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# A GENERALIZATION OF THE MOMENT PROBLEM TO A COMPLEX MEASURE SPACE AND AN APPROXIMATION USING BACKWARD MOMENTS

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Abstract. One traditionally considers positive measures in the moment problem. However, this restriction makes its theory and application limited. The main purpose of this paper is to generalize it to deal with complex measures. More precisely, the theory of the truncated moment problem is extended to include complex measures. This extended theory provides considerable flexibility in its applications. In fact, we also develop an approximation technique based on control of moments. The key idea is to use the heat equation as a link that connects the generalized moment problem and this approximation technique. The *backward moment* of a measure is introduced as the moment of a solution to the heat equation at a backward time and then used to approximate the given measure. This approximation gives a geometric convergence order as the number of moments under control increases. Numerical examples are given that show the properties of approximation technique.

1. **Introduction.** Let a doubly indexed complex sequence $\alpha_{ij} \in \boldsymbol{C}$ satisfy $\alpha_{ij} = \overline{\alpha}_{ji}$. Then the full complex $K$-moment problem related to a set $K \subset \boldsymbol{C}$ and this sequence is to find a positive Borel measure $\mu$ that is supported on $K$ and satisfies

$$\alpha_{ij} = \int \overline{z}^i z^j \, d\mu, \qquad i, j \geq 0. \tag{1}$$

Depending on the choice of $K$, the problem is called with the names Stieltjes ($K = \boldsymbol{R}^+$), Hamburger ($K = \boldsymbol{R}$), Hausdorff ($K = [a, b]$), and Toeplitz ($K = \mathbf{T}$) (see [1, 2, 20, 24]). However, if $K \subset \boldsymbol{R}$, then $\alpha_{ij} = \int x^{i+j} d\mu = \alpha_{ji} = \overline{\alpha}_{ij}$. Therefore, the doubly indexed sequence $\alpha_{ij}$ is actually a singly indexed one with real values (i.e., $\alpha_{i+j} := \alpha_{ij}$), and (1) becomes

$$\alpha_k = \int x^k d\mu, \qquad k \geq 0. \tag{2}$$

Hence the word 'complex' in the name of classical complex moment problem indicates that the support $K$ can be a subset of the complex plane $\boldsymbol{C}$. The main part of its theory is to determine necessary and sufficient conditions of the sequence for

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the existence and the uniqueness of such a nonnegative measure $\mu$. (Readers are referred to [3, 22, 23, 25] for the multidimensional full moment problem.)

The truncated moment problem is to find a positive Borel measure that satisfies (1) for $0 \le i, j < n$. In particular, a measure in an atomic representation $d\mu = \sum_{i=1}^n \rho_i \delta(x - c_i)dx$ is mostly considered. If $\mathrm{supp}(\mu) \subset \boldsymbol{R}$, then the truncated moment problem is finding $2n$ unknowns in $\rho_i \ge 0$ and $c_i \in \boldsymbol{R}$ that satisfy

$$\alpha_k = \sum_{i=1}^n \rho_i c_i^k, \quad 0 \le k < 2n. \tag{3}$$

This truncated moment problem has a direct impact on the full moment problem. Indeed, using a weak compactness arguments, one may solve the full problem from the solutions of truncated ones. However, a solution to the full moment problem does not give a one for the truncated one. For detailed discussions on the truncate moment problem readers are referred to [7, 8, 9, 15] for one dimensional cases and [10, 11, 16] for multidimensional cases.

It is well-known that if $\alpha_k$'s are the $k$-th moments of a nonnegative function, then both of the full and the truncated moment problems are solvable. For the case with $\mathrm{supp}(\mu) \subset \boldsymbol{R}$, the full moment problem (2) can be solved for any given sequence by constructing a singed or a complex measure that satisfies (2) (see [4, 13]). However, the truncated moment problem (3) is not solvable without certain positivity structure even if complex solutions are included. If $\mathrm{supp}(\mu) \subset \boldsymbol{C}$, even the solvability of the full moment problem is not known without the positivity restriction.

We have two purposes in this paper. The first one is to develop a theory to generalize the truncated moment problem and drop the positivity restriction. To do that we first introduce a nontrivial nonnegative function $\varrho_0(x) \ge 0$ for an imaginary part and take a sequence of complex numbers

$$m_k := \alpha_k + i\beta_k, \text{ where } \beta_k := \int x^k \varrho_0(x)dx. \tag{4}$$

The letter $i$ is to denote the imaginary unit which should be easily distinguished from the index $i$ from the context. The truncated moment problem related to the complex sequence $m_k$ in this paper is to find a complex measure $\mu$ such that

$$m_k = \int x^k d\mu, \quad \mathrm{supp}(\mu) \subset \boldsymbol{C}, \qquad 0 \le k < 2n. \tag{5}$$

We will only consider the discrete measure in an atomic representation $d\mu = \sum_{i=1}^n \rho_i \delta(z - c_i)dx$ with $\rho_i, c_i \in \boldsymbol{C}$ and, hence, (5) is written as

$$m_k = \sum_{i=1}^n \rho_i c_i^k, \quad 0 \le k < 2n. \tag{6}$$

We will show in Theorem 2.4 that there exists a nonnegative function $\varrho_0(x)$ such that the complex moment problem (6) is solvable. As a result we may conclude that, for any sequence $\alpha_k \in \boldsymbol{R}$, there exist $\rho_i$'s and $c_i$'s such that

$$\alpha_k = Re\left(\sum_{i=1}^n \rho_i c_i^k\right), \quad \rho_i, c_i \in \boldsymbol{C}, \quad 0 \le k < 2n,$$

where $Re(\cdot)$ takes the real part of a complex number.

The second purpose of this paper is to develop an approximation theory based on moments. We do this as an application of the generalized truncated moment problem. (In fact, this application was the motivation for the generalized theory.) Note that, due to the positivity restriction of the classical moment problem, only a positive solution to the heat equation was considered in [19]. However, the generalized theory developed in this paper gives an approximation for general sign-changing functions.

Consider the solution to the heat equation

$$u_t = u_{xx}, \quad u(x,0) = u_0(x), \qquad u, x \in \boldsymbol{R}, \tag{7}$$

where the initial value $u_0(x)$ has finite moments up to $2n$-th order, i.e., $x^{2n}u_0(x) \in L^1(\boldsymbol{R})$. One may consider $u_0(x)$ as the target function to be approximated. Let $\alpha_k(t)$ be the $k$-th order moment of the solution, i.e.,

$$\alpha_k(t) := \int_{-\infty}^{\infty} x^k u(x,t)dx, \qquad 0 \le k < 2n.$$

Note that the solution $u(x,t)$ is not defined for a backward time $t<0$ in general. However, the moment at the backward time $\alpha_k(-t_0)$ with $t_0 > 0$ is well defined using the relations in (24) and we call it a *backward moment*. Notice that this backward moment is not a moment of a nonnegative function even if $u_0(x)$ is positive. Therefore, the classical theory of the moment problem is not applicable even for a positive solution case if backward moments are considered.

However, the theory of the generalized moment problem, Theorem 2.4, gives us $\rho_i, c_i \in \boldsymbol{C}$ for any given $t_0 > 0$ that satisfy

$$Re\left(\sum_{i=1}^n \rho_i c_i^k\right) = \alpha_k(-t_0), \quad 0 \le k < 2n. \tag{8}$$

Let

$$\varphi_n(x,t) := Re\left(\sum_{i=1}^n \frac{\rho_i}{\sqrt{4\pi(t+t_0)}} e^{\frac{-(x-c_i)^2}{4(t+t_0)}}\right), \quad t>0,\ x\in\boldsymbol{R}. \tag{9}$$

Then, since the heat kernels in (9) are delta sequences with weights $\rho_i$'s as $t \to -t_0$,

$$\lim_{t\to -t_0} \int x^k \varphi_n(x,t)dx = Re(\sum_{i=1}^n \rho_i c_i^k) = \alpha_k(-t_0), \quad 0 \le k < 2n.$$

Since $\varphi_n(x,t)$ is also a solution of the heat equation, $\varphi_n(x,t)$ and $u(x,t)$ share the same moments up to order $2n-1$. This agreement of moments gives the following asymptotic convergence (see [14, 19]):

$$\|u(x,t) - \varphi_n(x,t)\|_p = O\left(t^{\frac{1}{2p} - \frac{2n+1}{2}}\right) \quad \text{as} \quad t \to \infty. \tag{10}$$

This convergence order indicates that $\varphi_n(x,t)$ is a good approximation of the solution $u(x,t)$ for $t>0$ large.

