WELL-POSEDNESS OF THE CONDUCTIVITY RECONSTRUCTION FROM AN INTERIOR CURRENT DENSITY IN TERMS OF SCHAUDER THEORY

YONG-JUNG KIM AND MIN-GI LEE

Abstract. We show the well-posedness of the conductivity image reconstruction problem with a single set of interior electrical current data and boundary conductivity data. Isotropic conductivity is considered in two space dimensions. Uniqueness for similar conductivity reconstruction problems has been known for several cases. However, the existence and the stability are obtained in this paper for the first time. The main tool of the proof is the method of characteristics of a related curl equation.

MSC: 78A30 (65N21)

1. Introduction

The purpose of this paper is to show the well-posedness in an isotropic conductivity reconstruction method from interior current density and boundary conductivity. Consider a linear elliptic equation in a simply connected bounded domain $\Omega \subset \boldsymbol{R}^n$,

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}u\big) = f, \quad \text{in } \Omega, \tag{1}$$

$$-\sum_{i,j=1}^{n} \big(a_{ij}(x)\frac{\partial}{\partial x_j}u\big)n_i = g, \quad \text{on } \partial\Omega, \tag{2}$$

where $\mathbf{n} = (n_1, \cdots, n_n)$ is the outward unit normal vector on the boundary. We assume that the coefficients are bounded and uniformly elliptic, i.e., there exist $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \quad \xi \in \boldsymbol{R}^n \backslash \{0\}.$$

The net flux through the boundary is assumed to be balanced with interior source, i.e.,

$$\int_\Omega f(x)\, dx = \int_{\partial\Omega} g\, dS.$$

The matrix tensor $\sigma := (a_{ij})$ is called anisotropic conductivity. In particular, if $a_{ij}(x) = a(x)\delta_{ij}$ for a function $a(x) \ge \lambda > 0$, it is called isotropic conductivity; if $a_{ij}(x) = a_i(x)\delta_{ij}$ for functions $a_i(x) \ge \lambda > 0$, it is called orthotropic conductivity.

In any case the tensor $a_{ij}(x)$ is symmetric and positive definite. The solution u is called voltage and $\mathbf{J} := -\sigma\nabla u$ is current density.

The motivation of this paper is from MREIT (Magnetic Resonance Electrical Impedance Tomography) problems which is possible by the MRI technology. It is an EIT type conductivity reconstruction method, which uses internal current data (see [21, 22, 24]). The internal magnetic field $\mathbf{B}$ is obtained using MRI technology and the internal current density data $\mathbf{J}$ is obtained by Ampere's Law

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}.$$

The aquifer identification problem is a related example that uses internal potential, but not current data (see [8] for a mathematical introduction). Since the internal potential data u is used for the reconstruction, we call it a *voltage problem* in comparison with the current problem of this paper. There are two mathematical approaches related to such a reconstruction problem. One may find a solution by optimizing an energy functional. When this idea is numerically implemented for the reconstruction, an iterative algorithm is typically used. The other one is to locally solve (1) for $a_{ij}(x)$ treating ∇u as a coefficient. This approach uses a local non-iterative computational algorithm and gives a simpler analysis (see [2, 20] for an isotropic voltage problem). Another recent example is a conductivity reconstruction method that uses internal power density $\mathbf{J}\cdot\nabla u$ (see [3, 16, 17, 18]). MREIT has been developed rather recently. The uniqueness and conditional stability have been obtained using a single set of current density (see [7, 9, 19]). In addition, many numerical algorithms have been suggested (see [4, 21, 22, 24]).

In this paper, we introduce a curl-based local approach. The governing equations are

$$\nabla \times (r\mathbf{J}) = 0, \quad \text{in } \Omega, \tag{3}$$

$$\nabla \cdot \mathbf{J} = f, \quad \text{in } \Omega, \tag{4}$$

$$\mathbf{J}\cdot\mathbf{n} = g, \quad \text{on } \partial\Omega, \tag{5}$$

where $r(x) := \sigma^{-1}(x)$ is the resistivity. The conductivity reconstruction problem based on this system will be called a *current problem* since current density is explicitly involved in the problem. This approach will provide a natural framework to work with current density. If the functions are sufficiently smooth, the two systems, (1)–(2) and (3)–(5), are equivalent. Both can be deduced from Maxwell's equations,

$$\nabla \times \mathbf{E} = 0, \qquad \text{(Faraday's Law)} \tag{6}$$

$$\nabla \cdot \mathbf{J} = f, \tag{7}$$

$$\sigma\mathbf{E} = \mathbf{J} \quad \text{or} \quad \mathbf{E} = r\mathbf{J}, \qquad \text{(Ohm's Law)} \tag{8}$$

by introducing a potential u so that $\mathbf{E} = -\nabla u$. Note that we do not impose $f = 0$ in (4) due to a possible noise, which is essential in a stability analysis. If $f = 0$, at least in 2-dimensions, we will see in the next section that the problem is reduced to the equivalent voltage problem. In other words, many theorems on isotropic problems in MREIT actually can be deduced directly from results of [20]. In higher dimensions, they are different however. We will explain this in detail in the next section.

The ideas related to the curl free property of the electrical field have been used implicitly or explicitly as a part of Ohm's law based theories (see for example [10]). However, in this paper, we develop a theory using the curl free equation (3) as the governing equation. This approach gives us a local and linear analysis in a way done in the voltage problem as in [20] and hence the well-posedness, too. Furthermore, this approach is free from regularization or optimization issues. The dual structure between the divergence and the curl equations let us obtain useful theories for our current problem from the ones for the voltage problem. Some of independently obtained MREIT theories can be also obtained from voltage problem results (e.g., [20]) using such a relation. New features developed in this paper are as follows. We have developed a boundary control method with detailed lemmas, Lemmas 3–5. The formulation of the theory is given with admissibility condition, Definition 1. Even though this paper is for the simplest case of two dimensional isotropic conductivity, the idea of using the curl free equation is extended to orthotropic or anisotropic cases (see [11, 12, 13]). The use of the curl free equation provides a new discretization method based on loop integrals, which has been developed into a network method (see [14]).

2. Preliminaries and Problem Description

2.1. **Voltage problems versus current problems.** One may construct a conductivity reconstruction problem using internal voltage u or internal current $\mathbf{J}$. We first show that they are equivalent to each other in two space dimensions but not in higher space dimensions under the condition $f = 0$ in (4). The equivalence in 2-dimensions is a well-known fact as shown in [1] for example.

