r}-f(u_\ell)=\frac{f(u_\ell)u_r-f(u_r)u_\ell}{u_\ell-u_r}\leq 0.$$

Therefore, the local maximum of the potential decreases if it meets a shock. One can easily check that $(d/dt)U(\xi(t),t)=0$ if the left or the right limit is zero. Suppose that the point $(\xi(t),t)$ has a local minimum point of the potential and u is discontinuous at the point (i.e., the entropy condition is not satisfied). Then,

$u_\ell \le 0 \le u_r$ and the local minimum decreases. Hence the entropy condition allows the local minimums only to increase. If the flux is assumed $f(u) \le 0$, then it will control the local maximum. Now we summarize the properties of the critical values of a potential in the following proposition:

Proposition 3.1. *Let $u(x,t)$ be the solution to (1.1)–(1.3) and $U(x,t)$ be its potential function, where the flux $f(u)$ satisfies (H). Then,*

(i) *If the potential U has a local minimum at $(\xi(t),t)$, then u is continuous at that point, $\xi'(t) = 0$ and $(d/dt)U(\xi(t),t) = 0$, i.e., the local minimum point and the value are constant as long as they survive.*

(ii) *If the potential U has a local maximum at $(\xi(t),t)$ and*

$$u_\ell := \lim_{y\uparrow\xi(t)} u(y,t) \ne u_r := \lim_{y\downarrow\xi(t)} u(y,t),$$

then $u_\ell \ge 0 \ge u_r$ and the maximum decreases as

$$\frac{d}{dt}U(\xi(t),t) = \frac{f(u_\ell)u_r - f(u_r)u_\ell}{u_\ell - u_r} \le 0. \tag{3.3}$$

*Remark **3.2**.* The non-negativity of the flux is essential for the invariance property. If $u = 0$ is not a global minimum point of the flux f, then the solution may have a discontinuity at the global minimum point of the potential U such that $u_\ell < 0 < u_r$. Then the derivative in (3.3) can be strictly positive and hence the global minimum of the potential may strictly increase. Therefore, one may conclude that the non-negativity of the flux is the essential part for the invariance of the global minimum point c and the value p of the potential in the case of one dimensional hyperbolic conservation laws.

4. THE ASYMPTOTIC CONVERGENCE ORDER $O(\max_x |n_{p,q}(x,t)|)$

Now we show Theorem 2.1(ii). Since the global minimum point of the potential U is invariant, we may assume U has its minimum at the origin ($c = 0$), i.e.,

$$U(0,t) = -p,\ U(x,t) \ge -p \quad \text{for all } x \in \mathbb{R},\ t > 0$$

after an appropriate space shift. Then, as discussed earlier, $u(x,t)$ is continuous at $x = 0$ and $u(0,t) = 0$ for all $t > 0$. We assume this throughout this section.

Consider the convex envelope of the flux given by

$$h(u) := \sup_{\eta\in A} \eta(u), \quad A := \{\eta : \eta''(u) \ge 0,\ \eta(u) \le f(u) \text{ for } u \in \mathbb{R}\}. \tag{4.1}$$

Since there are only a finite number of inflection points, the convex envelope is obtained by simply connecting the humps of the graph of the flux with tangent lines. The convex envelope $h(u)$ is continuously differentiable and is linear on intervals on which $f(u) \ne h(u)$.

It is clear that h' is not invertible. However, one may consider a function $g(x)$ given by an inverse relation

$$g(0) = 0, \quad h'(g(x)) = x, \quad x \in \mathbb{R}. \tag{4.2}$$

Then $g(x)$ is piecewise continuous. Since $f(0) = 0$ and $f(u) > 0$ for all $u \neq 0$, there exists a maximal open interval $0 \in (-a, b)$ such that

$$f(u) = h(u) \quad \text{for } -a < u < b, \text{ where } a, b > 0. \tag{4.3}$$

Since the long time asymptotics of a solution depend on the structure of the flux near the origin, the interval $(-a, b)$ will play an important role asymptotically and hence will appear several times in the rest of the paper.