In this paper we are more interested in the approximation of a general function, i.e., the initial approximation $u_0(x) \cong \varphi_n(x,0)$. Note that, if $t_0 = 0$, then $\lim_{t\to 0}\varphi_n(x,t)$ is simply a summation of delta distributions. Hence, it is important to take the backward moments with a positive backward time $t_0 > 0$ to obtain certain regularity. In fact, if the initial heat distribution $u_0$ is of age $t_0 > 0$ (see [21] or

Remark 1) and we take it as our backward time, then one may observe numerically that

$$\frac{\|u_0(x)-\varphi_{n+1}(x,0)\|_\infty}{\|u_0(x)-\varphi_n(x,0)\|_\infty}\to\frac{v}{v+2t_0}\quad\text{as}\quad n\to\infty,$$

where the constant $v>0$ may depend on the initial value. This geometric convergence indicates that $\varphi_n(x,0)$ is a good approximation of the target function $u_0(x)$. (Similar analysis has been given in [19] for the positive solutions with $t_0=0$ and $t>0$.) One may compare this approximation technique to the Fourier integrals. Several interesting properties are discussed in Section 6 (see Figures 1–4).

This paper consists as the following. In Section 2 the truncated moment problem is generalized to include complex measures. Then, arbitrary real sequence is embedded to a complex sequence that its corresponding complex moment problem is solvable. In Section 3 the relations for the backward moments of the heat equation are given. Approximate solutions are constructed in Section 4 using the backward moments and the generalized moment problem. Several properties of approximate solutions are given in Section 5 for the special case with $\beta_k=0$. In Section 6 the property of approximation technique is discussed with a few numerical examples. Finally, in Section 7, the remaining issues in the generalized moment problem and the approximation technique are discussed. Possible directions of the further investigation of this study are also discussed.

2. **Truncated moment problem with a complex density.** In this section we extend the theory of moment problem to complex measure space. The moment problem (3) with arbitrary real $\alpha_k$'s will be understood as the real part of a complex moment problem. Let $\beta_k$ be the $k$-th moment of a nonnegative function $\varrho_0(x)\ge 0$, i.e.,

$$\beta_k=\int x^k\varrho_0(x)dx,\qquad 0\le k<2n,$$

and $m_k$'s be a sequence of complex numbers given by

$$m_k:=\alpha_k+i\beta_k,\qquad 0\le k<2n.$$

We will follow the routine of the classical moment problem to solve a complex valued moment problem,

$$\sum_{i=1}^n\rho_i c_i^k=m_k,\qquad 0\le k<2n,\tag{11}$$

where we are looking for complex solutions $\rho_i$'s and $c_i$'s. Let a column vector $\mathbf{h}_k\in\boldsymbol{C}^{n\times1}$ and the Hankel matrix $H\in\boldsymbol{C}^{n\times n}$ be given by

$$\begin{array}{ll}\mathbf{h}_k:=(m_k,m_{k+1},\cdots,m_{k+n-1})^t, & 0\le k\le n,\\ H:=(m_{i+j}), & 0\le i,j<n.\end{array}\tag{12}$$

(Here, $\boldsymbol{C}^{m\times n}$ stands for the collection of $m\times n$ complex matrices.) Note that Hankel matrix is symmetric and that the $j$-th column is $\mathbf{h}_{j-1}$. Similarly, we take the Hankel matrices and column vectors corresponding to the real sequences $\alpha_k$'s and $\beta_k$'s:

$$\begin{array}{ll}\mathbf{a}_k:=(\alpha_k,\alpha_{k+1},\cdots,\alpha_{k+n-1})^t, & 0\le k\le n,\\ \mathbf{b}_k:=(\beta_k,\beta_{k+1},\cdots,\beta_{k+n-1})^t, & 0\le k\le n,\\ A\ :=(\alpha_{i+j}), & 0\le i,j<n,\\ B\ :=(\beta_{i+j}), & 0\le i,j<n.\end{array}\tag{13}$$

The Hankel matrices and the moment vectors satisfy

$$H = A + iB, \quad \mathbf{h}_k = \mathbf{a}_k + i\mathbf{b}_k, \quad k = 0, \cdots, n.$$

Let $0 \neq \mathbf{y} \in \boldsymbol{R}^{n\times 1}$. Then,

$$\begin{aligned} \mathbf{y}^t B\mathbf{y} = \sum_{i,j=0}^{n-1} y_i y_j \beta_{i+j} &= \int \Big( \sum_{i,j=0}^{n-1} y_i x^i y_j x^j \Big) \varrho_0(x) dx \\ &= \int \Big( \sum_{k=0}^{n-1} y_k x^k \Big)^2 \varrho_0(x) dx > 0. \end{aligned}$$

Hence, the matrix $B$ is positive definite.

**Lemma 2.1.** *Let $A$ and $B$ be $n \times n$ real symmetric matrices. If $B$ is positive definite, then*
*(i) The matrix $H := A + iB$ is non-singular.*
*(ii) For $\mathbf{z} \in \boldsymbol{C}^n$,*

$$\overline{\mathbf{z}}^t H\mathbf{z} = 0 \iff \mathbf{z} = 0.$$

*Proof.* Let $\mathbf{z} := \mathbf{x} + i\mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^{n\times 1}$ satisfy $H\mathbf{z} = 0$. Then, the linear system $H\mathbf{z} = 0$ can be written as

$$\Big( H_{2n} \Big) \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Big( H_{2n} \Big) := \begin{pmatrix} B & A \\ A & -B \end{pmatrix}. \tag{14}$$

Hence, if the $2n \times 2n$ matrix $H_{2n}$ is invertible, then so is the $n \times n$ complex matrix $H$. Since the matrix $B$ is positive definite, it is invertible and a block elimination gives

$$\begin{pmatrix} I & 0 \\ -AB^{-1} & I \end{pmatrix} \begin{pmatrix} B & A \\ A & -B \end{pmatrix} = \begin{pmatrix} B & A \\ 0 & -(B + AB^{-1}A) \end{pmatrix}.$$

Let $\tilde{\mathbf{x}} = B^{-1}A\mathbf{x}$ for any given $\mathbf{x} \in \boldsymbol{R}^{n\times 1}$. Then, $B\tilde{\mathbf{x}} = A\mathbf{x}$ and

$$\mathbf{x}^t AB^{-1}A\mathbf{x} = (B\tilde{\mathbf{x}})^t B^{-1} B\tilde{\mathbf{x}} = \tilde{\mathbf{x}}^t B\tilde{\mathbf{x}} \geq 0.$$

Therefore, $AB^{-1}A$ is at least semi-positive definite. Finally we have that the Schur martix $-(B + AB^{-1}A)$ is negative definite and

$$\det(H_{2n}) = -\det(B)\det(B + AB^{-1}A) < 0,$$

which completes the proof of the first part $(i)$.

Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ satisfy $\overline{\mathbf{z}}^t H\mathbf{z} = 0$, i.e.,

$$\overline{\mathbf{z}}^t H\mathbf{z} = (\mathbf{x}^t A\mathbf{x} + \mathbf{y}^t A\mathbf{y}) + i(\mathbf{x}^t B\mathbf{x} + \mathbf{y}^t B\mathbf{y}) = 0.$$

Since $B$ is positive definite, $\mathbf{x}^t B\mathbf{x} + \mathbf{y}^t B\mathbf{y} = 0$ implies $\mathbf{x} = 0 = \mathbf{y}$ and hence $\mathbf{z} = 0$. □ □

Since $H$ is invertible, there exists a vector $\Psi = (\psi_0, \cdots, \psi_{n-1})^t$ that satisfies

$$H\Psi = \mathbf{h}_n. \tag{15}$$

This can be written as $\sum_{j=0}^{n-1} \psi_j \mathbf{h}_j = \mathbf{h}_n$ or

$$m_{n+k} - \sum_{j=0}^{n-1} \psi_j m_{j+k} = 0, \quad 0 \leq k < n. \tag{16}$$

Introduce an auxiliary polynomial,

$$g_n(z) := z^n - \sum_{j=0}^{n-1} \psi_j z^j, \quad z \in \boldsymbol{C}. \tag{17}$$

Due to the fundamental theorem of algebra, there exist $n$ complex zeros of the polynomial $g_n(z)$ including multiplicities.

