Relative Newtonian potentials of radial functions and asymptotics in nonlinear diffusion*

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Abstract. The Newtonian potential is introduced in a relative sense for radial functions. In this way one may treat the potential theory for a larger class of functions in a unified manner for all dimensions $d \geq 1$. For example, Newton's theorem is given in terms of relative potentials, which is a simpler statement for all dimensions. This relative potential is then used to obtain the L^1-convergence order $O(t^{-1})$ as $t \to \infty$ for radially symmetric solutions to the porous medium and fast diffusion equations. The technique is also applied to radial solutions of the p-Laplacian equations to obtain the same convergence order.

1. Introduction

The fundamental solution of Laplace equation in $\mathbf{R}^d$ has three different shapes depending on the dimension. They are

$$\Phi(\mathbf{x}) := \begin{cases} \frac{-1}{(d-2)\omega_d}|\mathbf{x}|^{2-d}, & d \geq 3, \\ \frac{1}{\omega_d}\ln|\mathbf{x}|, & d = 2, \\ \frac{1}{\omega_d}|\mathbf{x}|^{2-d}, & d = 1, \end{cases} \tag{1}$$

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where $\omega_d := 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbf{R}^d$. The Newtonian potential of a Radon measure $v(\mathbf{x})$ is defined by

$$V(\mathbf{x}) := \int_{\mathbf{R}^d} \Phi(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\mathbf{y}, \tag{2}$$

which solves the Poisson equation $\Delta V = v$. The fundamental solution Φ itself is the Newtonian potential of the Dirac delta measure$\delta(x)$ and $\Delta\Phi = \delta$. Since Φ is locally integrable, the Newtonian potential is well-defined if $v(\mathbf{x})$ decays with order

$$|v(\mathbf{x})| = O(|\mathbf{x}|^{-2-\epsilon}) \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{3}$$

Due to the dimension dependency of the fundamental solution, one should consider the Newtonian potential separately for the three cases. Such a difference is an obstacle to obtain simple statements that work for all dimensions and makes certain analysis lengthy and complicated. The purpose of this paper is to propose a potential theory in a relative sense, which brings the properties of the Newtonian potential of dimensions $d \geq 3$ to all dimensions $d \geq 1$. We also give two examples to show that the new theory provides a unified approach to all dimensions. The first example is the Newton's theorem itself, which is:

Newton's Theorem(Theorem 9.7 in Lieb and Loss [24]) *Let $v(\mathbf{x}) \geq 0$ be a radial Radon measure satisfying the decay condition (3) with total mass $M = \int v(\mathbf{x})d\mathbf{x}$. Then, its Newtonian potential V satisfies*

$$|V(\mathbf{x})| \leq M|\Phi(\mathbf{x})| \quad \textit{for all} \ \ \mathbf{x} \in \mathbf{R}^d. \tag{4}$$

Furthermore, if the support of the measure v lies in a ball of radius $L > 0$ centered at the origin, i.e., $\mathrm{supp}(v) \subset B_L(0)$, *then*

$$V(\mathbf{x}) = M\Phi(\mathbf{x}) \quad \textit{if} \ \ |\mathbf{x}| > L. \tag{5}$$

Considering the signs of Φ and V, the theorem gives three scenarios depending on the dimension, which are given in the three diagrams in Figure 1. For dimension $d = 1$, the fundamental solution is non-negative and hence so is the Newtonian potential V given by (2). Therefore the inequality (4) gives $0 \leq V(\mathbf{x}) \leq M\Phi(\mathbf{x})$. If $d \geq 3$, the situation is opposite; Φ is non-positive and $M\Phi(\mathbf{x}) \leq V(\mathbf{x}) \leq 0$. The case $d = 2$ is a mixture of these two cases. Since $\ln(1) = 0$, the inequality (4) forces $V(\mathbf{x}) = 0$ if $|\mathbf{x}| = 1$ and furthermore, $M\Phi(\mathbf{x}) \leq V(\mathbf{x}) \leq 0$ for all $|\mathbf{x}| < 1$ and $0 \leq V(\mathbf{x}) \leq M\Phi(\mathbf{x})$ for all $|\mathbf{x}| > 1$.