Lemma 4.1 (No contact discontinuity for u small)**.** *Let $\xi(s)$, $s < t$, be a characteristic that emanates from a continuity point (x_0, t) with $u(x_0, t) \in (-a, b)$. Then, $\xi(s)$ does not intersect a discontinuity curve, i.e., it is global.*

Proof. Suppose that the backward characteristic $\xi(s)$ intersects a discontinuity (or shock) curve which connects the value $u(x_0, t)$ to $\bar{u}$ at the intersection point. If $\bar{u} > u(x_0, t)$, then the entropy condition implies that $u_r := u(x_0, t)$ and $u_\ell := \bar{u}$ are right and left hand side limits, respectively. The convexity of the flux on $(-a, b)$ also implies that

$$f'(u(x_0, t)) < \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell}.$$

Clearly, it is not possible that a slower backward characteristic intersects a faster shock curve from the right hand side. One can derive a similar contradiction if $\bar{u} < u(x_0, t)$ and hence one can conclude that $\xi(s)$ is global and does not intersect a discontinuity for all $0 < s < t$. $\square$

Now define N-waves like functions as

$$\tilde{n}_{p,q}(x, t) = \begin{cases} g\left(\dfrac{x}{t}\right), & -a_p(t) < x < b_q(t), \\[2ex] 0, & \text{otherwise}, \end{cases} \tag{4.4}$$

where $a_p(t)$, $b_q(t) > 0$ satisfy

$$p = -\int_{-a_p(t)}^{0} g\left(\frac{y}{t}\right) \mathrm{d}y, \quad q = \int_0^{b_q(t)} g\left(\frac{y}{t}\right) \mathrm{d}y. \tag{4.5}$$

One can easily check that $\tilde{n}_{p,q}(x, t)$ is a weak solution of both of the conservation laws $u_t + f(u)_x = 0$ and $u_t + h(u)_x = 0$. However, $\tilde{n}_{p,q}(x, t)$

does not satisfy the entropy condition (2.3) in general and hence it is not the solution. On the other hand, under Hypothesis (H), it is shown in [13] that $n_{p,q}(x,t) = \tilde{n}_{p,q}(x,t)$ for $0 < t \ll 1$ small or $t \gg 1$ large. In particular there exists $T > 0$ such that

$$n_{p,q}(x,t) = \tilde{n}_{p,q}(x,t) \in (-a,b) \quad \text{for all } x \in \mathbb{R},\ t > T. \tag{4.6}$$

Since our interest in this paper is the long time behavior of the solution, we employ the explicit formula for the N-wave like function $\tilde{n}_{p,q}(x,t)$ for $t > T$.

Lemma 4.2. *There exists $T > 0$ such that, for all $t > T$,*

$$\begin{cases} u(x,t) \le n_{p,q}(x,t), & 0 < x < b_q(t), \\ u(x,t) \ge 0, & x > b_q(t), \end{cases} \tag{4.7}$$

$$\begin{cases} u(x,t) \ge n_{p,q}(x,t), & -a_p(t) < x < 0, \\ u(x,t) \le 0, & x < -a_p(t), \end{cases} \tag{4.8}$$

$$\|n_{p,q}(t) - u(t)\|_1 \le 4\|N_{p,q}(t) - U(t)\|_\infty. \tag{4.9}$$

Proof. We take $T > 0$ that satisfies (4.6). Since $a_p(t), b_q(t) \to \infty$ as $t \to \infty$, we may assume $a_p(t), b_q(t) > L$ by taking larger $T > 0$ if needed. Consider a backward characteristic $\xi(s)$, $0 < s < t$, that emanates from a continuity point (x_0, t) with $0 < x_0 < b_q(t)$, $t > T$.

We first show $u(x_0,t) \le n_{p,q}(x_0,t)$. Suppose that $0 \le u(x_0,t) < b$. Then Lemma 4.1 implies that $\xi(s)$ does not intersect a shock curve. Furthermore, the invariance property implies that $x = 0$ is a characteristic line of $u(x,t)$ and hence $\xi(0) \ge 0$. Consider another backward characteristic $\bar{\xi}(s)$, $0 < s < t$, that emanates from the same point (x_0,t) related to the N-wave $n_{p,q}(x,t)$. Then $\bar{\xi}(s)$ is a line with speed $f'(n_{p,q}(x_0,t))$ and $\bar{\xi}(0) = 0$ since the N-wave is a rarefaction wave centered at $x = 0$. Therefore, $f'(u(x_0,t)) \le f'(n_{p,q}(x_0,t))$. Since f is convex on $(-a,b)$, we have $u(x_0,t) \le n_{p,q}(x_0,t)$.