The next step is to investigate the multiplicity of zeros of the auxiliary polynomial $g_n(z)$. To do that we consider a linear functional $S(f)$ defined on the polynomial space. For a given polynomial $f(z) = \sum_{i=0}^{l} f_i z^i$, $S(f)$ is defined by

$$S(f) := f_0 m_0 + \cdots + f_l m_l = \sum_{i=0}^{l} f_i m_i = \sum_{i=0}^{l} f_i(\alpha_i + i\beta_i). \tag{18}$$

One may easily see that this is the linear functional that gives the expectation of the polynomial if the sequence $m_i$'s are moments of a probability function. For example, if $m_k = \int x^k p(x)dx$ for all $k$, then

$$S(f) = \sum_{i=0}^{l} f_i m_i = \sum_{i=0}^{l} f_i \int x^i p(x) = \int f(x)p(x)dx. \tag{19}$$

Since we are interested in the application to the solutions of the heat equation in the real line, the moments $m_k$'s and the functional $S(f)$ are defined as a line integral along the real axis. In general one may take a line integral and define

$$S(f) := \oint_C f(z)p(z)dz = \int_0^1 f(z(t))p(z(t))z'(t)dt,$$

where $z = z(t)$ is a parametrization of a curve $C$. For example, the case with $C = T$, the unit circle in $\boldsymbol{C}$, is called Toeplitz.

We set the conjugate of the polynomial $f(z) = \sum_{i=0}^{l} f_i z^i$ as

$$\overline{f}(z) := \sum_{i=0}^{l} \overline{f}_i z^i.$$

In the followings we consider basic properties related to this functional and the zeros of the auxiliary polynomial $g_n(z)$.

**Lemma 2.2.** *Let the imaginary part of $m_k$'s be given by a positive density as in (4) and the polynomial $g_n(z)$ and the linear functional $S(f)$ be given by (17) and (18), respectively. Then,*
*(i) If $f \neq 0$, then $S(\overline{f}f) \neq 0$.*
*(ii) If $f(z) = (z - c_1)\cdots(z - c_k)$, then $\overline{f}(z) = (z - \overline{c}_1)\cdots(z - \overline{c}_k)$.*
*(iii) The auxiliary polynomial $g_n(z)$ given by (17) satisfies*

$$S(g_n(z)z^k) = 0, \qquad 0 \le k < n. \tag{20}$$

*Proof.* $(i)$ Let $0 \neq f(z) = \sum_{i=0}^{l} f_i z^i$. Then, $\overline{f}(z)f(z) = \sum_{i,j=0}^{l} \overline{f}_i f_j z^{i+j}$. Therefore, by Lemma 2.1$(ii)$,

$$S(\overline{f}f) = \sum_{i,j=0}^{l} \overline{f}_i f_j m_{i+j} = \overline{\mathbf{f}}^t H \mathbf{f} \neq 0,$$

where $\mathbf{f} = (f_0, f_1, \cdots, f_l)^t$.

$(ii)$ Let $f(z)=(z-c_1)\cdots(z-c_k)=\sum_{i=0}^k f_i z^i$. Then, the coefficients $f_i$'s are given by

$$f_i=\sum_{I\in A_i}\Big(\prod_{j\in I}c_j\Big),$$

where $A_i$ is the collection of all index sets consists of $k-i$ indices. Hence,

$$\overline{f}_i=\overline{\sum_{I\in A_i}\Big(\prod_{j\in I}c_j\Big)}=\sum_{I\in A_i}\Big(\prod_{j\in I}\bar{c}_j\Big).$$

In other words $\overline{f}(z)=(z-\bar{c}_1)\cdots(z-\bar{c}_k)$. The last claim $(iii)$ is obtained by comparing (16) and (17), i.e.,

$$S(g_n(z)z^k)=m_{n+k}-\sum_{j=0}^{n-1}\psi_j m_{j+k}=0,\quad 0\le k<n. \qquad \square$$

$\square$

**Lemma 2.3.** *Let the imaginary part of $m_k$'s be given by a positive density as in (4) and the polynomial $g_n(z)$ be given by (17). Then,*
*(i) If $c\in \boldsymbol{C}\setminus\boldsymbol{R}$ is a zero of $g_n(z)$, then its conjugate $\bar{c}$ is not.*
*(ii) There is no real zero of $g_n(z)$ with multiplicity two or higher.*
*(iii) If $\alpha_k=0$ for all $k$'s and $\varrho_0(x)\ge 0$ is non-trivial, then $g_n(z)$ has $n$-distinct complex zeros.*

*Proof.* $(i)$ Suppose that $c$ and its conjugate $\bar{c}$ are zeros of the polynomial $g_n(z)$. Then one may write

$$g_n(z)=(z-c)(z-\bar{c})(z-c_3)\cdots(z-c_n).$$

Let $h(z)=(z-c)(z-c_3)\cdots(z-c_n)$. Then,

$$g_n(z)(z-\bar{c}_3)\cdots(z-\bar{c}_n)=\bar{h}(z)h(z).$$

The linearity of the operator $S$, given by (18), and Lemma 2.2$(iii)$ imply that

$$S(g_n(z)(z-\bar{c}_3)\cdots(z-\bar{c}_n))=0.$$

However, Lemma 2.2$(i)$ implies that $S(\bar{h}h)\ne 0$. Therefore, if a $c\in\boldsymbol{C}$ is a zero of $g_n(z)$, then its conjugate $\bar{c}$ is not.

$(ii)$ Suppose that $g_n(z)$ has a real zero of multiplicity of two or higher, say $a\in\boldsymbol{R}$. Then we may write

$$g_n(z)=(z-a)^2(z-c_3)(z-c_4)\cdots(z-c_n).$$

Let $h(z)=(z-a)(z-c_3)\cdots(z-c_n)$. Then, since $\bar{a}=a$,

$$g_n(z)(z-\bar{c}_3)\cdots(z-\bar{c}_n)=\bar{h}(z)h(z).$$

The arguments in the previous step derive the same contradiction. Therefore, $g_n(z)$ has no real zero of multiplicity two or higher.

$(iii)$ Suppose that $c=a+ib$ is a complex zero of $g_n(z)$ with multiplicity two or higher. First, $b\ne 0$ from $(ii)$. Then we may write

$$g_n(z)=((z-a)-ib)^2h(z),\qquad h(z)=(z-c_3)\cdots(z-c_n).$$

Since $(z-a)\bar{h}(z)$ is a polynomial of degree $n-1$, the linearity of $S$ and Lemma 2.2($iii$) imply that $S(g_n(z)(z-a)\bar{h}(z))=0$. Since $a_k=0$ for all $k$, $m_k=i\beta_k$ and hence

$$S(g_n(z)(z-a)\bar{h}(z)) = i\int[(x-a)^3-b^2(x-a)-2ib(x-a)^2]\bar{h}(x)h(x)\varrho_0(x)dx.$$

The real part gives

$$\int 2b(x-a)^2\bar{h}(x)h(x)\varrho_0(x)dx=0.$$

Since the integrand is non-negative, it contradicts to the assumption that $\varrho_0(x)\geq 0$ is non-trivial. Hence, there is no zero of multiplicity two or higher. □ □

**Theorem 2.4.** *Let a sequence $\alpha_k\in\boldsymbol{R}$, $0\leq k<2n$, be given. Then, there exists a positive density function $\varrho_0(x)\geq 0$ such that, for the complex sequence*

$$m_k:=\alpha_k+i\beta_k \text{ with } \beta_k:=\int x^k\varrho_0(x)dx,\quad 0\leq k<2n,$$

*the truncated complex moment problem,*

$$\sum_{i=1}^n \rho_i c_i^k = m_k,\qquad 0\leq k<2n, \tag{21}$$

*has a solution set $\rho_i, c_i\in\boldsymbol{C}$ which is unique up to reordering. Hence, $\alpha_k = Re(\sum_{i=1}^n\rho_i c_i^k)$ for $0\leq k<2n$.*

*Proof.* First we show the following claim.

