Asymptotic agreement of moments and higher order contraction in the Burgers equation ⋆

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Abstract. The purpose of this paper is to investigate the relation between the moments and the asymptotic behavior of solutions to the Burgers equation. The Burgers equation is a special nonlinear problem that turns into a linear one after the Cole-Hopf transformation. Our asymptotic analysis depends on this transformation. In this paper an asymptotic approximate solution is constructed, which is given by the inverse Cole-Hopf transformation of a summation of n heat kernels. The k-th order moments of the exact and the approximate solution are contracting with order $O\big((\sqrt{t})^{k-2n-1+1/p}\big)$ in L^p-norm as $t \to \infty$. This asymptotics indicates that the convergence order is increased by a similarity scale whenever the order of controlled moments is increased by one. The theoretical asymptotic convergence orders are tested numerically.

Key words. truncated moment problem; heat equation; burgers equation; long time asymptotics; complex Gaussian; Cole-Hopf transformation; backward moment;

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1. Introduction

The main motivation of this paper is to investigate the relation between the agreement of moments and the asymptotic contraction orders of solutions to convection-diffusion equations. Let u and ψ be integrable real-valued solutions to

$$u_t + \nabla_x \cdot (F(u, \nabla u)) = 0, \tag{1}$$

where $x \in \boldsymbol{R}^d$ and $F : \boldsymbol{R}^{1+d} \to \boldsymbol{R}^d$. Then one may ask what decides the asymptotic contraction order $\gamma > 0$, i.e.,

$$\|u(x,t) - \psi(x,t)\|_r = O(t^{-\gamma}) \quad \text{as} \quad t \to \infty, \tag{2}$$

where $\| \cdot \|_r$ is the L^r-norm, $r \geq 1$, over the space domain $x \in \boldsymbol{R}^d$.

It is well-known that two solutions to the linear heat equation share the same moments all the time if they do initially. Using this property it has been shown that, if

$$\int x^k (u(x,0) - \psi(x,0)) dx = 0, \qquad 0 \leq k \leq m,$$

the asymptotic contraction order in (2) is $\gamma = \frac{m+2}{2} - \frac{1}{2r}$ (see [6,10, 13]). This one dimensional asymptotics is easily extended to multidimensional ones. However, nonlinear problems do not have such a nice property. For the porous medium equation (PME for brevity) case, only the total mass and the center of mass have such a property (i.e., for $k = 0, 1$). For the p-Laplacian equation case, even the center of mass do not have this property.

In this paper we consider bounded solutions to the (viscous) Burgers equation in one space dimension,

$$\begin{aligned} &u_t + uu_x = \mu u_{xx}, && t > 0,\ x \in \boldsymbol{R}, \\ &u(x,0) = u_0(x), && x \in \boldsymbol{R}, \end{aligned} \tag{3}$$

where $\mu > 0$ is the viscosity coefficient and the initial value u_0 is bounded and has finite moments up to order $2n$, i.e., $x^{2n} u_0(x) \in L^1(\boldsymbol{R})$. In this case the total mass ($k = 0$) is the only one that the initial agreement gives a permanent one. The reason why the Burgers equation is picked as an exemplary case is the Cole-Hopf transformation (see [9]), which makes an rigorous analysis possible. It is given by

$$\Phi(x,t) = e^{-\frac{1}{2\mu}\int_{-\infty}^x u(y,t)dy} - 1 =: H(u). \tag{4}$$

For notational convenience, we denote its space derivative as

$$\phi(x,t) := \partial_x \Phi(x,t) = -\frac{1}{2\mu} u(x,t) e^{-\frac{1}{2\mu}\int_{-\infty}^x u(y,t)dy}. \tag{5}$$

Then Φ and ϕ are solutions to the heat equation (11) and $\phi(x,t)$ has finite moments up to order $2n$, i.e., $x^{2n}\phi(x,0) \in L^1(\boldsymbol{R})$. Its inverse transformation is given by

$$u(x,t) = -2\mu \frac{\phi(x,t)}{1+\Phi(x,t)} =: H^{-1}(\Phi). \tag{6}$$

If Φ is a Cole-Hope transformation of a function $u(x)$, then

$$\Phi(x,t) + 1 = e^{-\frac{1}{2\mu}\int_{-\infty}^{x} u(y)dy} > 0,$$

and hence $H^{-1}(\Phi)$ is well-defined. However, for a general case, one should show that the denominator $1+\Phi(x,t)$ is strictly positive. The main theorem of this paper is the following:

Theorem 1. *Let $u(x,t)$ be the solution to the Burgers equation (3) with a bounded initial value $u_0(x)$ such that $x^{2n}u_0(x) \in L^1(\boldsymbol{R})$. Then, for any $t_0 \geq 0$, there exist $\rho_i, c_i \in \boldsymbol{C}$ and $T \geq 0$ such that $w_n := H^{-1}(\Psi_n)$ is well-defined for $t \geq T$, and, for $1 \leq r \leq \infty$ and $k \geq 0$,*

$$\|x^k(u(x,t) - w_n(x,t)\,)\|_r = O\Big(\big(\sqrt{t}\big)^{1/r-2n-1+k}\Big) \quad \text{as} \quad t \to \infty, \tag{7}$$

where

$$\Psi_n(x,t) := \int_{-\infty}^{x} \psi_n(y,t)dy, \tag{8}$$

$$\psi_n(x,t) := Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\mu\pi(t+t_0)}}\, e^{\frac{-(x-c_i)^2}{4\mu(t+t_0)}}\Big). \tag{9}$$

It is clear that $\psi_n(x,t)$ is a solution to the heat equation. The ρ_i's and c_i's are chosen to satisfy $2n$ equations of

$$\int x^k\phi(x,t)dx - \int x^k\psi_n(x,t)dx = 0, \qquad 0 \leq k < 2n. \tag{10}$$

The construction of $\psi_n(x,t)$ has been made in [13] for the positive solutions with $t_0 = 0$ using the classical truncated moment problem. For the general case in the theorem, the classical theory is not enough. However, we could construct $\psi_n(x,t)$ using a generalized moment problem in [10]. Using these techniques higher order convergence have been obtained [10,13]. One may also control the moments using the derivatives of the Gaussian as in (18). This technique has been used in [6] and obtained higher order asymptotics.

The inverse transformation w_n is a solution to the Burgers equation and valid for $t \geq T$. For the case with $t_0 = 0$ and $u_0 \geq 0$, it is proved that $T = 0$. For the general case in the theorem, we only have a numerical evidence for $T = 0$ which left conjectured. Note that, even if ϕ and ψ_n have same moments up to order $2n-1$, their inverse transformations do not, i.e., $\int x^k \big(u(x,t) - w_n(x,t)\big) dx \neq 0$. However, the asymptotic convergence order in (7) shows that they approach to each other asymptotically. In other words the moment setting after the Cole-Hopf transformation actually gives asymptotic moments agreement for the solutions to the Burgers equation and provides fine asymptotics. The higher order contraction indicates that the solution w_n is an excellent asymptotic approximation of the solution u (the case for $k = 0$).