Suppose that $u(x_0,t) \ge b$. Then since $u(x,t)$ is continuous at $x = 0$ and $u(0,t) = 0$, we have $\beta := \inf\{0 < x < b_q(t) : u(x,t) \ge b\} \in (0, b_q(t))$. Let u_r and u_ℓ be the right and the left hand side limits of $u(x,t)$ at $x = \beta$, respectively. Then $u_\ell \le n_{p,q}(\beta,t) < b \le u_r$, which is a discontinuity that violates the entropy condition. Hence $u(x_0,t) \le b$ and we obtained $u(x_0,t) \le n_{p,q}(x_0,t)$ for $0 < x < b_q(t)$. Since the characteristics for negative values have negative speed and $b_q(t) > L$ we have $u(x,t) \ge 0$ for $x > b_q(t)$, which completes (4.7).

One may similarly obtain (4.8) and we show (4.9) in the following. The relations in (4.7) and (4.8) imply that

$$\|u(t)-n_{p,q}(t)\|_1 = -\int_{-\infty}^{-a_p} u(x,t)\,dx + \int_{-a_p}^{0} \big(u(x,t)-n_{p,q}(x,t)\big)\,dx + \int_0^{b_q} \big(n_{p,q}(x,t)-u(x,t)\big)\,dx + \int_{b_q}^{\infty} u(x,t)\,dx.$$

Since $U(0,t)=N_{p,q}(0,t)=p$, one can easily see that

$$\|u(t)-n_{p,q}(t)\|_1 = -2\int_{-\infty}^{-a_p} u(x,t)\,dx + 2\int_{b_q}^{\infty} u(x,t)\,dx.$$

Since $-\int_{-\infty}^{-a_p} u(x,t)\,dx,\ \int_{\infty}^{b_q} u(x,t)\,dx \le \|U(t)-N_{p,q}(t)\|_\infty$, the inequality in (4.9) is now clear. ◻

Lemma 4.3 (Trapped between space translations)**.** *There exists $T>0$ such that, for all $x\in\mathbb{R}$ and $t>T$,*

$$N_{p,0}(x{+}L,\ t)+N_{0,q}(x{-}L,\ t) \le U(x,t) \le N_{p,q}(x,t). \tag{4.10}$$

Proof. One can easily check that

$$N_{p,0}(x+L,0)+N_{0,q}(x-L,0)\le U(x,0),$$

and hence the comparison principle gives the first inequality in (4.10). Now let

$$\bar p = \max_{x<0} U(x,0), \qquad \bar q = \max_{x>0} U(x,0).$$

Then, since $U(x,0)\to 0$ as $x\to-\infty$ and $U(x,0)\to M$ as $x\to\infty$, we clearly have $\bar p\ge 0$ and $\bar q\ge M$. Consider a summation of three N-waves

$$n_{0,\bar p}(x+L,t)+n_{\bar p+p,\bar q+p}(x,t)+n_{\bar q-M,0}(x-L,t).$$

Then the N-waves have disjoint supports for $t>0$ small, say $0<t<t_0$. Let $n(x,t)$ be a solution with this summation of N-waves as its initial value, and let $N(x,t)$ be its potential. Then clearly, $U(x,0)\le N(x,0)$ and hence the potential comparison principle implies that $U(x,t)\le N(x,t)$ for all $t>0$. Therefore, the second inequality in (4.10) is completed if it is shown that $n(x,t)=n_{p,q}(x,t)$ for all $t>T$, where $T>0$ is the one in Lemma 4.2 with $n(x,t)$ in the place of $u(x,t)$.