The proof of the theorem should be completed by doing the three cases separately, which is lengthy and complicate. In particular the case $d = 2$ requires more work. Furthermore, the essence of the theorem becomes less clear by those three different scenarios. In Section

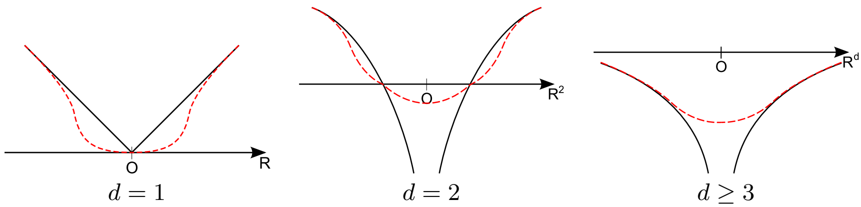


Fig. 1. These diagrams show the relation between the fundamental solution $M\Phi$ (solid lines), and the Newtonian potential V of a Radon measure $v \geq 0$ with mass $M > 0$ (dashed lines) given by the Newton's theorem.

2, we will observe that the source of this complication is to get the Newtonian potential well-defined and it is not avoidable as long as one wants to consider the potential itself. The key thing that one should remember is that it is the potential difference but not the potential itself that makes physics. For example, the electrical current is produced by the voltage difference, but not by the voltage. Hence it is desirable to define a potential in a relative sense from the beginning and develop a theory based on it. The Newton's theorem is also a comparison between two Newtonian potentials, one for $v \geq 0$ and the other for $M\delta$, where they share the same mass under the assumption $M = \int v(\mathbf{x})d\mathbf{x}$.

In Section 2 the relative potential of two radial Radon measures, say v_1 and v_2, is defined by

$$E(r; v_1, v_2) := -\int_r^\infty \Big(x^{1-d}\int_0^x y^{d-1}(v_1(y)-v_2(y))\,dy\Big)dx. \tag{6}$$

For dimensions $d \geq 3$, this relative potential is well-defined if $|v_1 - v_2|$ has the order in (3) for $|\mathbf{x}|$ large. Even if the Newtonian potentials are not well-defined for each one of v_1 and v_2, the relative potentials can be well-defined. For dimensions $d \leq 2$, this decay rate is not enough to get it well-defined. However, if v_1 and v_2 share the same mass, then it is. Therefore, if one compares two Radon measures of the same mass, which is the case of the Newton's theorem and of the potential comparison technique, the decay rate in (3) for the difference $|v_1 - v_2|$ is just enough. If the relative potential is well-defined, then, for all dimensions $d \geq 1$,

$$\Delta E(r; v_1, v_2)\Big(\equiv r^{1-d}(r^{d-1}E'(r; v_1, v_2))'\Big) = v_1(r) - v_2(r).$$

Now we can state our new version of the Newton's theorem using the relative Newtonian potential, which is proved in Section 2:

Theorem 1 (Newton's theorem in relative potentials). *Let $v(\mathbf{x}) \geq 0$ be a radial Radon measure satisfying the decay condition*

(3) with $M = \int v(\mathbf{x})d\mathbf{x}$. *Then, the relative Newtonian potential* E *satisfies*

$$E(|\mathbf{x}|; M\delta, v) \leq 0 \quad \textit{for} \quad |\mathbf{x}| > 0. \tag{7}$$

Furthermore, if $\mathrm{supp}(v) \subset B_L(0)$, *then*

$$E(|\mathbf{x}|; M\delta, v) = 0 \quad \textit{for} \quad |\mathbf{x}| > L. \tag{8}$$

Let $\Phi(\mathbf{x})$ *be the fundamental solution in (1) and* V *be the Newtonian potential of* v *in (2). Then, for dimensions* $d \geq 3$,

$$E(|\mathbf{x}|; M\delta, v) = M\Phi(\mathbf{x}) - V(\mathbf{x}). \tag{9}$$

In this new version of the Newton's theorem, there is no complication depending on the dimension and one may develop a potential theory that works for all dimensions in a unified way. The equality (9) indicates that the new theorem is identical to the original Newton's theorem for dimensions $d \geq 3$.

The second example is a study of long time asymptotics in nonlinear diffusion equations. In Section 3, the relative potentials are applied to obtain intermediate asymptotics of radial solutions to

$$u_t = \Delta(u^m), \quad u(\mathbf{x}, 0) = u^0(\mathbf{x}) \geq 0, \quad t > 0,\ \mathbf{x} \in \mathbf{R}^d, \tag{10}$$

where the exponent m is a positive constant. If $m < 1$, then the equation is called the fast diffusion equation, written here FDE for brevity. If $m > 1$, it is called the porous medium equation (PME). This equation is a nonlinear version of the heat equation with a temperature depending conductivity mu^{m-1}. This model has been used to describe various diffusion processes such as a gas flow through a porous media, heat radiation in plasmas, groundwater flow, curvature flow, and spreading species (see Chapter 2 in [29]).