The Cole-Hopf transformation has been a main tool to study the large-time behavior of the Burgers equation. It allows one to study the Burgers equation from the behavior of solutions to the heat equation ([12, 14, 16, 20]). For general nonlinear problems there is no such transformation. We only hope to glimpse the large-time behavior of a general nonlinear problem from the study of the Burgers case.

The solution to the Burgers equation has been played a prototype role in many problems such as traffic or fluid flows (see [19]). It has been shown that the asymptotic behavior of general systems of hyperbolic conservations laws are given as a solution to the Burgers equation ([4, 5, 15]). On the other hand asymptotic convergence to similarity solutions has been done for general convection-diffusion equations that may include the Burgers equation ([2, 7, 8, 21]). Special attention has been given to the study of asymptotics of the porous medium equation (shortly PME), $u_t = (u^m)_{xx}$, $m > 0$, last two decades (see[1]). One may find optimal convergence to the Barenblatt solution of similarity order $O(t^{-1/(m+1)})$ (see [3, 18]). The convergence orders in (7) indicate that the contraction order in (2) will be increased by the similarity scale if the order of asymptotically converging moments increases. A brief discussion about this relation is given in Section 4. There is a different kind of optimal convergence order $O(1/t)$ which was obtained for radially symmetric solutions or for a very fast diffusion case (see [11, 18]).

This paper is organized as the following. In Section 2, we construct an approximate solution $\psi_n(x,t)$ to the heat equation so that it share the same $2n$ moments with $\phi(x,t)$ in (5). In this construction the generalized moment problem for given backward moments is used. The decay order of $\|x^k(\psi_n(x,t) - \phi(x,t)\|_r$ as $t \to \infty$ is also derived. In Section 3, we show that this decay rate is transferred to the Burgers equation after the Cole-Hopf transformation and complete the proof of Theorem 1. In Section 4, we briefly discuss the relation between the asymptotic convergence order and the control of moments at $t = \infty$ for a general nonlinear problem. Finally, in Section 5 we provide several numerical examples to numerically test the convergence orders obtained in Section 3 and the role of the backward moments.

2. Large-time asymptotics in the heat equation

Consider the heat equation with a bounded and integrable initial value:

$$\begin{aligned} &v_t = \mu v_{xx}, && x \in \boldsymbol{R},\ t > 0, \\ &v(x,0) = v_0(x), && x \in \boldsymbol{R}. \end{aligned} \tag{11}$$

One usually sets the diffusion constant $\mu = 1$ after the time rescaling $t \to \mu t$. However, we leave μ in the equation to observe the dependency on the viscosity.

2.1. Approximate solutions to the heat equation

In this section, we decide the ρ_i's and c_i's in Theorem 1 using a generalized truncated moment problem developed in [10]. A similar construction for $\psi_n(x,t)$ is given in [13] for positive solutions. We briefly review this asymptotic approximation method based on a generalized moment problem.

Note that the Cole-Hopf transformation $\Phi(x,t)$ and its space derivative $\phi(x,t)$ are solutions to the heat equation (11) and $\psi_n(x,t)$ is constructed as an asymptotic approximation of $\phi(x,t)$. Set the moments of the solution $\phi(x,t)$ as

$$\alpha_k(t) = \int x^k \phi(x,t)dx, \qquad k \geq 0. \tag{12}$$

One may easily check that the moments of a solution to the heat equation (11) satisfy the following algebraic relations:

$$\begin{aligned} \alpha_{2k}(t) &= \textstyle\sum_{l=0}^{k} \frac{(2k)!}{(k-l)!(2l)!} t^{k-l} \alpha_{2l}(0), \\ \alpha_{2k+1}(t) &= \textstyle\sum_{l=0}^{k} \frac{(2k+1)!}{(k-l)!(2l+1)!} t^{k-l} \alpha_{2l+1}(0). \end{aligned}$$

These relations are valid for all $t \in \boldsymbol{R}$ as long as its backward solution exists. The first two moments, α_0 and α_1, are constant for all $t \in \boldsymbol{R}$, which are called the conservation of mass and its center. However, for $k \geq 2$, the moment $\alpha_k(t)$ are not constant anymore.

It is shown that, for any given real sequence α_k, there exists a real sequence β_k's such that the following $2n$ equations

$$\sum_{i=1}^{n} \rho_i c_i^k = \alpha_k + i\beta_k, \qquad 0 \leq k < 2n \tag{13}$$

have a solution set $\rho_i, c_i \in \boldsymbol{C}$, $i = 1, \cdots, n$, which is unique up to reordering. If one takes $\alpha_k(-t_0)$ in the place of α_k's for a given $t_0 > 0$, then

$$Re\Big(\sum_{i=1}^{n} \rho_i c_i^k\Big) = \alpha_k(-t_0), \tag{14}$$

where $Re(\cdot)$ takes the real part of a complex number. Note that ρ_i's and c_i's that satisfy (14) is not unique since they may depends on the choice of β_k's. In the numerical tests in Section 5, we simply took $\beta_k = 0$ as long as (13) is numerically solvable.

Finally, we take the approximate solution $\psi_n(x,t)$ as

$$\psi_n(x,t) \equiv Re\Big(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\mu\pi(t+t_0)}}\, e^{\frac{-(x-c_i)^2}{4\mu(t+t_0)}}\Big), \tag{15}$$

which is a solution to the heat equation (11). Then,

$$\begin{aligned} \lim_{t\to -t_0} \int x^k \psi_n(x,t)dx &= Re\Big(\textstyle\sum_{i=1}^{n} \rho_i \int x^k \delta(x-c_i)dx\Big) \\ &= Re\Big(\textstyle\sum_{i=1}^{n} \rho_i c_i^k\Big) = \alpha_k(-t_0). \end{aligned} \tag{16}$$

Therefore, $\psi_n(x,t)$ and $\phi(x,t)$ have the same moments up to order $2n-1$ at the time $t = -t_0$ and hence at all time $t \in \boldsymbol{R}$. Hence, $\psi_n(x,t)$ in Theorem 1 satisfies for all $t \in \boldsymbol{R}$ that

$$\int x^k(\phi(x,t) - \psi_n(x,t))dx = 0, \quad 0 \le k < 2n-1. \tag{17}$$

Remark 1. Note that ρ_i's and c_i's depend on the backward time $t_0 \ge 0$. We do not have a criterion to choose t_0 and left it as a free variable. If one may find t_0 that solves one more moment equation, i.e.,

$$\sum_{i=1}^{n} \rho_i c_i^{2n} = \alpha_{2n}(-t_0) + i\beta_{2n},$$

then one may obtain an extra asymptotic convergence order. Furthermore, more importantly, it will give a better initial approximation. However, its solvability seems beyond the scope of this paper. For the simplest case, $n = 1$, Miller and Bernoff [16] gave such an approximation for positive solutions. Using the complex heat kernel in this paper and the generalized moment problem one may extend the result for sign-changing solutions easily.