Suppose that there exists $x_0>b_q(t)$ such that $n(x_0,t)>0$ and $t>T$. Then, since $n(x_0,t)\in(-a,b)$, the backward characteristic $\xi(s)$, $0<s<t$,

that emanates from the point (x_0, t) does not intersect a shock curve (Lemma 4.1) and $\xi(0) \geq 0$. Since $n_{\bar{q}-M,0}(x-L,t)$ is negative and $n_{\bar{p}+p,\bar{q}+p}(x,t)$ has rarefaction waves centered at the origin, we have $\xi(0) = 0$. Therefore, for any characteristic $\bar{\xi}(s)$ that emanates from a point (x,t) with $0 < x < b_q(t)$, we have $\bar{\xi}(0) = 0$ and hence comparison of characteristic speed gives $n(x,t) = n_{p,q}(x,t)$ for all $0 < x < b_q(t)$. Therefore $\lim_{x\to\infty} N(x,t) > \lim_{x\to\infty} N_{p,q}(x,t) = M$, which is a contradiction. Therefore, $n(x,t) = 0$ for all $x > b_q(t)$ and hence $n(x,t) = n_{p,q}(x,t)$ for all $x > 0$. One can show the equality for $x < 0$ similarly and obtain $N(x,t) = N_{p,q}(x,t)$ for all $x \in \mathbb{R}$ and $t > T$. □

Notice that the inequality (4.9) transfers the convergence order between two potentials to the one between their derivatives. This is one of the essential steps that make the potential comparison technique work. Now we show the second part of Theorem 2.1 as a corollary of previous lemmas.

Proof of Theorem 2.1(ii). Let $t > T$. Then, Lemma 4.3 implies that

$$|N_{p,q}(x,t) - U(x,t)| \leq |N_{p,q}(x,t) - N_{p,0}(x+L,\, t) - N_{0,q}(x-L,\, t)|$$

$$= \begin{cases} \left| \displaystyle\int_x^{x+L} n_{p,0}(y,t)\,\mathrm{d}y \right|, & x < 0, \\ \left| \displaystyle\int_{x-L}^{x} n_{0,q}(y,t)\,\mathrm{d}y \right|, & x > 0. \end{cases}$$

Therefore,

$$\|N_{p,q}(t) - U(t)\|_\infty \leq L \max_x(n_{p,q}(x,t)), \quad t > T,$$

and (4.9) in Lemma 4.2 implies that

$$\|n_{p,q}(t) - u(t)\|_1 \leq 4L \max_x(n_{p,q}(x,t)), \quad t > T.$$

Remember that we are considering the problem assuming the global minimum point of $U(x,0)$ is $c = 0$ after a space translation. Hence, this estimate should be understood as

$$\|n^c_{p,q}(t) - u(t)\|_1 \leq 4L \max_x(n_{p,q}(x,t)), \quad t > T.$$

One can easily check using the previous arguments that

$$\|n_{p,q}(t) - n^c_{p,q}(t)\|_1 \leq 4|c| \max_x(n_{p,q}(x,t)).$$

Then the triangle inequality in the L^1 norm gives

$$\|n_{p,q}(t) - u(t)\|_1 \leq 4(L + |c|) \max_x(n_{p,q}(x,t)), \quad t > T.$$

Therefore, since $|c| < L$, the proof of Theorem 2.1(ii) is complete. □

5. The Asymptotic Convergence Order $1/t$

Now we show Theorem 2.1(iii). Remember that we assume U has its minimum at the origin after an appropriate space shift (i.e., $c = 0$). Furthermore, since the minimum point is unique, we may set

$$U(0,t) = -p, \quad U(x,t) > -p \quad \text{for all } x \neq 0,\ t > 0.$$

Lemma 5.1 (Trapped between time translations)**.** *Let u be the solution of* (1.1)*, $n_{p,q}$ be the N-wave satisfying* (1.4)*, and U, $N_{p,q}$ be their potentials, respectively. If the flux f satisfies* (H) *and the initial value $u(x,0)$ satisfies the conditions in* (2.8) *(with $c = 0$), then there exist T, $T_1 > 0$ such that, for all $t \geq T$,*

$$N_{p,q}(x, T_1 + t) \leq U(x,t) \leq N_{p,q}(x,t). \tag{5.1}$$

Proof. The second inequality in (5.1) has been shown in Lemma 4.3 and we show the first one in the following. Due to the invariance property in Theorem 2.1(i), $U(0,t) + p = N_{p,q}(0,t) + p = 0$ for all $t > 0$. Therefore, we may split the domain for $x > 0$ and $x < 0$ and show the inequality on each domains separately.