One of the essential structures of the equation is the radial symmetry. For example, if the initial data u^0 is radially symmetric, then the solution keeps the symmetry all the time $t > 0$. If the initial value is not radial, then the solution asymptotically converges to a fundamental solution of the same mass which is radial. In fact, the homogeneity of the problem allows a similarity structure and one may find the fundamental solution explicitly. This fundamental solution is called the Barenblatt solution and is given by

$$\rho(\mathbf{x}, t) = t^{-d\alpha}(C_M - k|\mathbf{x}t^{-\alpha}|^2)_+^{1/(m-1)}, \tag{11}$$

where $\alpha = \frac{1}{d(m-1)+2} > 0$, $k = \frac{\alpha(m-1)}{2m}$, and $C_M > 0$ is a constant decided by the total mass $\int \rho(\mathbf{x}, t)d\mathbf{x} = M$. Here, we denote $f_+ := \max(f, 0)$. The restriction $\alpha > 0$ indicates that we assume $m > (d-2)_+/d$. In fact this is the mass conservative regime and the explicit Barenblatt solution is valid only when the exponent m is in the regime. For the FDE regime $m < 1$, the coefficient k is negative and hence the inside of the parenthesis in (11) is strictly positive for all $\mathbf{x} \in \mathbf{R}$. Hence $\rho(\mathbf{x}, t)$ is strictly positive everywhere for all $t > 0$. For the PME regime, the inside is positive only in a disc and hence ρ is compactly supported.

In Section 4, the long time asymptotics of the p-Laplacian equation (PLE) is considered. For a fixed $p > 1$, the equation is given by

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad u(\mathbf{x}, 0) = u^0(\mathbf{x}) \geq 0, \quad t > 0, \ \mathbf{x} \in \mathbf{R}^d. \tag{12}$$

This problem also has a similarity structure and its fundamental solution is explicitly given by

$$\rho(\mathbf{x}, t) = t^{-d\alpha} (C_M - k|\mathbf{x}t^{-\alpha}|^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}, \tag{13}$$

where $\alpha = \frac{1}{d(p-2)+p} > 0$, $k = \frac{p-2}{p} \alpha^{1/(p-1)}$, and $C_M > 0$ is a constant decided by the total mass $\int \rho(\mathbf{x}, t) d\mathbf{x} = M$. This formula is valid for the mass conservative regime $p > 2d/(d+1)$. The solution is strictly positive everywhere if $p < 2$ and is compactly supported if $p > 2$ by the same mechanism of the earlier case.

Now we can state our second theorem of the paper, which is proved in Sections 3 and 4 using relative Newtonian potentials.

Theorem 2. *Let the initial value $u^0(\mathbf{x}) \geq 0$ be radially symmetric, compactly supported, and of mass $M = \int u^0(\mathbf{x}) d\mathbf{x} < \infty$. Then the solution $u(\mathbf{x}, t)$ to Eq. (10) with $m > (d-2)_+/d$ satisfies*

$$\|u(t) - \rho(t)\|_1 = O(t^{-1}) \quad as \quad t \to \infty, \tag{14}$$

where $\rho(\mathbf{x}, t)$ is the Barenblatt solution given by (11). If $u(\mathbf{x}, t)$ is the solution to Eq. (12) with $p > 2d/(d+1)$, then the same convergence order holds with the Barenblatt-type solution $\rho(\mathbf{x}, t)$ given in (13).

Here we denote the L^1-norm over the space variable as $\|\cdot\|_1$. We will prove this theorem using the relative Newtonian potential of a radial solution $u(\mathbf{x}, t)$ respect to the Barenblatt solution $\rho(\mathbf{x}, t)$ in a unified way for all dimensions $d \geq 1$. The theorem is proved for the FDE and PME case in Section 3 and then for PLE case in Section 4.