For a divergence free vector field $\mathbf{J}$, we introduce a stream function ψ that satisfies

$$\mathbf{J} = \nabla^{\perp}\psi, \quad \text{where } \nabla^{\perp} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}.$$

From (6) and (8), we have

$$\begin{aligned} \nabla \times \left(r\nabla^{\perp}\psi\right) &= 0, \quad \text{in } \Omega, \\ \nabla^{\perp}\psi \cdot \mathbf{n} &= g, \quad \text{on } \partial\Omega, \end{aligned}$$

which is written as

$$0 = \nabla \times \left(r\nabla^{\perp}\psi\right) = \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{:=S} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \psi = \nabla \cdot \left(S\nabla\psi\right),$$

$$g = \nabla^{\perp}\psi \cdot \mathbf{n} = \begin{pmatrix} \partial_y\psi & -\partial_x\psi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T(x) = \nabla\psi \cdot T(x), \tag{9}$$

where $T(x)$ is a counterclockwise unit tangent vector on $\partial\Omega$. If ℓ is an arc length parameter on $\partial\Omega$, (9) becomes a Dirichlet boundary condition,

$$\psi = G \quad \text{on } \partial\Omega, \quad \text{where } G := \psi(x(0)) + \int_0^{\ell} g(x(\tau))d\tau.$$

Therefore we have a voltage problem with Dirichlet boundary condition,

$$\nabla \cdot (S\nabla\psi) = 0, \quad \text{in } \Omega, \tag{10}$$

$$\psi = G, \quad \text{on } \partial\Omega. \tag{11}$$

There is a one-to-one correspondence between S and r and it is enough to obtain S instead of r.

A potential for a divergence free vector field in n-dimension is an $\binom{n}{2}$-dimensional quantity. This is the dimension of a space of 2-forms. For 3-dimensions, we know it is a vector potential,

$$\mathbf{J} = \nabla \times \mathbf{B}.$$

Then (8) becomes

$$\nabla \times \mathbf{B} = -\sigma\nabla u. \tag{12}$$

We take a divergence on (12) and obtain a single and linear equation for σ. This applies in any dimension. There are no obstacles in applying what one can do in 2-dimensions and indeed, [20] dealt with arbitrary dimensions.

However for a current problem, (12) gives us three equations with two unknowns u and σ. Thus for a real-valued σ, this is an over-determined problem. If we restrict ourselves to know only one component of $\mathbf{J}$ or $\mathbf{B}$, we will have a non-linear problem since the unknown σ and the unknown components of $\mathbf{J}$ or $\mathbf{B}$ will be multiplied together. Taking curl to have $\nabla \times (r\mathbf{J}) = 0$ does not help. Hence the properties of the voltage problems and the current problems are different in dimension $n \geq 3$.

If the current $\mathbf{J}$ is the given data, the curl equation (3) gives a direct way to compute the resistivity r. However, the divergence equation (1) only shows the requirement of the internal current data and the information for the conductivity reconstruction comes from Ohm's law (8). This is the reason why the inverse problem based on (1) becomes nonlinear even with internal data. Since the reconstruction process is based on two equations, an iteration method has been used. However, (3) is only a linear problem for r. See [11, 12, 13, 14, 15] for further conductivity reconstruction studies based on the curl equation.

2.2. **Problem description.** The vector field $\mathbf{J}$ is usually assumed to be divergence free but we prove the uniqueness and the existence without such an assumption since the current $\mathbf{J}$ may contain a noise and hence $\nabla \cdot \mathbf{J}$ might not be zero. To make it clear, we denote the current with a noise by $\mathbf{F}$ instead of $\mathbf{J}$. However, $\mathbf{F}$ cannot be an arbitrary vector field. We will first introduce a notion of an admissibility which is a sufficient condition for a unique solvability for conductivity.

Definition 1. *Consider a two dimensional vector field* $\mathbf{F} = (f^1, f^2) \in C^{1,\alpha}(\overline{\Omega})$ *for* $0 < \alpha < 1$. *Denote* $\Gamma^+ := \{\vec{x} \in \partial\Omega \,|\, \mathbf{F}^\perp \cdot \mathbf{n}(\vec{x}) > 0\}$, $\Gamma^- := \{\vec{x} \in \partial\Omega \,|\, \mathbf{F}^\perp \cdot \mathbf{n}(\vec{x}) < 0\}$, $\Gamma^0 := \{\vec{x} \in \partial\Omega \,|\, \mathbf{F}^\perp \cdot \mathbf{n}(\vec{x}) = 0\}$ *and* $\Omega' := \overline{\Omega} \setminus \Gamma^0$, *where* $\mathbf{F}^\perp := (-f^2, f^1)$. *The vector field* $\mathbf{F}$ *is called admissible in this paper if* $\mathbf{F} \neq 0$ *in* $\overline{\Omega}$ *and* $\Gamma^\pm$ *are connected.*

If the conductivity σ is $C^{1,\alpha}(\overline{\Omega})$ and the source f is $C^{0,\alpha}(\overline{\Omega})$, then it is well known that the voltage u is $C^{2,\alpha}(\overline{\Omega})$ and the current $\mathbf{F}$ is $C^{1,\alpha}(\overline{\Omega})$ (see Theorem 6.19 [6]). The regularity of $\mathbf{F}$ in the definition is consistent with classical Schauder theory.

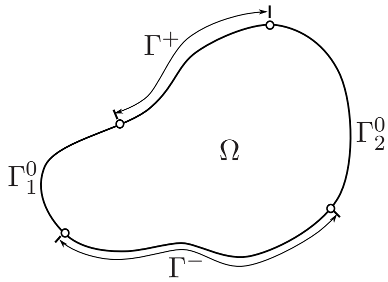


FIGURE 1

The part of boundary, Γ^0, consists of two components, $\Gamma^0 = \Gamma_1^0 \cup \Gamma_2^0$ and each of them can be a single point. However, in general, it can be more than a single point and we include such a case in our analysis (see Figure 1). The well-posedness of the conductivity reconstruction is stated in the following theorem using the notion of Definition 1:

Theorem 2. *Let Ω be a bounded simply connected open set with $C^{2,\alpha}$ boundary. Suppose that an admissible vector field $\mathbf{F} \in C^{1,\alpha}(\overline{\Omega})$ and a boundary resistivity $r_0 \in C^{0,\alpha}(\overline{\Gamma^-})$ are given. Then,*

(*i*) *There exists a unique $r \in C^{0,\alpha}_{loc}(\Omega') \cap C^0(\overline{\Omega})$ that satisfies*

$$\nabla \times (r\mathbf{F}) = 0 \quad in \quad \Omega, \tag{13}$$

$$r = r_0 \quad on \quad \overline{\Gamma^-} \subset \partial\Omega. \tag{14}$$

(*ii*) *Let $\tilde{r}$ be the solution for an admissible vector field $\tilde{\mathbf{F}}$ with $\tilde{\Gamma}^- = \Gamma^-$ and a $\tilde{r}_0 \in C^{0,\alpha}(\overline{\Gamma^-})$. Then, for any compact set $K \subset \Omega'$,*

$$\|r - \tilde{r}\|_{L^\infty(K)} \le C \left(\|r_0 - \tilde{r}_0\|_{L^\infty(\Gamma^-)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|^\alpha_{C^1(\overline{\Omega})} \right), \tag{15}$$

where $C = C\big(K, \|F\|_{C^{1,\alpha}(\overline{\Omega})}, \|\tilde{F}\|_{C^{1,\alpha}(\overline{\Omega})}, \|r_0\|_{C^{0,\alpha}(\overline{\Omega})}, \|\tilde{r}_0\|_{C^{0,\alpha}(\overline{\Omega})}\big)$.