One can clearly see that

$$p + N_{p,q}(L,t) = \int_0^L n_{p,q}(x,t)\,\mathrm{d}x \to 0 \quad \text{as } t \to \infty.$$

Therefore, there exists $T_1 > \alpha$ such that $p + N_{p,q}(L, T_1) \leq p + U(\varepsilon, 0)$. Since $U(x,0)$ has a unique minimum point at $x = 0$, we may assume that $U(x,0) \geq U(\varepsilon, 0)$ for all $x > \varepsilon$ by taking smaller $\varepsilon > 0$ if needed. Therefore, $N_{p,q}(x, T_1) \leq U(x,0)$ for all $0 < x < L$. Furthermore, since $p + U(x,0) = q$ for all $x \geq L$ and $p + N_{p,q}(x, T_1) \leq q$ for all $x \in \mathbb{R}$, we obtain $N_{p,q}(x, T_1) \leq U(x,0)$ for all $x > 0$. For $x < 0$ we may similarly obtain the estimate and obtain the initial comparison $N_{p,q}(x, T_1) \leq U(x,0)$. Therefore, the comparison principle completes that $N_{p,q}(x, T_1 + t) \leq U(x,t)$ for all $x \in \mathbb{R}$, $t > T$, which is the first inequality of the lemma. □

Theorem 2.1(iii) is obtained as a corollary of Lemma 5.1.

Proof of Theorem 2.1(iii). Using the comparison inequality (5.1) and the evolution equation for potentials (3.1), we obtain

$$\begin{aligned}|U(x,t) - N_{p,q}(x,t)| &\leq |N_{p,q}(x, T_1 + t) - N_{p,q}(x,t)| \\ &= \int_t^{t+T_1} f(n_{p,q}(x,s))\,\mathrm{d}s \leq T_1 \|f(n_{p,q}(t))\|_\infty.\end{aligned}$$

Since the right hand side is independent of $x \in \mathbb{R}$, the estimate is uniform. This uniform estimate is naturally transferred to the L^1 estimate of the difference between solutions using (4.9), i.e.,

$$\|u(x+c,t) - n_{p,q}(x,t)\|_1 \leq 4T_1 \|f(n_{p,q}(t))\|_\infty. \tag{5.2}$$

Therefore, the proof of Theorem 2.1(iii) is completed with $C = 4T_1$. □

In Lemma 5.1 the potential $U(x,t)$ has been sandwiched between $N_{p,q}(x,t)$ and its time delay $N_{p,q}(x, T_1 + t)$ with $T_1 \geq \alpha$. Basically we may take $T_1 = \alpha$ after taking larger $T > 0$ if needed. In what follows we give a brief sketch of it.

First we may assume

$$\max_x n_{p,q}(x,T) \leq \max_{0<x<\varepsilon} n_{p,q}(x,\alpha)$$

by taking larger $T > 0$ if needed. Consider a backward characteristic $\xi(s)$, $0 < s < t$, related to the N-wave $n_{p,q}(x, t + \alpha)$ that emanates from a continuity point (x_0, t) with $0 < x_0 < b_q(t)$ and $t > T$. As discussed in the proof of Lemma 4.3, it does not meet a discontinuity all the way to $s = 0$ and hence it is a straight line with speed $f'(n_{p,q}(x_0, t + \alpha))$. Similarly consider another backward characteristic $\bar{\xi}(s), 0 < s < t$, that emanates from the same point (x_0, t) related to the solution $u(x,t)$. Then $\bar{\xi}(s)$ is also a line with speed $f'(u(x_0,t))$. By taking larger $T > 0$ if needed we may expect that $0 \leq \bar{\xi}(0) \leq \varepsilon$ if $u(x_0, t) \neq 0$.