Remark 2. One can easily check that

$$\tilde{\psi}_n(x,t) \equiv \sum_{k=0}^{2n-1} \frac{(-1)^k \alpha_k(0)}{(k!)\sqrt{4\mu\pi t}}\, \partial_x^k\big(e^{\frac{-x^2}{4\mu t}}\big) \tag{18}$$

is a solution to the heat equation (11) and satisfies the relation (17) (see Duoandikoetxea and Zuazua [6]). Yanagisawa [20] applied this kind of approximation to obtain the higher order asymptotics in the Burgers equation. In the proof of Theorem 2, the choice of ψ_n does not matter as far as (17) is satisfied. However, the constants and hence the proof of Theorem 1 may depend on its choice. Furthermore, even if we obtain the same convergence order as $t \to \infty$, the convergence for $n \to \infty$ may show a different behavior. In fact, one may easily construct an example that $\tilde{\psi}_n(x,t)$ diverges as $n \to \infty$ (see [13]). One may improve this approach using the backward moments as we did in this paper, i.e.,

$$\tilde{\psi}_n(x,t) \equiv \sum_{k=0}^{2n-1} \frac{(-1)^k \alpha_k(-t_0)}{(k!)\sqrt{4\mu\pi(t+t_0)}} \partial_x^k (e^{\frac{-x^2}{4\mu(t+t_0)}}). \tag{19}$$

In this way one may obtain some initial regularity.

2.2. *Contraction rates of moments*

The agreement of the moments in (17) does not hold after the inverse Cole-Hopf transformation. However, the key observation is that the contraction order (21) in L^r-norm is transferred after the inverse transformation. Since $\phi(x,t)$ and $\psi_n(x,t)$ satisfies (17), they contract to each other having order $O(t^{\frac{1}{2r}-\frac{2n+1}{2}})$ in L^r-norm as $t \to \infty$ (see [6,10,13]). This contraction property is extended to a contraction of moments in this section.

Lemma 1. *Let $g \in L^1(\boldsymbol{R})$ satisfy $\int g(x)\,dx = 1$ and $g_\epsilon(x) := \epsilon^{-1} g(x/\epsilon)$. Suppose that $||hf||_p < \infty$ with $1 \le p < \infty$ and $\|h(f - f_\eta)\|_p \to 0$ as $\eta \to 0$, where $f_\eta(x) := f(x-\eta)$ is a space shift. Then,*

$$\|h(f * g_\epsilon) - hf\|_p \to 0 \quad \text{as} \quad \epsilon \to 0.$$

That is, $||h(f * g_\epsilon)||_p \to ||hf||_p$ as $\epsilon \to 0$.

Proof. The definition of the convolution and the Minkowski's inequality in an integral form give

$$\begin{aligned}
\|h(f * g_\epsilon) - hf\|_p &= \left(\int \left| \int h(x) f(x-y) g_\epsilon(y)\,dy - h(x)f(x) \right|^p dx \right)^{1/p} \\
&= \left(\int \left| \int h(x)(f(x-y) - f(x)) g_\epsilon(y)\,dy \right|^p dx \right)^{1/p} \\
&\le \int \left(\int |h(x)(f(x-y) - f(x)) g_\epsilon(y)|^p \,dx \right)^{1/p} dy \\
&= \int \|h(\cdot)(f(\cdot - y) - f(\cdot))\|_p |g_\epsilon(y)|\,dy \\
&= \int \|h(\cdot)(f(\cdot - \epsilon y) - f(\cdot))\|_p |g(y)|\,dy.
\end{aligned}$$

The lemma follows from the dominated convergence theorem. $\square$

Theorem 2. *Let $\phi(x,t)$ and $\psi(x,t)$ be solutions to the heat equation (11). Suppose that $\phi(x,0)$ is bounded, $x^q\phi(x,0)\in L^1(\boldsymbol{R})$ and*

$$\int x^k(\phi(x,t)-\psi(x,t))dx=0,\quad k=0,\cdots,q-1.$$

Then, there exists $e_q\in W^{q,1}(\mathbf{R})$ that satisfies $\partial_x^q e_q(x)=\phi(x,0)-\psi(x,0)$. Furthermore, for $1\le r\le\infty$ and $0\le k$,

$$\lim_{t\to\infty} t^{\frac{q+1-k}{2}-\frac{1}{2r}}\|x^k\big(\phi(x,t)-\psi(x,t)\big)\|_r=\frac{\|x^k\partial_x^q(e^{\frac{-x^2}{4\mu}})\|_r}{\sqrt{4\mu\pi}}\left|\int e_q(x)dx\right|. \tag{20}$$

In other words,

$$\|x^k\big(\phi(x,t)-\psi(x,t)\big)\|_r=O\Big(t^{\frac{1}{2r}-\frac{q+1-k}{2}}\Big)\quad as\ \ t\to\infty. \tag{21}$$

Proof. The existence of such $e_q\in W^{q,1}(\mathbf{R})$ is given in [6,13] and it depends on the relation (17). Let $e_q(x,t)$ be the solution to the heat equation with this initial value $e_q(x)$. Then, $\partial_x^q e_q(x,t)$ is a solution to the heat equation with initial value $\phi(x,0)-\psi(x,0)$ and hence $\partial_x^q e_q(x,t)=\phi(x,t)-\psi(x,t)$. The solution $e_q(x,t)$ can be explicitly written as

$$e_q(x,t)=\frac{1}{\sqrt{4\pi\mu t}}\int e^{-\frac{(x-y)^2}{4\mu t}}e_q(y)dy.$$

An integrable solution to the heat equation has the similarity scale $\sqrt{t}$, and $\sqrt{t}u(\sqrt{t}x,t)$ converges to a nontrivial bounded function as $t\to\infty$. Using the similarity variables

$$\xi=x/\sqrt{t},\quad \zeta=y/\sqrt{t},$$

the solution in similarity variable $\tilde e_q(\xi,t)=\sqrt{t}\,e_q(\sqrt{t}\xi,t)$ can be written as

$$\tilde e_q(\xi,t)=\frac{1}{\sqrt{4\mu\pi}}\int e^{-\frac{(\xi-\zeta)^2}{4\mu}}e_q(\sqrt{t}\zeta)d\zeta$$

and its q-th order derivative is given by

$$\partial_\xi^q\tilde e_q(\xi,t)=\partial_x^q e_q(x,t)(\partial_\xi x)^q=\partial_x^q e_q(x,t)(\sqrt{t})^q.$$