Long time asymptotic contraction to the Barenblatt solution has been intensively studied for the FDE and PME cases. Vázquez has shown that an L^1-contraction order is not generally expected among all $L^1(\mathbf{R}^n)$ initial data even if they share the same mass (see [31]). Hence extra restrictions such as finiteness of moments, entropy or relative entropy has been imposed to obtain certain contraction order throughout the literature (see [6,7]). There are two kinds of optimal contraction rates. The first one is of the similarity scale of $O(t^{-\alpha})$ as $t \to \infty$, which is the order of a space translation $\|\rho(t) - \rho_{x_0}(t)\|_1$ with

$\rho_{x_0}(x,t) = \rho(x+x_0,t)$. This rate has been shown for $(d-1)/d < m < 2$ in [7–9,11,27]. The contraction rates in other regimes obtained so far are lower than this optimal rate. A complete spectrum analysis of FDE was given by Denzler and McCann [13].

The other optimal contraction rate, which requires tuning the center of mass, is $O(t^{-1})$ as $t \to \infty$. This is the one in the theorem. Note that the center of mass is already tuned for a radial solutions at the origin. This contraction rate is the one of a time translation $\|\rho(t) - \rho(t+T)\|_1$ for a fixed $T > 0$. This rate has been shown for $(d-2)_+/d < m \le d/(d+2)$ in [19] and a similar rate $O(t^{-1+\epsilon})$ for $(d-1)/d \le m < 1$ in [26]. For radial solutions or for dimension one such a contraction rate has been shown for all $m > (d-2)_+/d$ using a potential comparison or a mass concentration comparison [8,19,20, 30,31]. Therefore, the contraction rate in Theorem 2 is not new for FDE and PME cases. However, the point is that the relative Newtonian potential gives a simple proof in a unified way for all dimensions $d \ge 1$ and all exponents $m > (d-2)_+/d$.

The PLE also has a similar story of the asymptotic convergence in the conservative regime $p > 2d/(d+1)$. The case with $p < 2$ corresponds to the FDE and the other one corresponds to the PME. However, the convergence order obtained is not optimal. Kamin and Vázquez [15] obtained L^1 contraction without an order. Del Pino and Dolbeault [12] obtained a convergence order for $(2d+1)/(d+1) \le p < d$, Agueh [1] extended it to $p > d$, and Agueh, Blanchet and Carrillo [2] filled the gap and included the whole conservative regime $p > 2d/(d+1)$.

The entropy dissipation method (see [5,9]) has been used to obtain intermediate long time asymptotics. Even if the entropy of a solution is not defined, the relative entropy can be defined by comparing it with another solution. In this way the relative entropy theory has enlarged the regime that the dissipation method is applicable. The relative Newtonian potential may do exactly the same role. Furthermore, the relative Newtonian potential introduced in this paper provides a unified approach for all dimensions $d \ge 1$, which seems a more significant contribution. Pierre [28] employed Newtonian potentials for dimensions $d \ge 3$ only. Kim and McCann [19] fully used them for all dimensions $d \ge 1$ case by case.

2. Relative potential of radial functions

Let V be the Newtonian potential of a Radon measure $v \ge 0$. In other words V satisfies $\Delta V = v$ in the sense of distributions. Under the radial symmetry assumption for V and v, one may write the relation as

$$\Delta V = r^{1-d}(r^{d-1}V'(r))' = v(r), \quad r = |\mathbf{x}|. \tag{15}$$

Then, a formal integration of (15) gives a natural candidate for the potential,

$$V(r) = \int_{r_0}^{r} \Big(x^{1-d} \int_0^x y^{d-1} v(y)dy \Big) dx, \quad 0 \le r_0 \le \infty.$$

Since the fundamental solution is the case with $v(y) = \delta(y)$, one may easily see that $r_0 = 0$ if $d = 1$, $r_0 = 1$ if $d = 2$ and $r_0 = \infty$ if $d \ge 3$ to recover the fundamental solutions in (1). This explains the three scenarios of the Newton's theorem given in Figure 1. Therefore, it is desirable to find a way to pick $r_0 = \infty$ for all dimension, which will unify the three scenarios.

Let $\mathcal{V}(r)$ be the mass concentration of v, which is the mass in $B_r(0)$, a ball of radius $r > 0$ centered at $\mathbf{x} = 0$. After setting $r_0 = \infty$, it seems reasonable to define

$$V(r) := -\frac{1}{\omega_d} \int_r^\infty x^{1-d} \mathcal{V}(x) dx, \quad \mathcal{V}(x) = \omega_d \int_0^x y^{d-1} v(y) dy. \quad (16)$$

Notice the hierarchy of notations. A small letter v is a given measure, a calibrated letter $\mathcal{V}$ is the mass concentration and a capital letter V is the potential. Here we are sharing the same notation with the original Newtonian potential (2), which will be justified by Theorem 1 and its proof.