Now we show the order between $\xi(0)$ and $\bar{\xi}(0)$. Suppose that $\xi(0) < \bar{\xi}(0)$. Then the speed of the characteristic lines should be ordered by $f'(u(x_0,t)) < f'(n_{p,q}(x_0, t+\alpha))$. Since f is convex near $u = 0$ (or $u \in (-a, b)$) and solutions are constant along the characteristics, we have $u(\bar{\xi}(0), 0) < n_{p,q}(\xi(0), \alpha)$. Since $n_{p,q}(x,\alpha)$ is an increasing function on the interval $(-a, b)$, we have

$$u(\bar{\xi}(0), 0) < n_{p,q}(\xi(0), \alpha) < n_{p,q}(\bar{\xi}(0), \alpha)$$

which contradicts the initial condition (2.8). Therefore we have $\xi(0) \geq \bar{\xi}(0)$ and hence $u(x_0, t) > n_{p,q}(x_0, t)$ if $u(x_0, t) \neq 0$. Therefore,

$$\int_0^x u(y,t)\,\mathrm{d}y \geq \int_0^x n_{p,q}(y, t+\alpha)\,\mathrm{d}y.$$

One may obtain a similar estimate for $x < 0$ and may complete the comparison

$$N_{p,q}(x, t+\alpha) \leq U(x,t).$$

Therefore, we may take $T_1 = \alpha$ which is reasonable in the sense that the α measures the age of the initial value and hence it should control the convergence speed.

In the following we compute the order of the supremum norm $\|f(n_{p,q}(t))\|_\infty$ for t large to obtain an algebraic convergence order, which turns out to be the order $1/t$. For that purpose we take a hypothesis

$$\text{(H1)} \qquad \liminf_{u\to 0} \frac{uf'(u)}{f(u)} = \gamma > 1.$$

Corollary 5.2 (Convergence order $O(1/t)$). *If the flux function f satisfies* (H1), *then*

$$\lim_{t\to\infty} t\|u(x,t) - n_{p,q}(x,t)\|_1 \le 4T_1 \max(p,q)/(\gamma - 1). \tag{5.3}$$

Proof. Since $|n_{p,q}(\cdot\,,t)|$ has its supremum at $x = -a_p(t)$ or $x = b_q(t)$ for t large, we only need to check the order of $|f(n_{p,q}(\cdot\,,t))|$ at these two points to estimate $\|f(n_{p,q}(t))\|_\infty$. Let $u_r = g(b_q(t)/t)$ and hence $f'(u_r) = b_q(t)/t$. One can easily check that $g(x/t)$ and $tf'(x)$ satisfy the inverse relation for any fixed t. Therefore,

$$\int_0^{b_q(t)} g\left(\frac{x}{t}\right) \mathrm{d}x + \int_0^{u_r} tf'(x)\,\mathrm{d}x = u_r b_q(t).$$

Using these relations one can easily see that

$$\begin{aligned} q &= \int_0^{b_q(t)} g\left(\frac{x}{t}\right) \mathrm{d}x = u_r b_q(t) - \int_0^{u_r} tf'(x)\,\mathrm{d}x \\ &= t(u_r f'(u_r) - f(u_r)) = t\left(\frac{u_r f'(u_r)}{f(u_r)} - 1\right) f(u_r). \end{aligned} \tag{5.4}$$

This equality shows that $u_r f'(u_r)/f(u_r) > 1$. Therefore, the flux that satisfies the assumptions in (H) satisfies $\liminf_{u\to 0}(uf'(u)/f(u)) =: \gamma \ge 1$. Under the extra hypothesis (H1), one obtains from (5.4) that

$$\lim_{t\to\infty} tf(u_r) \le \frac{q}{\gamma - 1}.$$

We may similarly estimate that

$$\lim_{t\to\infty} tf(u_\ell) \le \frac{p}{\gamma - 1} \quad \text{for } u_\ell = g(-a_p(t)/t)$$

and obtain

$$\lim_{t\to\infty} t\|f(n_{p,q}(t))\|_\infty \le \frac{\max(p,q)}{\gamma - 1}.$$