**Claim:** There exists a function $\varrho_0(x)\geq 0$ such that the auxiliary polynomial $g_n(z):=z^n-\sum_{i=0}^{n-1}\psi_i z^i$ has n-distinct zeros, where $\Psi:=(\psi_0,\cdots,\psi_{n-1})^t$ is the unique solution to (15):

Let $f(x)\geq 0$ be a non-trivial nonnegative function and

$$\gamma_k(f,\tau)=\tau\alpha_k+if_k,\qquad f_k:=\int x^kf(x)dx.$$

For a given $\tau\geq 0$, let $g_n(z;f,\tau)$ be the auxiliary polynomial decided by the moments $\gamma_k(f,\tau)$. (Hence, $g_n(z)$ in the theorem can be written as $g_n(z;\varrho_0,1)$.) We already know that $g_n(z;f,0)$ has n-distinct zeros, Lemma 2.3($iii$), and hence there exists $\tau_0>0$ such that $g_n(z;f,\tau_0)$ also has n-distinct zeros by the continuity argument or the implicit function theorem. Set $\varrho_0(x):=f(x)/\tau_0$. Then, since $\beta_k:=\int x^k\varrho_0(x)dx=f_k/\tau_0$,

$$\gamma_k(f,\tau_0)=\tau_0\alpha_k+if_k=\tau_0(\alpha_k+i\beta_k)=:\tau_0 m_k.$$

Therefore, the corresponding linear systems (15) to the two sequences $\gamma_k(f,\tau_0)$ and $m_k$ are identical and hence $g_n(z;\varrho_0,1)=g_n(z;f,\tau_0)$, which completes the proof of the claim.

Now we show the solvability of the complex moment problem (21). Let $c_i$'s be the $n$ distinct zeros of the polynomial $g_n(z)$ for $i=1,\cdots,n$. Since $c_i$'s are distinct, there exists a unique solution that solves the first $n$ equations in (21), i.e., for $0\leq k<n$. Now we complete the proof using inductive arguments. Let $0\leq l\leq n-1$. We will show that the equation in (21) holds for $k=n+l$ under the assumption that

the equations hold for all $0 \le k < n+l$. First observe that, since $c_i$'s are zeros of $z^l g_n(z), l \ge 0$,

$$c_i^{n+l} = \sum_{j=0}^{n-1} \psi_j c_i^{j+l} \qquad \text{for any} \quad 1 \le i \le n,\ k \ge 0.$$

Using the relations (16) and (21) for $k < n+l$, we obtain

$$m_{n+l} = \sum_{j=0}^{n-1} \psi_j m_{j+l} = \sum_{j=0}^{n-1} \psi_j \sum_{i=1}^{n} \rho_i c_i^{j+l} = \sum_{i=1}^{n} \rho_i \sum_{j=0}^{n-1} \psi_j c_i^{j+l} = \sum_{i=1}^{n} \rho_i c_i^{n+l}.$$

Hence, (21) holds for $k = n+l$ and hence for all $0 \le k < 2n$ by induction. □ □

In Lemma 2.3($iii$) the assumption $\alpha_k = 0$ was needed to show that $g_n(z)$ has $n$-distinct zeros. If one may show it without this assumption, then the solvability of the complex moment problem (21) is given for an arbitrary nontrivial imaginary part $\varrho_0(x) \ge 0$.

## 3. **Backward moments of solutions to the heat equation.**

Let $w(z,t)$ be the solution to the heat equation with a complex initial value, i.e.,

$$w_t = w_{xx}, \quad w(x,0) = w_0(x), \qquad t > 0,\ x \in \boldsymbol{R},\ w \in \boldsymbol{C}. \tag{22}$$

It is assumed that the initial value $w_0(x)$ decays fast enough as $|x| \to \infty$ to get its $k$-th order moment $\gamma_k(t)$ be well defined,

$$\gamma_k(t) = \int_{-\infty}^{\infty} x^k w(x,t) dx, \quad k = 0, \cdots, 2n-1 \tag{23}$$

at least for the initial time $t = 0$. One can easily show how these moments evolve as the time $t$ increases or decreases.

**Lemma 3.1.** *Suppose that the initial value $w_0(x)$ has finite moments up to $2n$-th order, say $x^{2n} w_0(x) \in L^1(\boldsymbol{R})$. Then the moments of the solution $w(x,t)$ at time $t \ge 0$ are given by*

$$\begin{aligned} \gamma_{2k}(t) &= \textstyle\sum_{l=0}^{k} \frac{(2k)!}{(k-l)!(2l)!} t^{k-l} \gamma_{2l}(0), \\ \gamma_{2k+1}(t) &= \textstyle\sum_{l=0}^{k} \frac{(2k+1)!}{(k-l)!(2l+1)!} t^{k-l} \gamma_{2l+1}(0). \end{aligned} \tag{24}$$

*Furthermore, the summations in (24) are well defined for all $t \in \boldsymbol{R}$ and identical to the moments of the solution to the backward heat equation if it is solvable up to the given backward time.*

*Proof.* Integrating by parts gives

$$\begin{aligned} \gamma_0'(t) &= \int w_t dx = \int w_{xx} dx = \big[w_x\big]_{-\infty}^{\infty} = 0, \\ \gamma_1'(t) &= \int x w_t dx = \int x w_{xx} dx = \big[x w_x - w\big]_{-\infty}^{\infty} = 0. \end{aligned}$$

Hence, $\gamma_0(t)=\gamma_0(0)$ and $\gamma_1(t)=\gamma_1(0)$ are constants which gives (24) for $k=0$ and 1. For $k\ge 2$,

$$\begin{aligned}\gamma_k'(t) &= \int x^k w_t dx = \int x^k w_{xx} dx\\ &= \left[x^k w_x - kx^{k-1}w\right]_{-\infty}^{\infty} + \int k(k-1)x^{k-2} w dx\\ &= k(k-1)\gamma_{k-2}(t).\end{aligned}$$

Hence, in summary, we have

$$\frac{d}{dt}\gamma_k(t) = \begin{cases} 0\,, & k=0 \text{ or } 1,\\ k(k-1)\gamma_{k-2}(t), & k\ge 2.\end{cases}$$

This relation shows that the even ordered moments and the odd ordered ones evolve independently and can be obtained inductively by integrating lower order moments. The formulas in (24) can be easily verified in that manner.

Consider a column vector $\mathbf{m}_{2n}(t)=(\gamma_0(t),\cdots,\gamma_{2n-1}(t))^t$ and the $2n\times 2n$ matrix $A(t)$ that consists of the coefficients in (24). Then,

$$\mathbf{m}_{2n}(t) = A(t)\mathbf{m}_{2n}(0).$$

One may easily check that the matrix multiplication $A(t)A(-t)$ gives the identity matrix for all $t>0$. Hence $A(t)$ is non-singular and the last sentence of the lemma is clear. □ □

**Remark 1.** A heat distribution $f(x)$ is called of *age* $t_0\ge 0$ if $t_0$ is the supremum of $\tau\in\boldsymbol{R}^+$ such that there exists a function $w_0(x)$, where the solution to the heat equation,

$$w_t = w_{xx},\quad w(x,0)=w_0(x),\qquad t>0,\ x\in\boldsymbol{R},$$

satisfies $w(x,\tau)=f(x)$. One may find an estimate of such an age for a positive case from [21]. However, Lemma 3.1 indicates that moments of the solution to the backward heat equation can be easily computed even if the backward problem itself is not solvable. This is not strange at all. Since for any given $t_0>0$ there may exist $W_0(x)$ such that it has an age of $t_0$ or older and shares the same moments up to order $2n-1$ with $w_0(x)$. Then the backward moments can be considered as the ones for the solution with $W_0(x)$ as its initial value.