Uniqueness has been shown for several reconstruction methods. However as far as authors know, the existence and the stability are obtained for the first time. One can find conditional stability in [19] for a equipotential line method, which contains certain stability structure obtained in the theorem. The proof of Theorem 2 is given in Section 3. The main technique of its proof is the method of characteristics because (3) will be a hyperbolic equation.

3. Existence, uniqueness, stability and regularity of the solution

3.1. Preliminary lemmas.

The construction of the resistivity r is based on an analysis of integral curves of the vector field $\mathbf{F}^\perp$. For any given $\mathbf{x}_0 \in \overline{\Omega}$, the integral curve is a solution of the ordinary differential equation (or ODE for brevity)

$$\frac{d}{dt}\mathbf{x}(t) = F^\perp(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \ -\infty < t < \infty. \tag{16}$$

In the following lemma we quickly summarize elementary properties of integral curves of a smooth vector field such that $\mathbf{F} \neq 0$ in $\overline{\Omega}$.

Lemma 3. *If* $\mathbf{F} \in C^1(\overline{\Omega})$ *and* $\mathbf{F} \neq 0$ *in* $\overline{\Omega}$*, then*

(i) *Integral curves of* $\mathbf{F}^\perp$ *do not touch other ones nor themselves.*
(ii) *The length of an integral curves of* $\mathbf{F}^\perp$ *is uniformly bounded.*
(iii) *Both ends of an integral curve of* $\mathbf{F}^\perp$ *are extendable to the boundary.*

Proof. Let $\mathbf{x}_0$ be a tangential or intersection point of different integral curves. This implies that there exist two solutions of (16) locally at $\mathbf{x}_0$. However, F is assumed to be smooth and hence it contradicts the existence of unique solutions to such ODEs and hence we obtained the first assertion.

The second assertion depends on the assumption $\mathbf{F}^\perp \neq 0$ in $\overline{\Omega}$. Suppose that there is an integral curve $\mathbf{x}(t)$ which is infinitely long. Then, since the domain Ω is bounded, there exist nonempty limit set $\omega(\mathbf{x})$. Since there is no critical point, Poincare-Bendixon implies that $\omega(\mathbf{x})$ is a periodic orbit. This implies that there exists a critical point in the interior of the orbit, which contradicts to the assumption $\mathbf{F}^\perp \neq 0$ in $\overline{\Omega}$. Therefore, all the integral curves are finitely long. Since $\overline{\Omega}$ is compact, they are uniformly bounded.

Since $\overline{\Omega}$ is compact and $|\mathbf{F}^\perp| > 0$ on $\overline{\Omega}$, there exists a lower bound $a > 0$ such that

$$|\mathbf{F}^\perp| \geq a > 0.$$

Suppose that an integral curve $\mathbf{x}(t)$ converges to an interior point $\mathbf{y} \in \Omega$ as $t \to \infty$. One can easily see that this is not possible since the speed of the curve is uniformly bounded from below, i.e., $|\mathbf{x}'(t)| = |\mathbf{F}^\perp(\mathbf{x}(t)| \geq a$, the curve cannot stay in a small neighborhood of $\mathbf{y}$ forever. Therefore, the integral curve $\mathbf{x}$ should connect two boundary points of $\partial\Omega$. □

We will see in the following lemma that, if the vector field is admissible in the sense of Definition 1, integral curves should connect the boundaries Γ^- and Γ^+.

Lemma 4. *If* $\mathbf{F}$ *is admissible, then the integral curve of* $\mathbf{F}^\perp$ *that passes through an interior point* $\mathbf{x}_0 \in \Omega$ *starts from* Γ^- *and ends at* Γ^+*. Furthermore, there exists* $T > 0$*, a uniform upper bound of the domain size of integral curves.*

Proof. Since the vector field $\mathbf{F}$ is assumed to be admissible, the boundary $\partial\Omega$ is divided into four parts, $\partial\Omega = \Gamma^- \cup \Gamma_1^0 \cup \Gamma^+ \cup \Gamma_2^0$, where $\mathbf{F}^\perp \cdot \vec{n}(\vec{x}) = 0$ on Γ_i^0 (see Figure 1).

Note that each Γ_i^0 is a single point or is an integral curve of $\mathbf{F}^\perp$ by the definition of admissibility. From Lemma 3, we know that the integral curve that passes through an interior point $\mathbf{x}_0$ is unique and has two end points on $\partial\Omega$, i.e., there exist $t_- < 0 < t_+$ such that

$$\mathbf{x}'(t) = \mathbf{F}^\perp(\mathbf{x}(t)) \text{ for } t_- < t < t_+, \quad \mathbf{x}(t_-), \mathbf{x}(t_+) \in \partial\Omega.$$

Since $\mathbf{x}'(t_-) \cdot \vec{n} \leq 0$ and $\mathbf{x}'(t_+) \cdot \vec{n} \geq 0$, we have $\mathbf{x}(t_-) \in \Gamma^- \cup \Gamma_1^0 \cup \Gamma_2^0$ and $\mathbf{x}(t_+) \in \Gamma^+ \cup \Gamma_1^0 \cup \Gamma_2^0$. If any of Γ_i^0's is not a single point, then they are integral curves by definition. Since two integral curves do not intersect with each other for admissible vector fields, $\mathbf{x}(t_-) \in \Gamma^-$ and $\mathbf{x}(t_+) \in \Gamma^+$.