Let $A_q:=|\int e_q(z)\,dz|$ and suppose $A_q\neq0$. Then

$$\begin{aligned}(\sqrt{t})^{q+1-k}|x^k\partial_x^q e_q(x,t)|&=(\sqrt{t})|\xi^k\partial_\xi^q\tilde e_q(\xi,t)|\\ &=\frac{A_q}{\sqrt{4\mu\pi}}\left|\xi^k\int f(\zeta)g_t(\xi-\zeta)\,d\zeta\right|,\end{aligned} \tag{22}$$

where $f(\xi):=\partial_\xi^q(e^{-\xi^2/4\mu})$ is smooth and $g_t(\xi):=\sqrt{t}\,e_q(\sqrt{t}\xi)/A_q$ is a delta-sequence as $t\to\infty$. Since $f(\xi)$ decays exponentially as

$|\xi| \to \infty$, the assumptions in Lemma 1 are satisfied with $h(\xi) = \xi^k$ for any $k > 0$. Taking $t \to \infty$ limit to (22) gives

$$\lim_{t\to\infty} (\sqrt{t})^{q-k+1} |x^k \partial_x^q e_q(x,t)| = \frac{A_q}{\sqrt{4\mu\pi}} |\xi^k f(\xi)|.$$

For $r = \infty$,

$$\lim_{t\to\infty} (\sqrt{t})^{q-k+1} \|x^k \partial_x^q e_q(x,t)\|_\infty = \frac{A_q}{\sqrt{4\mu\pi}} \|\xi^k f(\xi)\|_\infty.$$

For $1 \le r < \infty$,

$$\begin{aligned}
(\sqrt{t})^{q-k+1-1/r} & \| \; x^k \partial_x^q e_q(x,t)\|_r \\
&= \left(\int |(\sqrt{t})^{q-k+1} x^k \partial_x^q e_q(x,t)|^r \, d\left(\frac{x}{\sqrt{t}}\right)\right)^{1/r} \\
&= \left(\int |\sqrt{t}\, \xi^k \partial_\xi^q e_q(\xi,t)|^r \, d\xi \right)^{1/r} \\
&= \frac{A_q}{\sqrt{4\mu\pi}} \left(\int \left| \xi^k \int f(\zeta) g_t(\xi-\zeta)\, d\zeta \right|^r d\xi \right)^{1/r} \\
&= \frac{A_q}{\sqrt{4\mu\pi}} \|\xi^k (f * g_t)(\xi)\|_r.
\end{aligned}$$

Hence Lemma 1 gives

$$\lim_{t\to\infty} (\sqrt{t})^{q-k+1-1/r} \|x^k \partial_x^q e_q(t)\|_r = \frac{A_q}{\sqrt{4\mu\pi}} \|\xi^k \partial_\xi^q (e^{-\frac{\xi^2}{4\mu}})\|_r.$$

Now suppose $A_q = 0$. If e_0 is nontrivial, then there exists $l > q$ such that $\int_{-\infty}^{\infty} e_l(x) = \lim_{x\to\infty} e_{l+1}(x) \neq 0$ for some $l > q$ (see [13]). Let $e_l(x,t)$ be the solution with initial value $e_l(x)$. Then, since $\partial_x^l e_l(x) = e_0(x)$, we obtain for $1 \le r \le \infty$

$$\begin{aligned}
\lim_{t\to\infty} t^{\frac{l-k+1}{2} - \frac{1}{2r}} \|x^k \partial_x^q e_q(t)\|_r &= \lim_{t\to\infty} t^{\frac{l-k+1}{2} - \frac{1}{2r}} \|x^k \partial_x^l e_l(t)\|_r \\
&= \frac{A_l}{\sqrt{4\mu\pi}} \|\xi^k \partial_\xi^l (e^{-\frac{\xi^2}{4\mu}})\|_r < \infty.
\end{aligned}$$

Therefore, the convergence order in (20) still holds. In fact the convergence order is actually higher in this case. $\square$

3. Large-time asymptotics in the Burgers equation

Let $u(x,t)$ be the solution of the Burgers equation, i.e.,

$$\begin{array}{ll} u_t + uu_x = \mu u_{xx}, & x \in \boldsymbol{R},\ t>0, \\ u(x,0) = u_0(x), & x \in \boldsymbol{R}. \end{array}$$

Then, the Cole-Hopf transformation and its partial derivative,

$$\varPhi(x,t) = e^{-\frac{1}{2\mu}\int_{-\infty}^{x} u(y,t)dy} - 1 \quad \text{and} \quad \phi(x,t) = \varPhi_x(x,t),$$

are solutions to the heat equation

$$\begin{array}{ll} v_t = \mu v_{xx}, & x \in \boldsymbol{R},\ t>0, \\ v(x,0) = v_0(x), & x \in \boldsymbol{R}. \end{array}$$

The approximation $\psi_n(x,t)$ in Theorem 1(9) has been given in Section 2.1. Now consider an asymptotic approximate solution to the Burgers equation given by

$$w_n(x,t) := -2\mu \frac{\psi_n(x,t)}{1+\varPsi_n(x,t)} = H^{-1}(\varPsi_n), \tag{23}$$

where

$$\varPsi_n(x,t) := \int_{-\infty}^{x} \psi_n(y,t)\,dy. \tag{24}$$

It is needed to show that $w_n(x,t)$ is well defined since the denominator $1+\varPsi_n(x,t)$ can be zero. In the following lemma we will show that there is a time $T \ge 0$ such that this approximate solution $w_n(x,t)$ is well defined for $t \ge T$.

Lemma 2. *Let $M := \int u_0(x)dx$ and $a := \min\{1, e^{-\frac{M}{2\mu}}\} > 0$. Then for any $\epsilon > 0$, there exists $T > 0$ such that*

$$1+\varPsi_n(x,t) \ge a - \epsilon \quad \text{for } x \in \mathbf{R},\ t \ge T. \tag{25}$$

Proof. One can easily compare the boundary values at $x = \pm\infty$. First,

$$\lim_{x\to-\infty} \big(1+\varPsi_n(x,t)\big) = \lim_{x\to-\infty} \big(1+\varPhi_n(x,t)\big) = 1.$$

From the definition of the Cole-Hopf transformation (4) and the agreement of zeroth moments between ϕ and ψ_n, we have

$$\begin{aligned} e^{-\frac{M}{2\mu}} &= \lim_{x\to\infty} \big(1+\varPhi(x,t)\big) = 1 + \int_{-\infty}^{\infty} \phi(x,t)dx \\ &= 1 + \int_{-\infty}^{\infty} \psi_n(x,t)dx = \lim_{x\to\infty} \big(1+\varPsi_n(x,t)\big). \end{aligned}$$