4. **Asymptotic approximation using backward moments.** Let $u(x,t)$ be the solution to the heat equation with a real initial value $u_0(x)$ where $x^{2n}u_0(x)$ is integrable. Let $\alpha_k(t)$ be the $k$-th order moments, i.e.,

$$\alpha_k(t) := \int x^k u(x,t)dx,\qquad 0\le k<2n. \tag{25}$$

Then, for any $t_0>0$, the backward moment $\alpha_k(-t_0)$ is well defined by (24) (not by (25) ). Let $\varrho_0(x)\ge 0$ be a density function in Theorem 2.4 corresponding to the sequence $\alpha_k:=\alpha_k(-t_0)$ and $\varrho(x,t)$ be the solution of the heat equation

$$\varrho_t=\varrho_{xx},\quad \varrho(x,-t_0)=\varrho_0(x).$$

Then, Theorem 2.4 implies that, for

$$\beta_k(t) := \int x^k \varrho(x,t)dx,\qquad 0\le k<2n, \tag{26}$$

there exist $\rho_i, c_i \in \boldsymbol{C}$ that satisfy

$$\sum_{i=1}^{n} \rho_i c_i^k = m_k := \alpha_k(-t_0) + i\beta_k(-t_0). \tag{27}$$

Now we employ these $\rho_i$'s and $c_i$'s to construct an approximation

$$\Phi_n(z,t) \equiv \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}}\, e^{-(z-c_i)^2/4(t+t_0)}. \tag{28}$$

It is clear that this linear combination of complex heat kernels is also a solution to the heat equation. Let

$$w(x,t) = u(x,t) + i\varrho(x,t).$$

Then, due to the linearity of the heat equation, the complex valued function $w(x,t)$ is a solution to the heat equation

$$w_t = w_{xx}, \quad w(x,0) = u_0(x) + i\varrho(x,0), \quad t>0,\ x \in \boldsymbol{R}. \tag{29}$$

Since $\Phi_n(z,t) \to \sum_{i=1}^n \rho_i \delta(z-c_i)$ as $t \to -t_0$, the backward moments of $\Phi_n(x,t)$, $0 \le k < 2n$, are given by

$$\lim_{t\to -t_0} \int x^k \Phi_n(x,t)dx = \sum_{i=1}^{n} \rho_i c_i^k = \alpha_k(-t_0) + i\beta(-t_0). \tag{30}$$

Therefore, from the relations (24), we may conclude that $\Phi_n(x,t)$ and $w(x,t)$ share the same moments up to order $2n-1$. If the real parts are compared, then

$$\int x^k u(x,t)dx = \int x^k Re(w(x,t))dx = \int x^k Re(\Phi_n(x,t))dx,$$

and hence the solution $u(x,t)$ and the real part of the approximation $Re(\Phi_n(x,t))$ share the same moments up to order $2n-1$. Let

$$\varphi_n(x,t) := Re(\Phi_n(x,t)).$$

Then $\varphi_n(x,t)$ is the approximation of the solution $u(x,t)$ which is our candidate to replace the integral formula of the solution.

We summarize the results in the following theorem.

**Theorem 4.1.** *Let $u(x,t)$ be the solution to the heat equation with an initial value $u_0(x)$ such that $x^{2n}u_0(x)$ is integrable. Then, for any given $t_0 > 0$, there exist $\rho_i, c_i \in \boldsymbol{C}$ such that*

$$\int x^k \varphi_n(x,t)dx = \int x^k u(x,t)dx, \quad 0 \le k < 2n,$$

*where*

$$\varphi_n(x,t) := Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}}\, e^{-(x-c_i)^2/4(t+t_0)}\Big).$$

Note that $u(x,t)$ and $\varphi_n(x,t)$ share the same moments up to order $2n-1$ all the time and are getting similarly smeared as $t$ increases. Hence, it is natural to expect that $\varphi_n(x,t)$ approaches to $u(x,t)$ fast as $t \to \infty$. The convergence order in this situation has been obtained in [14, 19] using the agreement of moments, which is the one in the following theorem. The proof is same as the ones in [19] and omitted here.

**Theorem 4.2.** *Let $u(x,t)$ and $\varphi_n(x,t)$ be the ones in Theorem 4.1 under the same conditions. Then, for $1\le p\le\infty$,*

$$\lim_{t\to\infty} t^{\frac{2n+1}{2}-\frac{1}{2p}}\|\varphi_n(t)-u(t)\|_p = \frac{\|\partial_x^m(e^{\frac{-x^2}{4}})\|_p}{\sqrt{4\pi}}\Big|\int E_{2n}(x)dx\Big| < \infty, \tag{31}$$

*where*

$$E_0(x) := \varphi_n(x,0) - u(x,0)$$

*and*

$$E_k(x) := \int_{-\infty}^{x} E_{k-1}(y)dy,\ \ 0<k\le 2n. \tag{32}$$

This theorem gives a surprising conclusion. Even if the $n\times n$ Hankel matrix of a real valued solution $u(x,t)$ is singular, one may construct a complex valued approximation $\Phi_n(z,t)$ defined on the complex plane by choosing an imaginary part $\varrho_0(x)$. There are various ways to choose the imaginary part and a different imaginary part gives a different approximation. However, all of them share the same real part of the moments and show good behavior for $t>0$ and $n>0$ large. It is natural to ask a criterion to choose the best imaginary part $\varrho_0(x)\ge 0$ in a unique way. However, we do not have such a criterion.

5. **Truncated moment problem without an imaginary part.** In this section we consider the complex moment problem with zero imaginary part $\varrho_0(x)=0$. In other words we consider complex solutions $\rho_i, c_i\in\boldsymbol{C}$ that solve

$$\sum_{i=1}^{n}\rho_i c_i^k = \alpha_k \in \boldsymbol{R},\quad 0\le k<2n. \tag{33}$$

It is well-known that, if $\alpha_k$'s are moments of a nonnegative function, this problem has real solutions. Here we are interested in the case that $\alpha_k$'s are not necessarily moments of a nonnegative function. This moment problem is not solvable in general even if complex solutions are considered. In particular the Hankel $A$ given in (13) can be singular. In this section we consider the property of solutions of the moment problem and the approximation solution

$$\Phi_n(z,t)\equiv\sum_{i=1}^{n}\frac{\rho_i}{\sqrt{4\pi t}}\,e^{-(z-c_i)^2/4t}$$

for the case that (33) is solvable.

**Theorem 5.1.** *Let $\alpha_k$'s be real numbers. Suppose that the $n\times n$Hankel matrix $A=(\alpha_{i+j})$ is invertible and the auxiliary polynomial $g_n(z)$ has $n$ distinct zeros. (i) If $c_i$ is a complex zero of $g_n(z)$, then its conjugate $\bar c_i$ is also a zero. (ii) If $c_j=\bar c_i$, then $\rho_j=\bar\rho_i$. (iii) The restriction of $\Phi_n(z,t)$ to the real line is real valued.*

*Proof.* $(i)$ Since $\alpha_k$'s are real, the Hankel matrix $A$ and the vector $\mathbf{a}_n$ in (13) are real ones. Hence the solution $\Psi$ to the linear problem $A\Psi=\mathbf{a}_n$ consists of real numbers and hence the polynomial $g_n(z)=z^n-\sum_{k=0}^{n-1}\psi_k z^k$ is of real coefficients. Hence if $g_n(z)$ has a complex zero, its conjugate is also a zero.