Unfortunately, this definition is not well-defined in general. The total mass M should be the limit $\lim_{r\to\infty} \mathcal{V}(r) = M$. It is clear that, if $\mathcal{V}(r) = O(r^{d-2-\epsilon})$ for $r > 0$ large, then V is well-defined. Hence, if $d \ge 3$, the potential V is well-defined for any L^1 measure v. If $d \le 2$, then the potential V in (16) is not defined since the only non-negative function that has the mass concentration of order $\mathcal{V}(r) = O(r^{d-2-\epsilon})$ for $r > 0$ large is the trivial one. That is why it is forced to choose $r_0 = 0$ for dimension $d = 1$. For dimension $d = 2$, neither one of the two end points $r_0 = 0$ or $r_0 = \infty$ is not working and hence one should choose an interior point, $r_0 = 1$. Therefore, it is clear that, as long as the potential itself is considered, the three different scenarios in Figure 1 are not avoidable, even though they are basically talking about the same phenomenon.

However, if one wants to compare two potentials of the same mass, then the Newtonian potential of their difference can be considered from the beginning. Then, if the difference decays fast enough for $|\mathbf{x}|$ large, one may define its potential taking $r_0 = \infty$ for all $d \ge 1$. In other words, as long as the potential is understood in a relative sense, there is no difference depending on dimension.

Definition 1. *Let $v_1(r)$ and $v_2(r)$ be non-negative radial Radon measures in $\mathbf{R}^d$ with $d \ge 1$. The relative Newtonian potential $E(r; v_1, v_2)$*

between v_1 *and* v_2 *is defined by*

$$E(r;v_1,v_2) := -\int_r^\infty \left(x^{1-d}\int_0^x y^{d-1}(v_1(y)-v_2(y))\,dy\right)dx. \tag{17}$$

This relative potential is well-defined if the *relative mass concentration*

$$\mathcal{E}(r;v_1,v_2) := \omega_d \int_0^r y^{d-1}(v_1(y)-v_2(y))dy \tag{18}$$

in the ball of radius $r>0$ has order

$$\mathcal{E}(r;v_1,v_2) = O(r^{d-2-\epsilon}) \quad \text{as} \quad r\to\infty. \tag{19}$$

On the other hand, if the relative potential is well-defined, then the relative mass concentration should satisfy

$$\mathcal{E}(r;v_1,v_2) = o(r^{d-2}) \quad \text{as} \quad r\to\infty.$$

Therefore, when dimension $d\le 2$, the Radon measures v_1 and v_2 should have the same total mass, i.e., $\|v_1\|_1 = \|v_2\|_1$, to get their relative potential to be well-defined.

From the definition we have

$$\Delta E(r;v_1,v_2) = v_1 - v_2, \tag{20}$$
$$E(r;v_1,v_2) = -E(r;v_2,v_1). \tag{21}$$

If the Radon measures v_1, v_2 are clearly given from the context, then we simply denote the relative potential and relative mass concentration by $E(r)$ and $\mathcal{E}(r)$, respectively.

Newton's theorem is about a comparison between the fundamental solution of Laplace equation and the Newtonian potential of a radial function. One may consider it in terms of relative potentials in a unified way for all dimensions $d\ge 1$.

Lemma 1. *Let* v_i, $i=1,2$, *be non-negative radial Radon measures such that*

$$\mathrm{supp}(v_i) = [0,L_i] \ \textit{with}\ 0<L_1<L_2<\infty,\ \omega_d\int_0^{L_i} y^{d-1}v_i(y)dy = M,$$

and $E(r)(\equiv E(r;v_1,v_2))$ *be the corresponding relative potential, i.e.,*

$$E(r) := -\int_r^\infty x^{1-d}k(x)\,dx, \quad k(x) := \int_0^x y^{d-1}(v_1(y)-v_2(y))\,dy.$$

Then,

$$E(r)\le 0 \quad \textit{if} \quad L_1<r<L_2, \qquad E(r)=0 \quad \textit{if} \quad L_2<r.$$

Proof. Let

$$A(x) := \int_0^x y^{d-1} v_1(y)\, dy, \quad B(x) := \int_0^x y^{d-1} v_2(y)\, dy.$$