Suppose that Γ_1^0 is a single point and $\mathbf{x}(t_-) \in \Gamma_1^0$ as in Figure 2. (Notice that it is enough to show that this is not possible. Then, it implies $\mathbf{x}(t_-) \notin \Gamma_2^0$ by the same arguments and hence $\mathbf{x}(t_-) \in \Gamma^-$. The same arguments also give $\mathbf{x}(t_+) \in \Gamma^+$ and the first part of proof is complete.) Then, $\mathbf{x}(t_+) \in \Gamma^+ \cup \Gamma_2^0$. If Γ_2^0 is not an single point, then, by the same reason, $\mathbf{x}(t_+) \in \Gamma^+$. In any case, $\mathbf{x}(t_+) \in \overline{\Gamma^+} \setminus \Gamma_1^0$. Let $\mathbf{y}_0$ be an interior point of a region surrounded by the integral curve $\mathbf{x}(t)$, $t_- < t < t_+$, and Γ^+. The integral curve $\mathbf{y}(t)$ that passes through the point $\mathbf{y}_0$ should start from $\overline{\Gamma^-}$. Therefore, the integral curve $\mathbf{y}(t)$ should intersect the integral curve $\mathbf{x}(t)$, which contradicts to Lemma 3. Therefore $\mathbf{x}(t_-) \notin \Gamma_1^0$ even if Γ_1^0 is a single point. Similarly $\mathbf{x}(t_-) \notin \Gamma_2^0$ and hence $\mathbf{x}(t_-) \in \Gamma^-$. Similarly $\mathbf{x}(t_+) \in \Gamma^+$.

Since the $|\mathbf{F}^\perp|$ is uniformly bounded below away from zero and the length of an integral curve is uniformly bounded, there exists $T > 0$ such that the domain size of any integral curve is less than T, i.e.,

$$t_+ - t_- \leq T,$$

which completes the proof □

We will always consider an admissible vector field in Definition 1. The boundary Γ^- is assumed to be smooth, where the curve $\gamma : [0, L] \to \overline{\Gamma^-}$ is $C^{2,\alpha}$. We will write the whole set of integral curves appeared earlier into a mapping of two parameters, such that

$$\frac{\partial}{\partial t}\mathbf{x}(s,t) = \mathbf{F}^\perp(\mathbf{x}(s,t)), \quad \mathbf{x}(s,0) = \gamma(s), \quad 0 \leq s \leq L. \tag{17}$$

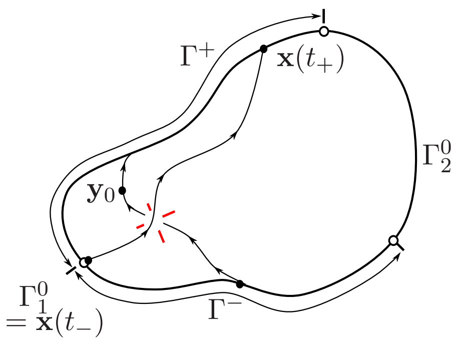


FIGURE 2. An illustration for the proof of Lemma 4.

The domain of the mapping $\mathbf{x}$ is a the closure of a bounded open subset $E \subset [0, L] \times [0, T]$. In the following lemma we will see that the mapping $\mathbf{x}$ gives a new coordinate system of the problem.

Lemma 5. *Let* $\mathbf{F}$ *be admissible.* (*i*) *The mapping* $\mathbf{x} : \overline{E} \to \overline{\Omega}$ *defined by the relation* (17) *is a homeomorphism.* (*ii*) *Furthermore, its restriction* $\mathbf{x} : E' \to \Omega'$ *is a* C^1*-diffeomorphism, where* $E' = \mathbf{x}^{-1}(\Omega')$.

Proof. Lemma 3 implies that the mapping $\mathbf{x} : \overline{E} \to \overline{\Omega}$ is one-to-one. If not, $\mathbf{x}(s,t) = \mathbf{x}(s',t')$ for some $(s,t) \neq (s',t')$. This implies that an integral curve is touched by another one, if $s \neq s'$, or by itself, if $s = s'$. Then, it contradicts Lemma 3(*i*). Lemma 4 implies that $\Omega' \subset \mathbf{x}(\overline{E})$. To show $\mathbf{x}$ is a surjection, it is enough to show that Γ_1^0 and Γ_2^0 are actually integral curves $\mathbf{x}(0,\cdot)$ and $\mathbf{x}(L,\cdot)$. If each of them is a single point, there is nothing to prove. If not, we already know from Definition 1 that they are.

Now we show that $\mathbf{x}$ is continuous. In fact we will show that it is Lipschitz. Consider

$$|\mathbf{x}(s,t)-\mathbf{x}(s',t')| \le |\mathbf{x}(s,t)-\mathbf{x}(s,t')| + |\mathbf{x}(s,t')-\mathbf{x}(s',t')|.$$

The first term is estimated by

$$|\mathbf{x}(s,t)-\mathbf{x}(s,t')| \le \|\partial_t\mathbf{x}\|_\infty |t-t'| \le \|F\|_\infty \; |t-t'|.$$

To estimate the second term, we first consider

$$\begin{aligned}\frac{\partial}{\partial t}|\mathbf{x}(s,t)-\mathbf{x}(s',t)|\Big|_{t=t'} &= |\mathbf{F}^\perp(\mathbf{x}(s,t')) - \mathbf{F}^\perp(\mathbf{x}(s',t'))| \\ &\le \|D\mathbf{F}\|_\infty \; |\mathbf{x}(s,t')-\mathbf{x}(s',t')|.\end{aligned}$$

Therefore, Gronwall's inequality gives, for $C = e^{T\|D\mathbf{F}\|_\infty}$,

$$\begin{aligned}|\mathbf{x}(s,t)-\mathbf{x}(s',t)| &\le C|\mathbf{x}(s,0)-\mathbf{x}(s',0)| \\ &= C|\gamma(s)-\gamma(s')| \\ &\le C\;\|\gamma'\|_\infty\;|s-s'|.\end{aligned}$$

Combining these estimates, we have, for some constant $C>0$,

$$|\mathbf{x}(s,t)-\mathbf{x}(s',t')| \le C|(s,t)-(s',t')|. \tag{18}$$

Furthermore, since $\mathbf{x}$ is a continuous bijection from a compact set to a compact set, its inverse is also continuous and hence $\mathbf{x}$ is homeomorphism.