Therefore, the estimate (5.2) gives the convergence order $O(1/t)$ in (5.3). □

Even if it is natural to ask whether or not the assumptions in (H) imply (5.3), we do not have a proof nor a counter example. However, there are many examples that satisfy (5.3). First, the power law $f(u) = |u|^\gamma$, $\gamma > 1$, is a typical example. Suppose that f is C^2 and $f''(0) \ne 0$. Then, using l'Hôpital's rule, one obtains

$$\lim_{u\to 0} \frac{uf'(u)}{f(u)} = 1 + \lim_{u\to 0} \frac{u}{f'(u)} f''(u) = 2.$$

Suppose f is C^2 and $f''(0) = 0$. Then, one can easily see that $f'(u)/u < f''(u)$ for $|u|$ small, i.e., $1 < uf''(u)/f'(u)$. Therefore, if the flux is C^2 and $f''(0) = 0$, then one has

$$\liminf_{u\to 0} \frac{uf'(u)}{f(u)} \geq 2.$$

If $f(u) = \exp(-1/|u|)$ for $|u| < 1$, then one can easily check that $uf'(u)/f(u) \to \infty$ as $u \to 0$. This example indicates that, if the flux f is very flat near the origin, the ratio $uf'(u)/f(u)$ may diverge. However, the hypothesis (H1) is satisfied and we still have the convergence order $O(1/t)$.

Acknowledgement This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF–2007–355–C00003).

REFERENCES

[1] D.P. BALLOU, *Solutions to nonlinear hyperbolic Cauchy problems without convexity conditions*, Trans. Amer. Math. Soc. **152** (1970), 441–460 (1971), http://dx.doi.org/10.2307/1995581. MR 0435615 (55 #8573)

[2] J.A. CARRILLO, M. DI FRANCESCO, and C. LATTANZIO, *Contractivity of Wasserstein metrics and asymptotic profiles for scalar conservation laws*, J. Differential Equations **231** (2006), 425–458, http://dx.doi.org/10.1016/j.jde.2006.07.017. MR **2287892 (2007k:**35315)

[3] J.A. CARRILLO, A. JÜNGEL, P.A. MARKOWICH, G. TOSCANI, and A. UNTERREITER, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math. **133** (2001), 1–82, http://dx.doi.org/10.1007/s006050170032. MR **1853037 (2002j:**35188)

[4] J.A. CARRILLO and G. TOSCANI, *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J. **49** (2000), 113–142, http://dx.doi.org/10.1512/iumj.2000.49.1756. MR **1777035 (2001j:**35155)

[5] J.A. CARRILLO and J.L. VÁZQUEZ, *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations **28** (2003), 1023–1056, http://dx.doi.org/10.1081/PDE-120021185. MR **1986060 (2004a:**35118)

[6] C.M. DAFERMOS, *Regularity and large time behaviour of solutions of a conservation law without convexity*, Proc. Roy. Soc. Edinburgh Sect. A **99** (1985), 201–239. MR **785530 (86j:**35107)

[7] J. DENZLER and R.J. MCCANN, *Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology*, Arch. Ration. Mech. Anal. **175** (2005), 301–342, http://dx.doi.org/10.1007/s00205-004-0336-3. MR **2126633 (2005k:**35214)

[8] R.J. DIPERNA, *Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws*, Indiana Univ. Math. J. **24** (1975), 1047–1071, http://dx.doi.org/10.1512/iumj.1975.24.24088. MR 0410110 (53 #13860)

[9] J. DOLBEAULT and M. DEL PINO, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) **81** (2002), 847–875. MR **1940370 (2003h:**35051) (English, with English and French summaries)

[10] J. DOLBEAULT and M. ESCOBEDO, *L^1 and L^∞ intermediate asymptotics for scalar conservation laws*, Asymptot. Anal. **41** (2005), 189–213. MR **2127996 (2006i:**35233)

[11] Y.-J. KIM, *Asymptotic behavior of solutions to scalar conservation laws and optimal convergence orders to N-waves*, J. Differential Equations **192** (2003), 202–224, http://dx.doi.org/10.1016/S0022-0396(03)00058-5. MR **1987091 (2004e:**35147)

[12] ———, *Potential comparison and asymptotics in scalar conservation laws without convexity*, J. Differential Equations **244** (2008), 40–51, http://dx.doi.org/10.1016/j.jde.2006.08.013. MR **2373653 (2009d:**35220)

[13] Y.-J. KIM and Y. HA, *Fundamental solutions of a conservation law without convexity*, http://amath.kaist.ac.kr/papers/Kim/16.pdf.