Then, $k(x) = A(x) - B(x)$. If $x > L_2$, then $A(x) = B(x) = M/\omega_d$ and hence $k(x) = 0$. Therefore, $E(r) = 0$ for all $r > L_2$. If $L_1 < x < L_2$, then $A(x) = M/\omega_d$ and $B(x) < M/\omega_d$. Hence, $k(x) > 0$. Therefore, for $L_1 < r < L_2$, $E(r) = -\int_r^{L_2} k(x)dx < 0$. $\square$

Newton's theorem is a special case of Lemma 1 that $v_1 = M\delta$. However, the proof of the lemma was almost trivial. It is because of the definition of the relative potential which is designed for radial ones. The nontrivial part is to show that it is actually the original one given in (2) at least for $d \geq 3$. In the following proof, we will show that part using the original Newton's theorem.

Proof of Theorem 1. The first two parts of Theorem 1 are clear from Lemma 1, where v_1 is replaced by the Dirac delta measure multiplied by $M > 0$. We show the last part (9). For a dimension $d \geq 3$, set

$$A(\mathbf{x}) := M\Phi(\mathbf{x}) - E(|\mathbf{x}|; M\delta, v).$$

Then,

$$\Delta A = M\Delta\Phi - \Delta E = M\delta - (M\delta - v) = v = \Delta V,$$

where V is the Newtonian potential given in (2). Therefore, $A - V$ is harmonic. The second part of the theorem (8) implies that $A = M\Phi$ for all $|\mathbf{x}| > L$. The original Newton's theorem implies that $V = M\Phi$ for all $|\mathbf{x}| > L$. Therefore, $A - V$ is a bounded harmonic function with a compact support. The only such a harmonic function is a trivial one, which shows $A = V$. $\square$

Remark 1. In the proof of the theorem we actually showed that

$$\int_{\mathbf{R}^d} \Phi(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\mathbf{y} = -\int_r^\infty x^{1-d}\Big(\int_0^x y^{d-1}v(y)dy\Big)dx,$$

where $d \geq 3$ and the Radon measure v is radial, which justifies the use of the same symbol in (2) and (16).

Remark 2. The definition of the relative Newtonian potential given in this paper is for the radial case. In particular, at the origin, one may write it as

$$E(0; v_1, v_2) = -\frac{1}{\omega_d}\int_0^\infty r^{1-d}\Big(\int_{B_r(0)} (v_1(\mathbf{y}) - v_2(\mathbf{y}))d\mathbf{y}\Big)dr.$$

Therefore, for non-radial Radon measures, the relative Newtonian potential can be defined in a unified way for all $d \geq 1$ by

$$E(\mathbf{x}; v_1, v_2) = -\frac{1}{\omega_d}\int_0^\infty r^{1-d}\Big(\int_{B_r(\mathbf{x})}(v_1(\mathbf{y}) - v_2(\mathbf{y}))d\mathbf{y}\Big)dr.$$

3. The porous medium and fast diffusion equation

In this section we prove Theorem 2 for the solutions to the PME and FDE. Since the solution $u(\mathbf{x}, t)$ and the initial value $u^0(\mathbf{x})$ are radially symmetric, one may rewrite the equation as

$$u_t = r^{1-d}(r^{d-1}(u^m)_r)_r,\ \ u(r, 0) = u^0(r) \geq 0,\ \ u_r(0, t) = 0, \tag{22}$$

where $r = |\mathbf{x}|$, $m > (d-2)_+/d$ and $d \geq 1$. Notice that we are slightly abusing notation by writing $u(\mathbf{x}, t) = u(r, t)$, $u^0(\mathbf{x}) = u^0(r)$. The initial value u^0 is assumed to be compactly supported and has total mass M, i.e.,

$$\int u^0(\mathbf{x})d\mathbf{x} = \omega_d \int_0^\infty r^{d-1}u^0(r)dr = M, \quad \mathrm{supp}(u^0) \subset B_L(0). \tag{23}$$

The Barenblatt solution can be written in the radial variable r which is

$$\rho(r, t) = t^{-d\alpha}(C_M - k(rt^{-\alpha})^2)_+^{1/(m-1)}, \tag{24}$$

where $\alpha = 1/(d(m-1) + 2) > 0$ and $k = \alpha(m-1)/(2m)$.