Differentiability of the mapping $\mathbf{x}(s,t)$ in s and t variables in E' is well-known from ODE theory (see Theorem 7.5 in [5] on pp.30 and remark on pp.23). We now show the differentiability of $\mathbf{x}^{-1}$ on Ω'. To do that it is enough to show that the determinant of the Jacobian matrix $D\mathbf{x}(s,t)$ is not zero on E'. Differentiation of (17) with respect to t and s gives

$$\begin{aligned}\partial_t\partial_s\mathbf{x}(s,t) &= D\mathbf{F}^\perp(\mathbf{x}(s,t))\partial_s\mathbf{x}(s,t), \\ \partial_t\partial_t\mathbf{x}(s,t) &= D\mathbf{F}^\perp(\mathbf{x}(s,t))\partial_t\mathbf{x}(s,t),\end{aligned}$$

which can be written in terms of Jacobian matrix as

$$\partial_t D\mathbf{x}(s,t) = D\mathbf{F}^\perp(\mathbf{x}(s,t))D\mathbf{x}(s,t).$$

Therefore, the determinant of the Jacobian matrix is given by

$$\big|D\mathbf{x}(s,t)\big| = \big|D\mathbf{x}(s,0)\big| \exp\Big(\int_0^t tr\big(D\mathbf{F}^\perp(\mathbf{x}(s,\tau))\big)\,d\tau\Big),$$

(see Theorem 7.3 in [5], pp.28). On the other hand,

$$\big|D\mathbf{x}(s,0)\big| = \big|[\partial_s\mathbf{x}(s,0), \partial_t\mathbf{x}(s,0)]\big| = \gamma'(s)\times\mathbf{F}^\perp(\gamma(s)).$$

Since $\mathbf{F}^\perp(\gamma(s))\cdot\vec{n} < 0$ for $\gamma(s)\in\Gamma^-$ and $\gamma'(s)\cdot\vec{n}=0$, $\mathbf{F}^\perp(\gamma(s))$ and $\gamma'(s)$ are not parallel to each other. Therefore, $\big|D\mathbf{x}(s,0)\big| \ne 0$ and hence $\big|D\mathbf{x}(s,t)\big| \ne 0$ for all $t>0$ for all $(s,t)\in E'$. □

3.2. **Proof of Theorem 2.** In this section we will show the well-posedness of the inverse problem of finding r that satisfies (13-14) for given $\mathbf{F}$ and r_0.

Proof of Theorem 2. Let $\mathbf{x} : \overline{E} \to \overline{\Omega}$ be the homeomorphism in Lemma 5. Then for any $\mathbf{x}_0 \in \overline{\Omega}$ there exist $0 \le s_0 \le L$ and $0 \le t_0 \le T$ such that $\mathbf{x}_0 = \mathbf{x}(s_0, t_0)$, i.e.,

$$\frac{\partial}{\partial t}\mathbf{x}(s_0, t) = \mathbf{F}^{\perp}(\mathbf{x}(s_0, t)),\ 0 \le t \le T,$$
$$\mathbf{x}(s_0, 0) \in \Gamma^-, \quad \mathbf{x}(s_0, t_0) = \mathbf{x}_0.$$

If r is smooth, then we have following equivalence relations.

$$\begin{aligned} \nabla \times (r\mathbf{F}) = 0 &\iff (rf^2)_x - (rf^1)_y = -\mathbf{F}^{\perp} \cdot \nabla r + (f^2_x - f^1_y)r = 0 \\ &\iff -\frac{d}{dt}r(\mathbf{x}(s,t)) + (\nabla \times \mathbf{F})r = 0 \\ &\iff \frac{\frac{d}{dt}r(\mathbf{x}(s,t))}{r(\mathbf{x}(s,t))} = \nabla \times \mathbf{F}(\mathbf{x}(s,t)). \end{aligned} \tag{19}$$

Therefore, the resistivity r at $\mathbf{x}_0 = \mathbf{x}(s_0, t_0)$ should be given by

$$r(\mathbf{x}_0) = r(\mathbf{x}(s_0, 0)) \exp\Big(\int_0^{t_0} \nabla \times \mathbf{F}(\mathbf{x}(s_0, \tau))d\tau\Big). \tag{20}$$

Since the relations are equivalent this is the unique weak solution.

In the following, we will first show that $\big(r \circ \mathbf{x}\big)(s,t)$ has the regularity of $C^{0,\alpha}(\overline{E})$. Then the Lemma 5 will imply $r(x,y) \in C^{0,\alpha}(\Omega') \cap C^0(\overline{\Omega})$ as in statement of Theorem 2 because $\mathbf{x}^{-1}(x,y)$ is continuous in $\overline{\Omega}$ and differentiable in Ω'.

Let $\mathbf{x}_i \in \overline{\Omega}$ and $\mathbf{x}(s_i, t_i) = \mathbf{x}_i$ for $i = 1, 2$. First $\big(r \circ \mathbf{x}\big)(s,t)$ is differentiable with respect to the variable t by (19). Also, $\mathbf{x}(s,t)$ is Lipschitz and $r_0(s)$ is Hölder continuous on the boundary Γ^- with respect to the variable s, hence their composition map $s \to r(\mathbf{x}(s, 0))$ is also Hölder continuous with respect to s. Similarly, the map $s \to e^{\left(\int_0^{t_0} \nabla\times\mathbf{F}(\mathbf{x}(s,\tau))d\tau\right)}$ is Hölder continuous and hence r in (20) is Hölder continuous with respect to s because it is given by the product of those two maps. Therefore $r \circ \mathbf{x} \in C^{0,\alpha}(\overline{E})$ and hence $r = r \circ \mathbf{x} \circ \mathbf{x}^{-1} \in C^{0,\alpha}(\Omega') \cap C^0(\overline{\Omega})$.

Now we show stability, the second part of Theorem 2. Let $\tilde{\mathbf{F}}$ be another admissible vector field and $\tilde{\mathbf{x}} : \tilde{E} \to \Omega$ and $\tilde{r} : \overline{\Omega} \to \boldsymbol{R}$ be the corresponding diffeomorphism and resistivity, respectively. We assume $\Gamma^- = \tilde{\Gamma}^-$ and $\mathbf{x}(s, 0) = \tilde{\mathbf{x}}(s, 0)$ for $s \in [0, L]$ for a simpler representation. We will show (15), for a fixed compact subset $K \subset \Omega'$. Let $\mathbf{x}_0 \in K$ be fixed and $\mathbf{x}_0 = \mathbf{x}(s_0, t_0) = \tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0)$ where $\Delta t := \tilde{t}_0 - t_0 \ge 0$ (see Figure 3 for an illustration). Consider, for $t \in [0, t_0]$,

$$\begin{aligned} |\partial_t\mathbf{x}(s_0,&t_0 - t) - \partial_t\tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0 - t)| \\ &= | - \mathbf{F}^{\perp}(\mathbf{x}(s_0, t_0 - t)) + \tilde{\mathbf{F}}^{\perp}(\tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0 - t))| \\ &\le | - \mathbf{F}^{\perp}(\mathbf{x}(s_0, t_0 - t)) + \tilde{\mathbf{F}}^{\perp}(\mathbf{x}(s_0, t_0 - t))| \\ &+ | - \tilde{\mathbf{F}}^{\perp}(\mathbf{x}(s_0, t_0 - t)) + \tilde{\mathbf{F}}^{\perp}(\tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0 - t))| \\ &\le \|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty + \|D\tilde{\mathbf{F}}\|_\infty|\mathbf{x}(s_0, t_0 - t) - \tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0 - t)|. \end{aligned}$$

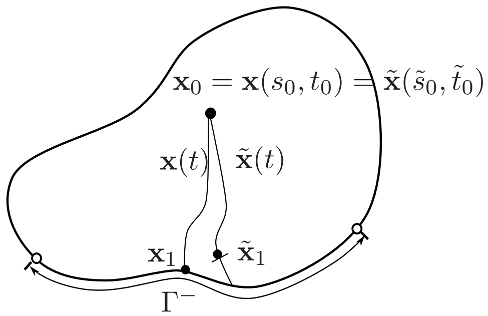


FIGURE 3. This figure is used as an illustration in the stability proof.