[14] Y.-J. KIM and Y. LEE, *Structure of fundamental solutions of a conservation law without convexity*, http://amath.kaist.ac.kr/papers/Kim/17.pdf.

[15] Y.-J. KIM and R.J. MCCANN, *Potential theory and optimal convergence rates in fast nonlinear diffusion*, J. Math. Pures Appl. (9) **86** (2006), 42–67. MR **2246356 (2007f:**35163) (English, with English and French summaries)

[16] P.D. LAX, *Hyperbolic systems of conservation laws. II*, Comm. Pure Appl. Math. **10** (1957), 537–566, http://dx.doi.org/10.1002/cpa.3160100406. MR 0093653 (20 #176)

[17] C. LEDERMAN and P.A. MARKOWICH, *On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass*, Comm. Partial Differential Equations **28** (2003), 301–332, http://dx.doi.org/10.1081/PDE-120019384. MR **1974458 (2004a:**35125)

[18] T.-P. LIU, *Invariants and asymptotic behavior of solutions of a conservation law*, Proc. Amer. Math. Soc. **71** (1978), 227–231, http://dx.doi.org/10.2307/2042838. MR **500495 (81c:**35085)

[19] T.-P. LIU and M. PIERRE, *Source-solutions and asymptotic behavior in conservation laws*, J. Differential Equations **51** (1984), 419–441, http://dx.doi.org/10.1016/0022-0396(84)90096-2. MR **735207 (85i:**35094)

[20] R.J. MCCANN and D. SLEPČEV, *Second-order asymptotics for the fast-diffusion equation*, Int. Math. Res. Not. (2006), Art. ID 24947, 22, http://dx.doi.org/10.1155/IMRN/2006/24947. MR **2211152 (2006k:**35150)

[21] O.A. OLEĬNIK, *Discontinuous solutions of non-linear differential equations*, Amer. Math. Soc. Transl. (2) **26** (1963), 95–172. MR 0151737 (27 #1721)

[22] F. OTTO, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26** (2001), 101–174, http://dx.doi.org/10.1081/PDE-100002243. MR **1842429 (2002j:**35180)

[23] J.L. VÁZQUEZ, *Asymptotic beahviour for the porous medium equation posed in the whole space*, J. Evol. Equ. **3** (2003) (Dedicated to Philippe Bénilan), 67–118, http://dx.doi.org/10.1007/s000280300004. MR **1977429 (2004d:**35138)

[24] T.P. WITELSKI and A.J. BERNOFF, *Self-similar asymptotics for linear and nonlinear diffusion equations*, Stud. Appl. Math. **100** (1998), 153–193, http://dx.doi.org/10.1111/1467-9590.00074. MR **1491842 (99d:**35081)

[25] K. ZUMBRUN, *N-waves in elasticity*, Comm. Pure Appl. Math. **46** (1993), 75–95, http://dx.doi.org/10.1002/cpa.3160460105. MR **1193344 (93k:**35173)

[26] ———, *Decay rates for nonconvex systems of conservation laws*, Comm. Pure Appl. Math. **46** (1993), 353–386, http://dx.doi.org/10.1002/cpa.3160460304. MR **1202961 (94b:**35174)

Department of Mathematical Sciences
KAIST (Korea Advanced Institute of Science and Technology)
373-1 Guseong-dong, Yuseong-gu
Daejeon, 305-701, South Korea
E-MAIL: MIJOUNG KIM: mijoungkim@kaist.ac.kr
E-MAIL: YONG-JUNG KIM: yongkim@kaist.edu

KEY WORDS AND PHRASES: asymptotics in time, characteristics, convergence order, N-wave, potential comparison, similarity.

2000 MATHEMATICS SUBJECT CLASSIFICATION: Author: Please supply.

Received: *November 30th, 2007; revised: November 18th, 2008.*

Article electronically published on April 10th, 2009.