$(ii)$ After reordering the sequence one may assume that $c_{2j-1}=\overline{c_{2j}}$ for $j=1,\cdots,l$ and $c_i$'s are real numbers for $j>2l$. Let

$$a_{k,2j-1} := c_{2j-1}^{k-1}+c_{2j}^{k-1},\quad a_{k,2j} := i(c_{2j-1}^{k-1}-c_{2j}^{k-1}).$$

Then, since $c_{2j-1} = \overline{c_{2j}}$, these $a_{k,i}$'s are real numbers for $0 < i \le 2l$. We will show that the solution $\rho_i$'s are given in the form of $\rho_{2j-1} = x_{2j-1} + ix_{2j}$ and $\rho_{2j} = x_{2j-1} - ix_{2j}$ with $x_{2j}, x_{2j-1} \in \boldsymbol{R}$ for $j = 1, \cdots, l$. If so and the first $n$ equations in (33) are written in terms of $x_i$'s, then one obtains

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,2l} & c_{2l+1}^0 & \cdots & c_n^0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{2l,1} & \cdots & a_{2l,2l} & c_{2l+1}^{2l-1} & \cdots & c_n^{2l-1} \\ a_{2l+1,1} & \cdots & a_{2l+1,2l} & c_{2l+1}^{2l} & \cdots & c_n^{2l} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,2l} & c_{2l+1}^{n-1} & \cdots & c_n^{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{2l} \\ \rho_{2l+1} \\ \vdots \\ \rho_n \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{2l-1} \\ \alpha_{2l} \\ \vdots \\ \alpha_{n-1} \end{pmatrix}. \quad (34)$$

Furthermore, the solutions to (34) gives the solution to (33) by simply setting $\rho_{2j-1} = x_{2j-1} + ix_{2j}$ and $\rho_{2j} = x_{2j-1} - ix_{2j}$ for $j = 1, \cdots, l$. Therefore, the existence of such $\rho_i$'s is equivalent to the solvability of (34).

One may easily check that the real matrix is obtained from the $n \times n$ Vandermonde matrix that gives the first $n$-equations in (33) by simply adding two columns or subtracting one from another. (Remember that the $ij$-component of the Vandermonde matrix is $c_j^{i-1}$.) One may also easily show that the $n \times n$ matrix in (34) is invertible since the Vandermonde is invertible. In other words there exist $\rho_i$'s and $c_i$'s satisfying the claims of the theorem. Since the solution to the moment problem is unique, these are the ones.

$(iii)$ Let $x$ be a real number. Then

$$\frac{\overline{\rho}}{\sqrt{4\pi t}}\, e^{-(x-\overline{c})^2/4t} = \overline{\frac{\rho}{\sqrt{4\pi t}}\, e^{-(x-c)^2/4t}}$$

Therefore,

$$\frac{\rho_{2j-1}}{\sqrt{4\pi t}}\, e^{-(x-c_{2j-1})^2/4t} = \overline{\frac{\rho_{2j}}{\sqrt{4\pi t}}\, e^{-(x-c_{2j})^2/4t}}, \quad 1 \le j \le l.$$

It is now clear that the restriction of $\Phi_n(z,t)$ to the real line is real valued. □ □

Even if the moment problem (33) is not solvable in general, such a case is very rare in the sense that it is of measure zero case. Hence it is important to include complex solutions. In the approximation of a general function, the complex heat kernel is used. The use of complex heat kernels makes the control of moments possible. The extension of the heat equation to the complex field seems to be natural to understand the mechanism well. Note that, even if $\rho_i$'s and $c_i$' are complex numbers, the restriction of $\Phi_n(z,t)$ to the real numbers has real values.

## 6. **Structure of the approximation.**

In this section we numerically investigate the property of the approximation

$$\varphi_n(x,t) := Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}}\, e^{-(z-c_i)^2/4(t+t_0)}\Big),$$

which was constructed in previous sections. This approximation is uniquely given after a choice of the backward time $t_0$ and an imaginary part $\varrho_0(x)$. We do not have a criterion to choose better $t_0$ and $\varrho_0$. In this section we just observe how these choices may make a difference.

6.1. **Approximation using a single heat kernel.** The case that clearly shows the benefit of using the complex moment problem over the real one is the single heat kernel case ($n = 1$). A fundamental solution is frequently considered as a canonical solution of various problems. In many cases fundamental solutions are given explicitly and play key roles in the analysis of general solutions. For the heat equation case it is given by the Gaussian which is also called the heat kernel. The real valued heat kernel is a signed function and hence it could not play as a canonical solution for a sign-changing ones. In the following examples we will see how a single heat kernel in the complex plane can show the behavior of sign-changing solutions.

The initial value of the first example is

$$u_0(x) := \frac{1}{\sqrt{4\pi}} e^{\frac{-(x+1)^2}{4}} - \frac{1}{\sqrt{8\pi}} e^{\frac{-x^2}{8}}. \tag{35}$$

Then the first two moments of the solution $u(x,t)$ are given by

$$\alpha_0(t) = 0, \quad \alpha_1(t) = 1.$$

Since they are constants, the backward moments are also given by $\alpha_0(-t_0) = 0$ and $\alpha_1(-t_0) = 1$ for any backward time $t_0 > 0$. For the case $n = 1$, the complex function $\Phi_1(z,t) = \frac{\rho_1}{\sqrt{4\pi(t+t_0)}} e^{\frac{-(z-c_1)^2}{4(t+t_0)}}$ is obtained by solving the following two moment equations

$$\rho_1 = 0, \qquad c_1\,\rho_1 = 1.$$

However, in this case, the corresponding $1 \times 1$ Hankel matrix $A$ is the zero matrix which is singular. It is clear that this moment problem is not solvable even if the complex solutions are allowed. Therefore, one should introduce an imaginary part to control two moments. Let $w(x,t)$ be a complex valued solution to the heat equation with an initial value

$$w_0(x) := u_0(x) + i\varrho_0(x), \quad \varrho_0(x) := \phi(x + 0.5, 1),$$

where $\phi(x,t)$ is the heat kernel. Then the real part of the complex solution $w(x,t)$ is just $u(x,t)$ for any choice of the imaginary part $\varrho_0(x)$. However, the real part of its approximation $\Phi_1(x,t)$ depends on the choice of $\varrho_0(x)$. Under the above choice of $\varrho_0(x)$, the first two moments are given by $\alpha_0(t) = i$ and $\alpha_1(t) = 1 - 0.5i$. Then the corresponding moment equations are

$$\rho_1 = i, \qquad c_1\,\rho_1 = 1 - 0.5i.$$

The solution of the moment problem is

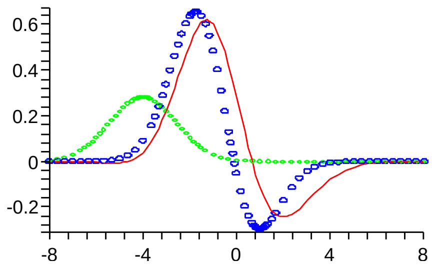
$$\rho_1 = i, \qquad c_1 = -0.5 - i.$$

Let $\varphi_1(x,t)$ be the restriction of the real part of the approximation $\Phi_1(z,t)$ to the real line, i.e., $\varphi_1(x,t) := Re(\frac{\rho_1}{\sqrt{4\pi(t+t_0)}} e^{-(x-c_1)^2/4(t+t_0)})$. Then the optimal convergence order is obtained even for the case that the corresponding Hankel matrix is singular. The convergence order is written as

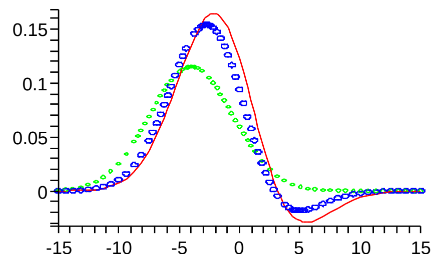
$$\lim_{t\to\infty} t^{(\frac{3}{2}-\frac{1}{2p})} \|u(t) - \varphi_1(t)\|_p = \frac{\|\partial_\xi^2 (e^{-\frac{1}{4}\xi^2})\|_p}{\sqrt{4\pi}} \Big| \int_{-\infty}^{\infty} E_2(x)dx \Big| < \infty,$$

where $1 \le p \le \infty$ and

$$E_2(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} [\varphi_1(s,0) - u_0(s)] ds dy.$$




(a) Initial approximation




(b) Approximations at time $t = 5$

FIGURE 1. The exact solution is given in a line. Single heat kernel using a real Gaussian is given in dots. However, the complex Gaussian with an imaginary part, circles, gives a sign-changing behavior and a better approximation.