Therefore, Gronwall's inequality gives, for $0 < t < t_0$,

$$|\mathbf{x}(s_0, t_0 - t) - \tilde{\mathbf{x}}(\tilde{s}_0, \tilde{t}_0 - t)| \leq C\|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty, \tag{21}$$

where $C = t_0 e^{t_0 \|D\tilde{F}\|_\infty}$.

Denote $\mathbf{x}_1 := \mathbf{x}(s_0, 0) \in \Gamma^-$, $\tilde{\mathbf{x}}_1 := \tilde{\mathbf{x}}(\tilde{s}_0, \Delta t) \in \Omega$, $h(t) := \nabla \times \mathbf{F}(\mathbf{x}(s_0, t))$ and $\tilde{h}(t) := \nabla \times \tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde{s}_0, t + \Delta t))$. Then, from (20),

$$r(\mathbf{x}_0) = r(\mathbf{x}_1) e^{\int_0^{t_0} h(t)\, dt}, \quad \tilde{r}(\mathbf{x}_0) = \tilde{r}(\tilde{\mathbf{x}}_1) e^{\int_0^{t_0} \tilde{h}(t+\Delta t) dt}.$$

Hence,

$$\begin{aligned}
&|r(\mathbf{x}_0) - \tilde{r}(\mathbf{x}_0)| \\
&\quad \leq \left| r(\mathbf{x}_1) e^{\int_0^{t_0} h\, dt} - r(\mathbf{x}_1) e^{\int_0^{t_0} \tilde{h}\, dt} \right| + \left| r(\mathbf{x}_1) e^{\int_0^{t_0} \tilde{h}\, dt} - \tilde{r}(\tilde{\mathbf{x}}_1) e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\quad \leq \|r_0\|_{C^0(\Gamma^-)} \left| e^{\int_0^{t_0} h\, dt} - e^{\int_0^{t_0} \tilde{h}\, dt} \right| + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \left| e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\quad \leq \|r_0\|_{C^0(\Gamma^-)} \max\left(e^{\int_0^{t_0} h\, dt}, \ e^{\int_0^{t_0} \tilde{h}\, dt} \right) \left| \int_0^{t_0} h - \tilde{h}\, dt \right| + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \left| e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\quad \leq C\Big(\|h - \tilde{h}\|_\infty + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \Big),
\end{aligned}$$

where C depends on the same quantities that the coefficient in (15) does. Now we estimate the two terms separately.

First, we have

$$\begin{aligned}
|r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| &\leq |r(\mathbf{x}_1) - \tilde{r}(\mathbf{x}_1)| + |\tilde{r}(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)|. \\
&\leq \|r_0 - \tilde{r}_0\|_\infty + [\tilde{r}]_{C^{0,\alpha}(K')} |\mathbf{x}(s_0, 0) - \tilde{\mathbf{x}}(\tilde{s}_0, \Delta t)|^\alpha \\
&\leq \|r_0 - \tilde{r}_0\|_\infty + [\tilde{r}]_{C^{0,\alpha}(K')} (C_1 \|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty)^\alpha,
\end{aligned}$$

where, in the second inequality, K' is a compact set containing $\mathbf{x}_1$ and $\tilde{\mathbf{x}}_1$ and hence $[\tilde{r}]_{C^{0,\alpha}(K')}$ is bounded. Also we used the fact that $\mathbf{x}_1 = \mathbf{x}(s_0, 0) = \tilde{\mathbf{x}}(s_0, 0) \in \Gamma^-$. Equation (21) is used in the last inequality.

The other term is estimated by

$$\begin{aligned}|h(t)-\tilde h(t)| &\le |\nabla\times\mathbf{F}(\mathbf{x}(s_0,t))-\nabla\times\tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde s_0,t+\Delta t))|\\ &\le |\nabla\times\mathbf{F}(\mathbf{x}(s_0,t))-\nabla\times\tilde{\mathbf{F}}(\mathbf{x}(s_0,t))|\\ &+|\nabla\times\tilde{\mathbf{F}}(\mathbf{x}(s_0,t))-\nabla\times\tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde s_0,t+\Delta t))|\\ &\le \|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}+[D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})}|\mathbf{x}(s_0,t)-\tilde{\mathbf{x}}(\tilde s_0,t+\Delta t)|^\alpha\\ &\le \|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}+[D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})}(C_1\|\mathbf{F}-\tilde{\mathbf{F}}\|_\infty)^\alpha,\end{aligned}$$

where estimate (21) is used again. Therefore we have

$$\begin{aligned}|r(\mathbf{x}_0)-\tilde r(\tilde{\mathbf{x}}_0)| &\le C_4\Big((C_1^\alpha[D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})}+1)+(C_1^\alpha[\tilde r]_{C^{0,\alpha}(K')}+1)\Big)\times\\ &\qquad\Big(\|r_0-\tilde r_0\|_\infty+\|\mathbf{F}-\tilde{\mathbf{F}}\|_\infty^\alpha+\|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}\Big)\\ &\le C\Big(\|r_0-\tilde r_0\|_\infty+\|\mathbf{F}-\tilde{\mathbf{F}}\|^\alpha_{C^1(\overline{\Omega})}\Big).\end{aligned}\tag{22}$$

$[\tilde r]_{C^{0,\alpha}(K')}$ depends on $\tilde{\mathbf{F}}$, $\tilde r_0$ and K. So $C=C(\mathbf{F},\tilde{\mathbf{F}},r_0,\tilde r_0,K)$. Here we assumed $\|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}<1$ so that $\|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}<\|\mathbf{F}-\tilde{\mathbf{F}}\|^\alpha_{C^1(\overline{\Omega})}$. Note that $[\tilde r]_{C^{0,\alpha}(K')}$ in (22) is not bounded as $\mathbf{x}_0$ approaches to $\mathbf{x}(0,t)$ or $\mathbf{x}(L,t)$ in Γ^0 thus the estimate (22) holds only for $K\subset\subset\Omega'$. □

3.3. **The optimal regularity of r.** We have obtained in Theorem 2 that $r\in C^{0,\alpha}_{loc}(\Omega')\cap C^0(\overline{\Omega})$. The same regularity is true for σ if r is away from 0. If $r_0>0$, the exponential term in (20) does not alter the sign, hence $r>0$ in $\overline{\Omega}$ and r has minimum in the compact domain thus is away from 0. Thus we will freely use r or σ in the following discussions.