Therefore, for any fixed time $T_0 > 0$, there exists $L > 0$ such that

$$(1 + \Psi_n)(x, T_0) \geq a - \epsilon/2 \quad \text{for } |x| > L$$

and $\frac{1}{\sqrt{4\pi\mu T_0}} \int_{-2L}^{0} e^{-\frac{y^2}{4\mu T_0}} dy \geq 1/4$. Let $m := \min\limits_{|x|\leq L} (1 + \Psi_n)(x, T_0)$ and define

$$G(x,t) := \frac{4(|m| + a)}{\sqrt{4\pi\mu t}} \int_{x-L}^{x+L} e^{-\frac{y^2}{4\mu t}} dy.$$

Then, $G(x,t)$ satisfies the heat equation,

$$G(-L, T_0) = G(L, T_0) = \frac{4(|m| + a)}{\sqrt{4\pi\mu T_0}} \int_{-2L}^{0} e^{-\frac{y^2}{4\mu T_0}} dy \geq |m| + a,$$

and, hence,

$$(1 + \Psi_n + G)(x, T_0) \geq m + |m| + a \geq a \quad \text{for } |x| \leq L.$$

Therefore, the maximum principle (see [17]) gives

$$(1 + \Psi_n + G)(x, t) \geq a - \epsilon/2 \quad \text{for } x \in \mathbf{R}, t \geq T_0.$$

On the other hand, since $G(x,t)$ is a bounded L^1 solution to the heat equation, there exists a large time $T \geq T_0$ such that

$$G(x,t) \leq \epsilon/2 \quad \text{for } x \in \mathbf{R},\ t \geq T.$$

Finally we obtain the conclusion

$$(1 + \Psi_n)(x,t) \geq a - \epsilon/2 - G(x,t) \geq a - \epsilon \quad \text{for } x \in \mathbf{R},\ t \geq T. \qquad \square$$

Remark 3. Suppose that the initial value u_0 is negative, $u_0(x) \leq 0$. Then, $\phi(x, 0)$, which is given by (5), is positive. The truncated moment problem for a positive measure, without using backward moments ($t_0 = 0$), implies that $\rho_i > 0$ for all i. Therefore, the denominator

$$1 + \Psi_n(x, 0) \equiv 1 + \int_{-\infty}^{x} \sum_{i=1}^{n} \rho_i \delta(x - c_i) dx$$

is monotone and hence $1 + \Psi(x, 0) > 0$ for all $x \in \boldsymbol{R}$. Therefore, one may take $T = 0$ in Lemma 2. One may obtain the same conclusion if u_0 is positive. If the backward time is positive or if the initial value is not signed, then $1 + \Psi_n(x, 0)$ is not monotone in general. However, our numerical examples always give $1 + \Psi_n(x, 0) > 0$, i.e., $T = 0$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The moment differences between $u(x,t)$ and $w_n(x,t)$ are estimated using (6) and (23) that is

$$\begin{aligned}
|x^k\big(u(x,t)-w_n(x,t)\big)| &= 2\mu\left|\frac{x^k\phi(x,t)}{1+\Phi(x,t)}-\frac{x^k\psi_n(x,t)}{1+\Psi_n(x,t)}\right| \qquad (26)\\
&= 2\mu\left|\frac{x^k\phi(x,t)+x^k\phi(x,t)\Psi_n(x,t)-x^k\psi_n(x,t)\big(1+\Phi(x,t)\big)}{(1+\Phi(x,t))(1+\Psi_n(x,t))}\right|\\
&\le 2\mu\frac{|\phi(x,t)|\,|x^k\big(\Phi(x,t)-\Psi_n(x,t)\big)|}{|1+\Phi(x,t)|\,|1+\Psi_n(x,t)|}\\
&\qquad +2\mu\frac{|1+\Phi(x,t)|\,|x^k\big(\phi(x,t)-\psi_n(x,t)\big)|}{|1+\Phi(x,t)|\,|1+\Psi_n(x,t)|}.
\end{aligned}$$

Let $U_0(x)=\int_{-\infty}^x u_0(y)dy$. Then, $A=-\inf_x U_0(x)$ and $B=\sup_x U_0(x)$ are non-negative. Since $1+\Phi(x,t)$ is a solution to the heat equation, the maximum principle gives uniform bounds

$$0<e^{-\frac{B}{2\mu}}\le 1+\Phi(x,t)\le e^{\frac{A}{2\mu}}<\infty, \qquad t\ge 0. \tag{27}$$

Let $T>0$ be the one in Lemma 2 and take $\epsilon=a/2$, then we obtain a uniform lower bound

$$\frac{1}{2}\min\{1,e^{-M/2\mu}\}\le 1+\Psi_n(x,t), \quad t\ge T. \tag{28}$$

Therefore, the denominators are uniformly bounded below away from zero. Now we show the convergence order of nominators to obtain (7). First, since ϕ is an L^1 solution to the heat equation, we have

$$\|\phi\|_\infty = O(t^{-\frac{1}{2}}) \quad \text{as} \quad t\to\infty. \tag{29}$$

The L^r-norm estimates of $x^k\big(\psi_n(x,t)-\phi(x,t)\big)$ and $x^k\big(\Psi_n(x,t)-\Phi(x,t)\big)$ are obtained similarly using Theorem 2. Recall that

$$\Psi_n(x,t)=\int_{-\infty}^x \psi_n(y,t)\,dy \quad \text{and} \quad \Phi(x,t)=\int_{-\infty}^x \phi(y,t)\,dy.$$

The approximation ψ_n was constructed to satisfy

$$\int_{-\infty}^\infty x^k(\phi(x,0)-\psi_n(x,0))dx=0, \quad \text{for } 0\le k\le 2n-1.$$

Then, for $0 \le k \le 2n-2$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^k(\Phi(x,0)-\Psi_n(x,0))dx \\
&\qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{x} x^k(\phi(y,0)-\psi_n(y,0))\,dy\,dx \\
&\qquad = \int_{-\infty}^{\infty}\int_{y}^{\infty} x^k(\phi(y,0)-\psi_n(y,0))\,dx\,dy \\
&\qquad = -\frac{1}{k+1}\int_{-\infty}^{\infty} y^{k+1}(\phi(y,0)-\psi_n(y,0))\,dy = 0.
\end{aligned}$$

Therefore, by Theorem 2, we obtain

$$\|x^k(\phi(x,t)-\psi_n(x,t))\|_r = O(t^{\frac{1}{2r}-\frac{2n+1-k}{2}}) \quad \text{as} \quad t\to\infty, \tag{30}$$
$$\|x^k(\Phi(x,t)-\Psi_n(x,t))\|_r = O(t^{\frac{1}{2r}-\frac{2n-k}{2}}) \quad \text{as} \quad t\to\infty. \tag{31}$$