If $\alpha_0 \neq 0$, then the approximation of the optimal order can be obtained without using an imaginary part. For the second example, consider such a case with an initial value

$$u_0(x) := \frac{4}{\sqrt{4\pi}}\, e^{\frac{-(x+1)^2}{4}} - \frac{3}{\sqrt{8\pi}}\, e^{\frac{-x^2}{8}}. \tag{36}$$

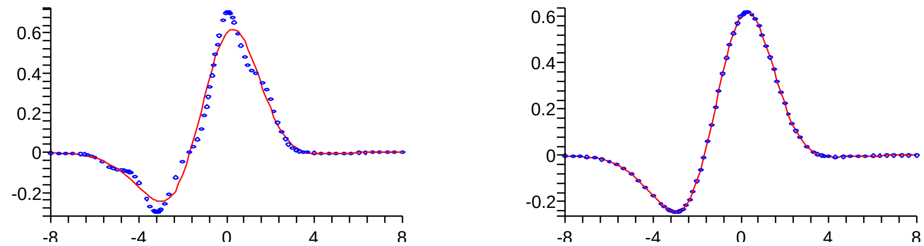
This initial value is given in Figure 1(a) with a solid line (red). Then, the zero-th moment is $\alpha_0 = 1$ and hence the Hankel matrix $A$ is non-singular. The approximation without the imaginary part has been computed using a backward time $t_0 = 1$ which is given in Figure 1(a) in dots. In this case the real valued Gaussian is a nonnegative function. On the other hand an approximation using the following imaginary part,

$$\varrho_0(x) = 4\phi(x + 0.5, 1)\Big( = \frac{4}{\sqrt{4\pi}}\, e^{\frac{-(x+0.5)^2}{4}}\Big),$$

is given in circles. Notice that, since the backward moments were used, the initial approximations are not spiky and the initial difference is not so big even if only one heat kernel is used. Furthermore, the case with an imaginary part, the behavior of sign-change is also observed using only a single heat kernel. In Figure 1(b) the evolution of the single heat kernels are given at time $t = 5$ with the exact solution.

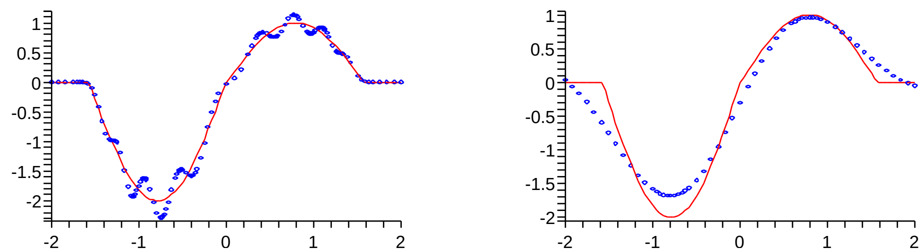
6.2. **Initial approximations using many heat kernels.** The asymptotic convergence order in Theorem 4.2 indicates that $\varphi_n(x, t)$ is a good approximation of the solution $u(x, t)$ for $t \geq 0$ large. In fact, a similar approximation showed an excellent asymptotic behavior in numerical tests in [19]. If one also obtains a good initial approximation using backward moments, then it will complete the approximation. In the following tests we mostly consider the initial approximation using backward moments at $t_0 > 0$. Note that this initial approximation is not actually related to the heat equation. One should consider it as a nonlocal approximation technique based on a moment control.

6.2.1. *Continuous initial values.* Consider the smooth initial value in (36) as the first example in this section. Using this initial value the backward heat equation can be solved up to backward time $t_0 = 1$. Hence one may say that the age of this initial value is 1, and it seems that taking backward time $t_0 = 1$ will give the best result. In Figure 2(a) an approximation using backward moment with the backward time




(a) backward time $t_0 = 0.2$ (b) backward time $t_0 = 0.4$

Figure 2. Initial approximations with $n = 10$ agree very well if the age of the initial heat distribution is not zero and the backward time is relatively close to it. In this example the age of the initial heat distribution (36) is 1.




(a) backward time $t_0 = 0.01$ (b) backward time $t_0 = 2.0$

Figure 3. The non-smooth initial value (37) is approximated using 10 heat kernels. This example is to compare a small and a large backward time. $t_0 = 0.1$ gives a better fit.

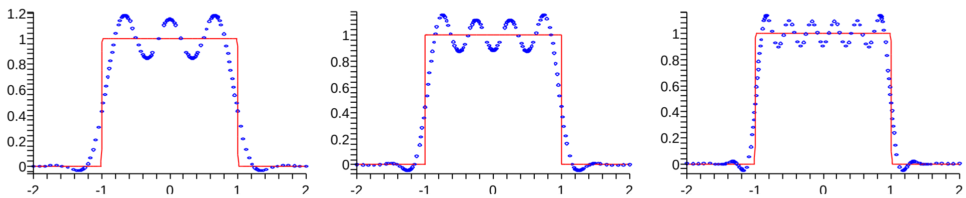
$t_0 = 0.2$ and $n = 10$ is given. Its imaginary part was not taken in this example. One may observe a little bit of wiggling in this case. If the backward time approaches to the maximum backward time $t_0 = 1$, then the approximation agrees with the initial value completely. In Figure 2(b) an approximation with $t_0 = 0.4$ is given. Even if the backward time is increased to $t_0 = 4$, the initial approximation gives a perfect match.

The initial value for the second example is

$$u_0(x) = \begin{cases} 2\sin(x) & , \quad -\frac{\pi}{2} < x < 0, \\ \sin(x) & , \quad 0 < x < \frac{\pi}{2}, \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{37}$$

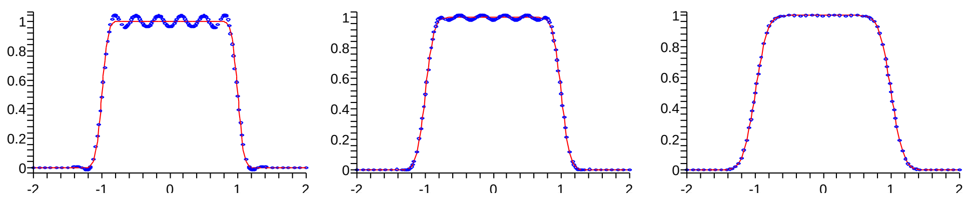
which is continuous, but not differentiable. Graphs of the approximations are given in Figure 3. The initial value $u_0(x)$ is given in solid (red) lines. Approximations obtained from moment problem using the real Hankel matrixes are given in dots. In Figure 3(a) a backward time $t_0 = 0.01$ is used. One may observe certain oscillations in the smooth regions. However, this approximation gives pretty correct approximation at the cusps.

In Figure 3(b) a bigger backward time, $t_0 = 2.0$, is used. There is no oscillation at all for this case. However, it gives a poor approximation for the cusps. This approximation is too smooth to get it right. From this example, one may see that some parts of the initial value requires small backward time and other parts larger



(a) $n = 10$ (b) $n = 20$ (c) $n = 40$

FIGURE 4. Initial approximations of a discontinuous function show oscillations. The backward time is $t_0 = 0.2$. The size of the oscillation is decreasing as $n \to \infty$.



(a) $t = 0.0025$ (b) $t = 0.005$ (c) $t = 0.01$

FIGURE 5. The initial oscillations disappear quickly. In this computation we set $n = 40$ and $t_0 = 0.2$. The exact solution and its approximation are given in lines and dots, respectively.

ones. Hence it seems desirable to develop a technique to take several backward time (see Remark 5). Note that the figures are not to show the best fits. In fact the backward time $t_0 = 0.1$ gives a better fit.

6.2.2. *Discontinuous initial values.* Approximation of a discontinuous function gives extra difficulties. In this section we consider a discontinuous initial value

$$u_0(x) = \begin{cases} 1 & , \quad -1 < x < 1, \\ 0 & , \quad \text{otherwise}. \end{cases} \tag{38}$$

Then the exact solution $u(x,t)$ is given by

$$u(x,t) = \int_{-\infty}^{x} [\phi(y+1,t) - \phi(y-1,t)]dy. \tag{39}$$

In Figure 4 one may observe that the initial approximation has an oscillating behavior. In these examples a different kind of Gibb's phenomenon is observed. One can clearly see that the maximum error near the discontinuity is decreasing as $n$ increases. The pattern of the oscillation is also different. However, in Figure 5, the oscillation disappears as time increases and the approximation agrees with the exact solutions almost completely at $t = 0.01$. One can also say that, if the initial value has regularity corresponding to the Figure 5(c), then the approximation using corresponding backward time gives a perfect initial match.