We will show that the regularity cannot be improved. For a forward elliptic problem, $\sigma\in C^{1,\alpha}(\overline{\Omega})$ guarantees $\mathbf{J}\in C^{1,\alpha}(\overline{\Omega})$ and $\sigma\in C^{1,\alpha}(\Omega)$ guarantees $\mathbf{J}\in C^{1,\alpha}(\Omega)$ without a boundary estimate. We will show that the above conditions are not necessary ones. One may lose one degree of interior regularity and even the Hölder continuity on the boundary since a less regular conductivity may produce a more regular current. This can be observed in the following examples.

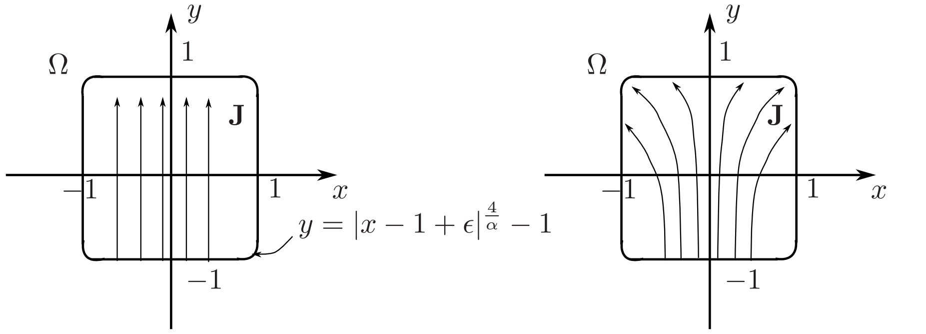
First, we will show that we lose Hölder continuity of σ on boundary, i.e., $r\notin C^{0,\alpha}(\overline{\Omega})$ in general. Consider an example,

$$r(x,y):=f(y)>0,\quad u(x,y):=-\int_0^y f(y')\,dy'.$$

This is an example of one dimensional electrical current in two space dimensions and one can easily check that the electrical current is

$$\mathbf{J}=-\sigma\nabla u=\begin{pmatrix}0\\1\end{pmatrix},$$

which is a real analytic function. Consider a domain given as in Figure 4(a), where a part of its boundary is along the line $y=-1$. According to Definition 1, this part of boundary belongs to Γ^0. Set $f(y)=1+|y+1|^{\frac{\alpha}{2}}$. This certainly does not belongs to $C^{0,\alpha}(\overline{\Omega})$ but belongs merely to $C^{0,\alpha}_{loc}(\Omega')$. Note that $r_0\in C^{0,\alpha}(\Gamma^-)$, provided



(a) domain of first example (b) domain of second example

FIGURE 4. These illustrations are used to show the optimality in regularity theory.

the curve at the corner of boundary is set as in Figure 4(a). One may consider a discontinuous f even if this case is excluded by the assumption $r_0 \in C^{0,\alpha}(\Gamma^-)$ in Theorem 2 since we are employing a classical Schauder theory and here $r \notin C^{0,\alpha}_{loc}(\Omega')$.

In the next example we will see one may lose one degree of interior regularity, i.e., $r \notin C^{0,\beta}_{loc}(\Omega)$ for any $\beta > \alpha$. Let the domain be given as in Figure 4 (b) and let

$$r(x,y) := \frac{1}{\left(1+|x|^{\frac{1}{2}}(1+y)\right)^3} > 0, \quad u(x,y) := \frac{-x}{\left(1+|x|^{\frac{1}{2}}(1+y)\right)^2}.$$

Then, the electrical current is

$$\mathbf{J} = -\sigma\nabla u = \begin{pmatrix} 1 \\ -2x|x|^{\frac{1}{2}} \end{pmatrix},$$

which is $C^{1,\alpha}(\overline{\Omega})$. However $r \in C^{0,\alpha}(\Omega')$ but $r \notin C^{0,\beta}(\Omega')$ for any $\beta > \alpha$.

One might think that the assumption $r_0 \in C^{0,\alpha}(\overline{\Gamma^-})$ in Theorem 2 is the reason to loose regularity. However

$$r(\mathbf{x}_0) = r(\mathbf{x}(s_0,0)) \exp\Big(\int_0^{t_0} \nabla\times\mathbf{F}(\mathbf{x}(s_0,\tau))d\tau\Big),$$

and the regularity of r depends also on the the vector field $\mathbf{F}$, hence increasing the boundary regularity of r_0 to $C^{k,\alpha}(\overline{\Gamma^-})$ for $k \geq 1$ does not improve the regularity.

3.4. **Voltage construction.** Now let us construct the voltage u from blackthe constructed r. It is well-defined up to an addition of a constant. If $r \in C^1(\Omega)$, then the existence of u that satisfies

$$-\nabla u = r\mathbf{F} \quad \text{in} \quad \overline{\Omega} \tag{23}$$

is clear. Even if $r \in C^{0,\alpha}(\Omega)$ as in our case, the existence theory of such a $u \in H^1(\Omega)$ is classical (see Weyl [23]). Since $-\nabla u = r\mathbf{F}$ in Ω, we conclude $u \in C^{1,\alpha}(\Omega')\cap C^1(\overline{\Omega})$.

We can also directly construct u. Define $\tilde{u} : \overline{E} \to \boldsymbol{R}$ by

$$\tilde{u}(s,0) := -\int_0^s r_0\big(\gamma(\tau)\big)\mathbf{F}\big(\gamma(\tau)\big)\cdot\gamma'(\tau)d\tau,$$
$$\tilde{u}(s,t) := \tilde{u}(s,0),$$

and $u : \overline{\Omega} \to \boldsymbol{R}$ by $u = \tilde{u} \circ \mathbf{x}^{-1}$. Then, one can easily see that $-r\mathbf{F} = \nabla u$ in $\overline{\Omega}$. We have obtained an optimal answer to the inverse Schauder solvability for σ and u.

4. Boundary control and admissibility

Theorem 2.8 in [1] says exactly that one can construct an admissible $\mathbf{J}$ by controlling the Neumann boundary condition. For a completeness we quote the theorem here.