Then, for $1 \le r \le \infty$ and $t \ge T$, taking the L^r-norm on (26) gives

$$\begin{aligned}
\|x^k(u(x,t)-w_n(x,t))\|_r \le C_1\|\phi(x,t)\|_\infty\|x^k(\Psi_n(x,t)-\Phi(x,t))\|_r \\
+ C_2\|x^k(\phi(x,t)-\psi_n(x,t))\|_r,
\end{aligned}$$

where constants $C_1, C_2 > 0$ are from the uniform estimates (27) and (28). Combining the asymptotic convergence orders in (29),(30) and (31) gives

$$\|x^k(u(t)-w_n(t))\|_r = O\big(t^{\frac{1}{2r}-\frac{2n+1-k}{2}}\big) \quad \text{as} \quad t\to\infty,$$

which completes the proof of Theorem 1. $\square$

4. Fine asymptotics and the similarity scale

There are many studies on the asymptotic analysis for various problems. The porous medium equation is one of the examples that such a study has been done intensively. We start our discussion with a brief review of it. Let $m > 0$ and u be a L^1 solution one space dimension

$$u_t = (u^m)_{xx}, \qquad u(x,0) = u_0(x), \tag{32}$$

where the initial value u_0 is bounded and integrable. Let $u(x,t) = av(ax, a^{m+1}t)$. Then v satisfies the equation and preserves the L^1-norm of u. The invariance property under this specific dilation is called the L^1-*similarity structure* of the problem. Variables and solutions that are also invariant under the dilation is called *similarity* variables and solutions, respectively. The Barenblatt solution, say $\rho(x,t)$, and variable ξ are the ones, where $\alpha = 1/(m+1)$,

$$\rho(x,t) = t^{-\alpha}\Big(A - \frac{m-1}{2m(m+1)}\big(xt^{-\alpha}\big)^2\Big)_+^{\frac{1}{m-1}} \quad \text{and} \quad \xi = xt^{-\alpha}.$$

Note that the constant A is a free parameter that decides the total mass and that the similarity variable $\xi = xt^{-\alpha}$ show how the support of the solution expands asymptotically. We say that t^α is the *asymptotic scale* for space distribution of the solution.

If an equation contains more terms, say

$$u_t + (u^q)_x = (u^m)_{xx} + (|u_x|^{p-2}u_x)_x,$$

then the problems has no similarity structure anymore. However, in general, there may exist an *asymptotic scale* t^α that gives the propagating speed of the solution distribution. In the last case the asymptotic scale is given by $\alpha := \max\{1/q, 1/p, 1/(m+1)\}$. Then, $t^\alpha u(t^\alpha x, t)$ converges to a L^1 function as $t \to \infty$. It seems that the asymptotic scale exists for more general kind of problems.

In the literature, two kinds of optimal asymptotic convergence rates appear. They actually correspond to the time and space shifts. In L^1-norm they can be written as, for $t > 0$ large,

$$\|\rho(x,t) - \rho(x-c,t)\|_1 = O(t^{-\alpha}), \tag{33}$$

$$\|\rho(x,t) - \rho(x,t+T)\|_1 = O(t^{-1}). \tag{34}$$

The mechanism of these two asymptotics are different. The first one in (33) is actually related to the convergence orders in Theorem 1. This convergence order corresponds to the one with zeroth moment agreement. The other one (34) is not actually related. In the following we formally investigate the relation between the asymptotic convergence orders and the control of moments at $t = \infty$. Even though we do not have a rigorous proof, it seems reasonable to put this formal arguments in this paper since the convergence order of moments in Theorem 1 motivates them.

Let v be another solution with initial value v_0. Set

$$e(x,t) := u(x,t) - v(x,t).$$

Suppose that

$$\||x|^N e(x,t)\|_1 = \int |x|^N |e(x,t)|dx = O(1) \quad \text{as} \quad t \to \infty. \tag{35}$$

We want to derive the decay rate of $\|e(x,t)\|_1$ as $t \to \infty$. Change the space variable using

$$x = t^\beta y, \qquad dx = t^\beta dy.$$

Then,

$$\int |x|^N |e(x,t)| dx = t^{N\beta} \int |y|^N \, |e(t^\beta y,t)| t^\beta dy.$$

If one obtains two positive constants c and C that are independent of the time $t>0$ and satisfy

$$c\int |e(t^\beta y,t)| t^\beta dy \le \int |y|^N \, |e(t^\beta y,t)| t^\beta dy \le C\int |e(t^\beta y,t)| t^\beta dy, \tag{36}$$

then the correct convergence order for $\|e(x,t)\|_1$ is obtained. Suppose that t^α is the similarity scale or the asymptotic scale that gives the propagating speed of the support of the solution. Then, if $\beta>\alpha$, then $t^\beta|e(t^\beta y,t)|$ behaves like a delta sequence and hence lower bound in (36) should fail. Similarly, if $\beta<\alpha$, then the support of $t^\beta|e(t^\beta y,t)|$ expands as $t\to\infty$ and hence the upper bound of (36) is not expected. Hence, $\beta=\alpha$ is the only case that one may obtain the correct convergence order for $\|e(x,t)\|_1$. Then, there exists $c^*=c^*(t)>0$ such that $c\le c^*\le C$ and

$$\int |e(t^\beta y,t)| t^\beta dy = c^* \int |y|^N \, |e(t^\beta y,t)| t^\beta dy.$$

Using these relations, one obtains

$$\begin{aligned} t^{N\alpha}\|e(x,t)\|_1 &= t^{N\alpha}\int |e(t^\alpha y,t)| t^\alpha dy \\ &= c^* t^{N\alpha}\int |y|^N \, |e(t^\beta y,t)| t^\beta dy = O(\||x|^N e(x,t)\|_1) = O(1) \end{aligned}$$

as $t\to\infty$. Hence, the decay of the N-th order moment at $t=\infty$ in (35) gives

$$\|e(x,t)\|_1 = \int |e(x,t)| dx = O(t^{-N\alpha}) \quad \text{as} \quad t\to\infty. \tag{37}$$

In summary, if one may show (36), then the following claim is obtained.

Fine asymptotics and the similarity scale: *Let $u(x,t)$ and $v(x,t)$ be integrable solutions to a nonlinear problem given in (1) in one space dimension. Suppose that $e:=u-v$ satisfies (35) and t^α, $\alpha>0$, is the asymptotic scale of the problem. Then, for $0\le k\le N$,*

$$\||x|^k e(x,t)\|_r = O(t^{(k-N-1+1/r)\alpha}) \quad \textit{as} \;\; t\to\infty. \tag{38}$$

This convergence order is the one corresponding to (7). One may say that the convergence order (34) is not of this kind. However, the one in (33) is this kind with $N=1$.

Remark 4. For the fast diffusion case, $0 < m < 1$, the similarity scale t^α is with $\alpha = 1/(m+1) > 1/2$. Hence, if $\||x|^2(\rho(x,t)-\rho(x,t+T))\|_1 = O(1)$ as $t \to \infty$, then above discussion implies that $\|\rho(t)-\rho(t+T)\|_1 = O(t^{-2\alpha})$. However, the order (34) is an optimal one and hence one should expect that $\||x|^2(\rho(x,t) - \rho(x,t+T))\|_1 \to \infty$ as $t \to \infty$. In fact, this is true and one may check that using the explicit formula of the Barenblatt solution.