An asymptotic convergence test is given in Table 1 that compares approximation error for four cases doubling the time from $t = 0.002$ to $t = 65.536$. Two different node numbers of $n = 10$ and $n = 20$ and two different backward times of $t_0 = 0.01$ and $t_0 = 0.2$ are tested. The errors are given in the uniform norm. One can clearly observe the asymptotic convergence order given in Theorem 4.2. The convergence

TABLE 1. An asymptotic convergence test is given with the initial value (38). The errors are in the uniform norm. One may observe the asymptotic convergence order given in Theorem 4.2.

| | $n = 10$, $t_0 = 0.01$ | | $n = 10$, $t_0 = 0.2$ | | $n = 20$, $t_0 = 0.01$ | | $n = 20$, $t_0 = 0.2$ | |
|---|---|---|---|---|---|---|---|---|
| t | error | order | error | order | error | order | error | order |
| 0.002 | 1.33e-01 | 0.34 | 2.65e-01 | 0.23 | 3.21e-03 | 0.22 | 9.53e-02 | 0.23 |
| 0.004 | 8.31e-02 | 0.68 | 1.94e-01 | 0.45 | 1.22e-03 | 1.39 | 6.93e-02 | 0.46 |
| 0.008 | 3.17e-02 | 1.39 | 1.05e-01 | 0.88 | 7.78e-05 | 3.97 | 3.66e-02 | 0.92 |
| 0.016 | 4.79e-03 | 2.73 | 3.28e-02 | 1.68 | 3.05e-07 | 7.99 | 1.03e-02 | 1.83 |
| 0.032 | 1.86e-04 | 4.69 | 4.00e-03 | 3.03 | 6.00e-11 | 12.31 | 8.85e-04 | 3.54 |
| 0.064 | 1.79e-06 | 6.70 | 1.40e-04 | 4.84 | 1.12e-15 | 15.71 | 1.21e-05 | 6.19 |
| 0.128 | 5.86e-09 | 8.25 | 1.38e-06 | 6.66 | 4.57e-21 | 17.90 | 1.69e-08 | 9.49 |
| 0.256 | 9.59e-12 | 9.25 | 4.95e-09 | 8.13 | 7.91e-27 | 19.14 | 2.23e-12 | 12.89 |
| 0.512 | 1.05e-14 | 9.84 | 8.76e-12 | 9.14 | 8.64e-33 | 19.80 | 4.01e-17 | 15.76 |
| 1.024 | 9.16e-18 | 10.16 | 1.01e-14 | 9.76 | 7.44e-39 | 20.15 | 1.76e-22 | 17.80 |
| 2.048 | 7.14e-21 | 10.33 | 9.07e-18 | 10.12 | 5.67e-45 | 20.32 | 3.26e-28 | 19.04 |
| 4.096 | 5.24e-24 | 10.41 | 7.18e-21 | 10.30 | 4.06e-51 | 20.41 | 3.71e-34 | 19.74 |
| 8.192 | 3.73e-27 | 10.46 | 5.31e-24 | 10.40 | 2.83e-57 | 20.46 | 3.27e-40 | 20.11 |
| 16.384 | 2.61e-30 | 10.48 | 3.80e-27 | 10.45 | 1.94e-63 | 20.48 | 2.53e-46 | 20.30 |
| 32.768 | 1.82e-33 | 10.49 | 2.67e-30 | 10.47 | 1.32e-69 | 20.49 | 1.83e-52 | 20.40 |
| 65.536 | 1.26e-36 | 10.49 | 1.86e-33 | 10.49 | 8.90e-76 | 20.49 | 1.27e-58 | 20.45 |

order at time $t > 0$ is computed using the following relation:

$$\text{asymptotic order} \cong \frac{\ln(\|u(t/2) - \varphi_n(t/2)\|_\infty / \|u(t) - \varphi_n(t)\|_\infty)}{\ln(1/2)}.$$

7. **Discussions.** In summary, the truncated moment problem has been generalized in this paper to deal with complex measures. This extended theory helped us to solve a truncated moment problem for any given real sequence by considering them as the real parts of complex moments. The approximation theory for the solutions to the heat equation is now completed for general sign-changing solutions using this generalized theory. To obtain regularity in the initial approximation this method has been developed by taking backward moments. As a result we have obtained an approximation method for a function. There are several questions and issues remaining related to this work. We discuss them in the following remarks.

**Remark 2** (choice of the imaginary part $\varrho_0(x)$)**.** The construction of the asymptotic approximation $\varphi_n(x,t)$ depends on the choice of $\varrho_0(x)$. Theorem 2.4 gives the existence of an imaginary part $\varrho_0(x)$ that allows the solvability of (21). On the other hand it is clear that there are various choices of such an imaginary part. However, we do not have any criterion to choose a better $\varrho_0(x)$.

**Remark 3** (further theory for generalized moment problem)**.** In this paper a generalized moment problem was introduced and only a minimum amount of its theory was developed which was needed in the approximation technique. Certain analogies of classical theory for the positive measure case should be developed. For example the solvability of (21) for an arbitrarily given non-trivial nonnegative imaginary part is left conjectured.

**Remark 4** (choice of the backward time $t_0 > 0$)**.** The construction of $\varphi_n(x,t)$ also depends on the choice of the backward time $t_0 > 0$. The approximation

$$\varphi_n(x,t) := Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}}\, e^{-(x-c_i)^2/4(t+t_0)}\Big).$$

has some regularity even for the initial time $t = 0$ thanks to the backward time $t_0 > 0$. This improves the initial approximation, in particular, if the initial value $u_0$ is smooth. It is natural to ask what is the optimal $t_0$ to obtain the best approximation result. One may consider the backward time $t_0$ as an unknown and solve $2n + 1$ equations

$$\alpha_k(-t_0) = Re\left(\sum_{i=1}^{n} \rho_i c_i^k\right), \quad \rho_i, c_i \in \boldsymbol{C}, \quad 0 \le k \le 2n.$$

The solvability of this highly nonlinear problem and the positivity of the backward time $t_0 > 0$ are interesting questions. The use of age of the initial profile is introduced and used in [21, 27].

**Remark 5** (distinct backward times $t_i \ge 0$)**.** It is observed from Figure 3 that certain regions of the initial approximation fit with small backward time and others with larger ones. Hence one may give a freedom in choosing the backward time. Consider

$$\varphi_n(x,t) := Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_i)}}\, e^{-(x-c_i)^2/4(t+t_i)}\Big).$$

Here we have $3n$ freedom of choices in $\rho_i$'s, $c_i$'s and $t_i$'s. Hence it is natural to ask if one may solve the following $3n$ equations:

$$\lim_{t\to 0} \varphi_n(x,t) = Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t_i}}\, e^{-(x-c_i)^2/4t_i}\Big) = \alpha_k, \quad 0 \le k < 3n.$$

We do not have any clue to attack this problem. However, if this problem is solved, then one might obtain an approximation that will fit both of smooth and non-smooth regions.

**Remark 6** (higher order asymptotics for nonlinear cases)**.** The relation between moments and asymptotic contraction order is well studied for the solutions to the heat equation (see [14, 18, 19]). For nonlinear diffusion cases there is no such a detailed results. However, asymptotic $L^1$-contraction order of the similarity scale and the order of $O(t^{-1})$ have been obtained by setting the center of mass or taking a function space with finite second or higher order moments (see [5, 17]). One challenging goal is to extend the higher asymptotic convergence order (31) to nonlinear problems. The Burgers equation is a special case that one may control the moments using the Cole-Hopf transformation. In fact the same convergence order corresponding to (31) has been obtained for the Burgers case in [6, 26]. Note that $\sqrt{t}$ is the similarity scale of the heat equation and the Burgers equation. It seems that the agreement of an extra order of moment gives an extra asymptotic convergence order of the similarity scale of the problem. A discussion about this relation is given in [6].

**Remark 7.** (heat equations in other forms) The heat equation on the whole real line was considered and hence the moment problem of the Hamburger's case has been studied in this paper, i.e., $K = \boldsymbol{R}$. The heat equation with a boundary can be related to the moment problem of the Stieltjes or the Hausdorff cases. It seems that the complex moment problem has a close relation to the heat equation on the complex plane.

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