Theorem 6 (Alessandrini et al.)**.** *Let $g \in H^{-1/2}(\partial\Omega)$ be such that $\partial\Omega$ can be split into $2M$ closed arcs $\Gamma_1, ..., \Gamma_{2M}$ such that $(-1)^j g \geq 0$ on $\Gamma_j, j = 1, ..., 2M$, in the sense of distributions. Let $u \in W^{1,2}(\Omega)$ be a solution of (1) and satisfying the Neumann condition (2) on $\partial\Omega$. Then, the geometric critical points of u in Ω, when counted according to their indices, are at most $M - 1$.*

By considering a case $M = 1$, we can easily obtain an admissible current density. For $\mathbf{J} \in C^{1,\alpha}(\overline{\Omega})$ the geometric critical point is simply the usual critical point.

Appendix. Comparison between conductivity and resistivity

If $r \neq 0$ or r is invertible, the conductivity σ is given by $\sigma = r^{-1}$, the inverse of the resistivity. Then, one can easily see that

$$-\operatorname{div}(\sigma\nabla u) = \operatorname{div}(\mathbf{F}).$$

If $\mathbf{F}$ is an electrical current without noise, then $\operatorname{div}(\mathbf{F}) = 0$. If a noise is included, $\mathbf{F}$ is not divergence free in general. Therefore, the above relation is what we can expect and the curl equation for resistivity is naturally connected to the divergence equation for conductivity with a forcing term. Theorem 2 and previous discussion imply that the resistivity r and voltage u are well-defined if a boundary resistivity r_0 and an admissible $\mathbf{F}$ are given.

The curl-based resistivity formulation and the divergence-based conductivity formulation show a difference when one considers degenerate elliptic operators with $\sigma = 0$ or $r = \infty$. Remember that the positivity of r_0 is not assumed in Theorem 2. Even if r_0 changes its sign, r is well defined by the relation (20). However, if $\sigma = 0$, or $r = \infty$, then our resistivity formulation does not work. If so, $\mathbf{J} = -\sigma\nabla u = 0$ at some points and hence the electrical current is not admissible and Theorem 2 is not applicable. On the other hand, for a case with $\sigma = \infty$ in a region, the curl equation $\nabla \times (r\mathbf{J}) = 0$ with the corresponding resistivity $r = 0$ in the region may handle the situation.

The equivalence between (1),(2) and (3),(4),(5) gives an implication that, if $0 < r < \infty$, the conductivity and resistivity formulations are equivalent. However, if $\sigma = \infty$, then it will be a better choice to work with resistivity r or vice versa.

References

[1] G. Alessandrini and Rolando Magnanini, *Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions*, SIAM Journal on Mathematical Analysis **25** (1994), no. 5, 12591268.

[2] Giovanni Alessandrini, *An identification problem for an elliptic equation in two variables*, Annali di matematica pura ed applicata **145** (1986), no. 1, 265295.

[3] G. Bal, E. Bonnetier, F. Monard, and F. Triki, *Inverse diffusion from knowledge of power densities*, ArXiv e-prints (2011).

[4] G. Bal, C. Guo, and F. Monard, *Inverse anisotropic conductivity from internal current densities*, ArXiv e-prints (2013).

[5] Earl A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[6] David Gilbarg and Neil S Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[7] Yong Jung Kim, Ohin Kwon, Jin Keun Seo, and Eung Je Woo, *Uniqueness and convergence of conductivity image reconstruction in magnetic resonance electrical impedance tomography*, Inverse Problems **19** (2003), no. 5, 12131225.

[8] Ian Knowles and Robert Wallace, *A variational solution for the aquifer transmissivity problem*, Inverse problems **12** (1996), no. 6, 953.

[9] Ohin Kwon, June-Yub Lee, and Jeong-Rock Yoon, *Equipotential line method for magnetic resonance electrical impedance tomography*, Inverse Problems **18** (2002), no. 4, 10891100.

[10] June-Yub Lee, *A reconstruction formula and uniqueness of conductivity in MREIT using two internal current distributions*, Inverse Problems **20** (2004), no. 3, 847858.

[11] Min Gi Lee, *Network approach to conductivity recovery*, **MS thesis, KAIST** (2009).

[12] Min Gi Lee and Yong-Jung Kim, *Existence and uniqueness in anisotropic conductivity reconstruction with Faraday's law*, preprint.

[13] Min Gi Lee, Min-Su Ko, and Yong-Jung Kim, *Orthotropic conductivity reconstruction with virtual resistive network and Faraday's law*, preprint.

[14] ———, *Virtual resistive network and conductivity reconstruction with Faraday's law*, Inverse Problems **30** (2014), no. 12, 125009, 21. MR 3291123

[15] Tae Hwi Lee, Hyun Soo Nam, Min Gi Lee, Yong Jung Kim, Eung Je Woo, and Oh In Kwon, *Reconstruction of conductivity using the dual-loop method with one injection current in MREIT*, Physics in Medicine and Biology **55** (2010), no. 24, 7523.

[16] F. Monard and G. Bal, *Inverse anisotropic conductivity from power densities in dimension* $n \geq 3$, ArXiv e-prints (2012).

[17] Franccois Monard and Guillaume Bal, *Inverse anisotropic diffusion from power density measurements in two dimensions*, Inverse Problems **28** (2012), no. 8, 084001, 20.

[18] Francois Monard and Guillaume Bal, *Inverse diffusion problems with redundant internal information*, Inverse Problems and Imaging **6** (2012), no. 2, 289313.

[19] Adrian Nachman, Alexandru Tamasan, and Alexandre Timonov, *Conductivity imaging with a single measurement of boundary and interior data*, Inverse Problems **23** (2007), no. 6, 25512563.

[20] Gerard R. Richter, *An inverse problem for the steady state diffusion equation*, SIAM Journal on Applied Mathematics **41** (1981), no. 2, 210221.

[21] Jin Keun Seo, Ohin Kwon, and Eung Je Woo, *Magnetic resonance electrical impedance tomography (MREIT): conductivity and current density imaging*, Journal of Physics: Conference Series, vol. 12, 2005, p. 140.

[22] Jin Keun Seo and Eung Je Woo, *Magnetic resonance electrical impedance tomography (MREIT)*, SIAM Review **53** (2011), no. 1, 40–68.

[23] Hermann Weyl, *The method of orthogonal projection in potential theory*, Duke Mathematical Journal **7** (1940), 411444.

[24] E. J Woo and J. K Seo, *Magnetic resonance electrical impedance tomography (MREIT) for high-resolution conductivity imaging*, Physiological Measurement **29** (2008), R1.

(Yong-Jung Kim)
Department of Mathematical Sciences, KAIST, Daejeon 305-701, Republic of Korea and
National Institute of Mathematical Sciences, Daejeon 305-811, Republic of Korea
E-mail address: yongkim@kaist.edu

(Min-Gi Lee)
Computer, Electrical and Mathematical Sciences & Engineering, 4700 King Abdullah University of Science & Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia
E-mail address: mgleemail@gmail.com