5. Numerical Examples

In this section, we test the asymptotic convergence orders obtained in Theorem 1 numerically. The effect of the backward moments is also tested. Note that the convergence order in (7) is for $t > 0$ large and hence one should wait certain amount of time to observe such a convergence order. However, at that stage, the error $\|u(t) - w_n(t)\|_r$ can be very small. Hence it is important to compute the exact solution,

$$u(x,t) = H^{-1}\Big(\int_{-\infty}^{x}\Big(\frac{1}{\sqrt{4\pi t}}\int \phi(y-z,0)e^{-z^2/4t}dz\Big)dy\Big),$$

with an error smaller than this asymptotic approximation error. However, it is unrealistic to do the required integrations with such a small tolerance. Hence one should test the convergence order with a case that an explicit solution exists. One easy way to do that is to set $\phi(x,t)$ first (not $u(x,t)$). Let

$$\begin{aligned}\phi(x,t) :=\ & \frac{5}{\sqrt{4\pi\mu(t+2)}}e^{-\frac{(x+1.8)^2}{4\mu(t+2)}} + \frac{20}{\sqrt{4\pi\mu(t+1)}}e^{-\frac{(x+0.5)^2}{4\mu(t+1)}} \\ & - \frac{16}{\sqrt{4\pi\mu(t+0.5)}}e^{-\frac{(x-0.5)^2}{4\mu(t+0.5)}} - \frac{9}{\sqrt{4\pi\mu(t+2)}}e^{-\frac{(x-1.2)^2}{4\mu(t+2)}},\end{aligned} \tag{39}$$

and $u(x,t)$ be the inverse Cole-Hopf transformation of

$$\Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy.$$

Remember that from the definition of the Cole-Hopf transformation,

$$\int_{-\infty}^{x} \phi(y,0)\,dy = \Phi(x,0) = e^{-\frac{1}{2\mu}\int_{-\infty}^{x} u_0(y)\,dy} - 1 > -1. \tag{40}$$

Hence one should choose $\phi(x,0)$ that satisfies (40) for all $x \in \boldsymbol{R}$. The one given in (39) satisfies it.

The numerical test in this section has two purposes. The first one is to observe approximation properties of the method suggested in this paper. In Figures 1 and 2 we have compared the approximations to the exact one varying the backward time and the number of kernels.

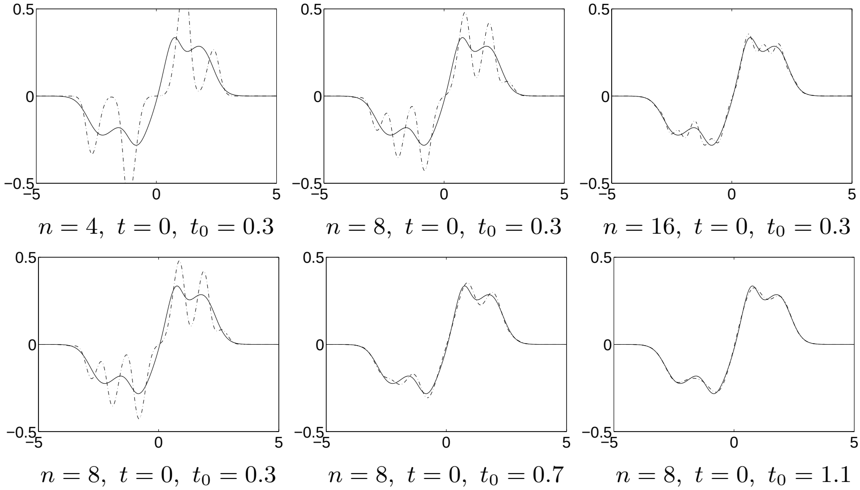

Fig. 1. The initial data u_0 (solid line) and its approximation (dashed line) are figured. The three figures in the first row show the convergence as n increases. The backward time is fixed with $t_0 = 0.3$. The three figures in the second row show the role of the backward time t_0. One may see that a better backward time gives better results for a given n. In the example with the given initial value and $n = 8$, the backward time $t_0 = 1.1$ seems a limit. After this limit of backward time the error of the approximation increases suddenly (see Table 1).

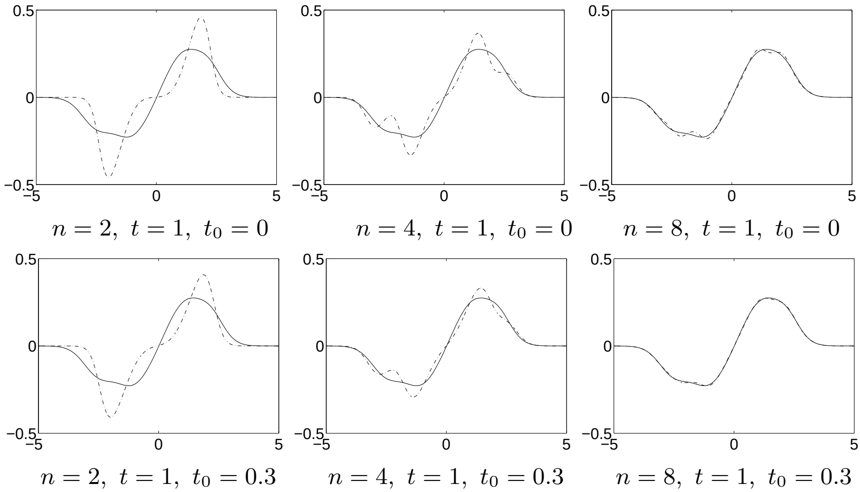

Fig. 2. The solution to the Burgers equation at time $t = 1$ are given in solid lines. Approximate solutions are given in dashed lines. The three figures in the first row are without using backward moments $t_0 = 0$. The others are with $t_0 = 0.3$. In both cases one may observe convergence as n increases.

The graph of the initial value $u(x,0)$, which is the inverse Cole-Hopf transformation of $\Phi(x,0)$, is given in Figure 1 in solid lines. For a given backward time $t_0 \geq 0$, the approximate solution $\psi_n(x,t)$ to the heat equation is given by (9). The approximation $w_n(x,t)$ in Theorem 1 is the inverse Cole-Hopf transformation of $\Psi_n(x,t) := \int_{-\infty}^{x} \psi_n(y,t)dy$. The initial approximation $w_n(x,0)$ are given in Figure 1 in dashed lines. The three figures in the first row show the convergence as n increases with a fixed backward time $t_0 = 0.3$. If the backward moments are not used, i.e., $t_0 = 0$, then the approximation is just a collection of delta distributions. Hence, initial convergence as $n \to \infty$ is not expected without using backward moments.

The three figures in the second row of Figure 1 show the role of the backward time t_0. One may see that a better backward time gives better results for a fixed n. In this example, the backward time $t_0 = 1.1$ seems the best. One should not be mislead that the approximation converges as $t_0 \to \infty$. In fact, the approximation error increases suddenly for $t_0 > 1.1$. This behavior is related to the initial value $u(x,0)$ given by (39) and the number of heat kernels $n = 8$. To verify this property an error comparison is given in Table 1 for $n = 2, 4, 8$ and 16 as increasing t_0. One may observe that the backward time improves the approximation only up to certain limit and, after that, the performance becomes poor suddenly. For a bigger n, the best backward time becomes smaller. This property seems related to the age of the initial heat distribution $\phi(x,0)$ in (39). In this example, the age is $t = 0.5$, and the best backward time t_0 seems to approach to this age as $n \to \infty$. However, we only have numerical experiments for this argument.

In Figure 2, the solution to the Burgers equation at time $t = 1$ is given in solid lines. Approximate solutions are given in dashed lines. The three figures in the first row are without using backward moments, i.e., $t_0 = 0$. The others are with $t_0 = 0.3$. In both cases one may observe convergence as n increases. One may also see that the effects of the backward time $t_0 > 0$ becomes smaller as t increases (compare Tables 2 and 3).

The second purpose of this section is to test the asymptotic convergence order in (7). In the followings we only test the zeroth moment in the uniform norm, i.e., the L^∞-contraction order between u and w_n. The convergence rate $\gamma_n(t)$ is computed using the formula

$$\gamma_n(t) := \ln\left(\frac{\|u(x,t) - w_n(x,t)\|_\infty}{\|u(x,t/2) - w_n(x,t/2)\|_\infty}\right) \Big/ \ln\left(\frac{1}{2}\right). \tag{41}$$

In Table 2 the error and the convergence rates are given in the uniform norm. The approximate solutions w_n are constructed for $n = 2, 4, 8$ and zero backward time $t_0 = 0$. One may roughly observe that the convergence order increases to $n + 0.5$ which is given by Theorem

Table 1. Approximation errors $e_n(t) := \|u(t)-w_n(t)\|_\infty$ at time $t=0.1$ are given increasing the backward time $t_0 \geq 0$. The error decreases as t_0 increases up to certain limit and then it blows up suddenly. Errors in small font are invalid ones.

t_0	$e_2(0.1)$	$e_4(0.1)$	$e_8(0.1)$	$e_{16}(0.1)$
0	1.26e-00	8.72e-01	4.69e-01	2.73e-01
0.1	8.41e-01	5.41e-01	2.49e-01	1.08e-01
0.2	6.55e-01	3.95e-01	1.56e-01	4.61e-02
0.3	5.45e-01	3.07e-01	1.10e-01	1.98e-02
0.4	4.70e-01	2.47e-01	7.75e-02	7.97e-03
0.5	4.15e-01	2.03e-01	5.37e-02	2.97e-03
0.6	3.72e-01	1.69e-01	3.68e-02	1.40e-03
0.7	3.38e-01	1.42e-01	2.85e-02	7.13e-04
0.8	3.11e-01	1.20e-01	2.37e-02	9.44e-02
0.9	2.87e-01	1.02e-01	2.04e-02	4.74e-02
1.0	2.68e-01	8.73e-02	1.75e-02	6.82e-02
1.1	2.51e-01	7.52e-02	1.47e-02	6.56e-02
1.2	2.37e-01	6.52e-02	6.57e+02	7.10e-01
1.3	2.24e-01	5.69e-02	1.11e+03	7.82e-02
1.4	2.13e-01	5.00e-02	4.51e-01	5.62e+15
1.5	2.03e-01	4.42e-02	4.01e-02	3.32e+02
1.6	1.94e-01	2.35e+02	1.88e-01	2.20e+02

Table 2. Approximation error without backward moments, i.e., $t_0 = 0$. The numerical order $\gamma_n(t)$ computed by (41) converges to the theoretical one as $t \to \infty$.

t	$e_2(t)$	$\gamma_2(t)$	$e_4(t)$	$\gamma_4(t)$	$e_8(t)$	$\gamma_8(t)$
0.125	1.11e-00		7.51e-01		3.89e-01	
0.25	7.23e-01	0.62	4.47e-01	0.75	2.00e-01	0.96
0.5	4.48e-01	0.69	2.36e-01	0.92	8.95e-02	1.16
1	2.53e-01	0.82	1.05e-01	1.17	2.19e-02	2.03
2	1.23e-01	1.04	3.14e-02	1.74	2.12e-03	3.37
4	4.77e-02	1.37	5.71e-03	2.46	7.36e-05	4.85
8	1.41e-02	1.76	6.35e-04	3.17	1.04e-06	6.15
16	3.33e-03	2.08	4.78e-05	3.73	7.97e-09	7.03
32	6.81e-04	2.29	3.03e-06	3.98	4.07e-11	7.61
64	1.31e-04	2.38	1.72e-07	4.14	1.66e-13	7.94
128	2.46e-05	2.41	9.34e-09	4.20	6.18e-16	8.07

Table 3. Approximation error and contraction order with backward time $t_0 = 0.3$. The error in this case is smaller than the case with $t_0 = 0$.

t	$e_2(t)$	$\gamma_2(t)$	$e_4(t)$	$\gamma_4(t)$	$e_8(t)$	$\gamma_8(t)$
0.125	5.22e-01		2.88e-01		1.02e-01	
0.25	4.31e-01	0.28	2.15e-01	0.42	6.94e-02	0.56
0.5	3.19e-01	0.43	1.38e-01	0.64	3.26e-02	1.09
1	2.03e-01	0.65	6.69e-02	1.04	8.16e-03	2.00
2	1.06e-01	0.94	2.09e-02	1.68	8.42e-04	3.28
4	4.24e-02	1.32	4.00e-03	2.39	3.19e-05	4.72
8	1.25e-02	1.76	4.66e-04	3.10	4.95e-07	6.01
16	3.07e-03	2.03	3.58e-05	3.70	4.01e-09	6.95
32	6.39e-04	2.26	2.30e-06	3.96	2.14e-11	7.55
64	1.24e-04	2.37	1.33e-07	4.11	8.95e-14	7.90
128	2.38e-05	2.38	7.32e-09	4.18	3.54e-16	7.98

1. In Table 3 the same comparisons are given for the approximate solutions using backward moment $t_0 = 0.3$. For a small time $t > 0$, the result is considerably better if a backward time is used. However, as $t \to \infty$, both of them become similar. The asymptotic convergence order in (7) is for $t > 0$ large. In conclusion, the approximate solution w_n constructed in this paper well behaves for a small time, too. This is partly due to the use of backward time. The approach using the derivatives of Gaussian as in (18) can be also improved by using the backward time as in (19).

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