Lemma 2. *Let $u_1(r, t)$ and $u_2(r, t)$ be solutions to (22) with compactly supported initial values $u_1^0(r)$ and $u_2^0(r)$ of the same mass $M > 0$. Then the corresponding relative Newtonian potential,*

$$E(r, t) := -\int_r^\infty \Big(x^{1-d}\int_0^x y^{d-1}(u_1(y, t) - u_2(y, t))\,dy\Big)dx, \tag{25}$$

is well-defined for all $d \geq 1$, $m > (d-2)_+/d$, and $t > 0$. Furthermore,

$$\frac{\partial}{\partial t}E(r, t) = u_1^m(r, t) - u_2^m(r, t). \tag{26}$$

Proof. For the PME regime, $m > 1$, the solutions are compactly supported and the mass concentration $\mathcal{E}(r, t) := \omega_d \int_0^r y^{d-1}(u_1(y, t) - u_2(y, t))dy$ becomes identically zero for $r > 0$ large. Hence, this relative potential is well-defined for all dimensions for the PME regime. For the fast diffusion regime, $(d-2)_+/d < m < 1$, it is well known that the solution u_i has the same decay rate for r large as the one of the Barenblatt solution. Since $\mathcal{E}(r,t) \to 0$ as $r \to \infty$, we have

$$\begin{aligned} |\mathcal{E}(r,t)| &= \omega_d \Big| \int_0^r y^{d-1}(u_1(y,t) - u_2(y,t))dy \Big| \\ &= \omega_d \Big| \int_r^\infty y^{d-1}(u_1(y,t) - u_2(y,t))dy \Big| \\ &\le C\omega_d \Big| \int_r^\infty y^{d-1} y^{\frac{2}{m-1}} dy \Big| = O(r^{\frac{2+d(m-1)}{m-1}}) \end{aligned}$$

as $r \to \infty$. Since $\frac{2+d(m-1)}{m-1} - (d-2) = \frac{2m}{m-1} < 0$, the relative potential is well-defined. If $m = 1$, then it is the heat equation case and one may conclude the lemma easily using the exponential decay of the solution as $r \to \infty$.

A formal proof of (26) can be given as

$$\begin{aligned} E_t &= -\int_r^\infty \Big(x^{1-d} \int_0^x y^{d-1}(u_1 - u_2)_t \, dy \Big) dx \\ &= -\int_r^\infty \Big(x^{1-d} \int_0^x y^{d-1} \Big(y^{1-d}(y^{d-1}(u_1^m - u_2^m)_y)_y \Big) \, dy \Big) dx \\ &= u_1^m - u_2^m. \end{aligned}$$

For the PME case, $m > 1$, taking the derivative inside the integration is simple since the integration is on a compact set. For the FDE case with a dimension $d \ge 3$, this relation is the one with the original Newtonian potential, which was given in [19]. The FDE case with a dimension $d \le 2$ can be similarly dealt as in the proof of Proposition 10 in [19]. $\square$

It has been pointed out in [19] that the conservative regime of the FDE is exactly the limit to get the Newtonian potential well-defined. Therefore, it is no wonder that the well-definedness of the relative Newtonian potential was obtained without a delicate tail analysis. Since we always compare solutions with same initial mass $\omega_d \int_0^\infty r^{d-1} u^0(r) \, dr = M$, we have

$$\omega_d \int_0^\infty r^{d-1} u(r,t) \, dr = M \quad \text{for all } t > 0 \tag{27}$$

in the mass conservative regime. An application to the nonconservative regime will provide an example that the relative Newtonian potential really extends the potential theory.

Proposition 1 (Comparison Principle). *Let $u_1(r,t)$ and $u_2(r,t)$ be solutions to (22) with compactly supported initial values $u_1^0(r)$ and*

$u_2^0(r)$ of the same mass $M > 0$ and $E(r,t;u_1,u_2)$ be their relative Newtonian potential given by (25). If there exists $t_0 \ge 0$ such that

$$E(r,t_0;u_1,u_2) \ge 0 \quad \text{for all } r > 0$$

then

$$E(r,t;u_1,u_2) \ge 0 \quad \text{for all } r > 0,\ t \ge t_0.$$

Proof. Using the relations in (20) and (26) one may write

$$E_t = a(\mathbf{x},t)\Delta E, \qquad a(\mathbf{x},t) := (u_1^m - u_2^m)/(u_1 - u_2) \ge 0. \tag{28}$$

First consider the PME case $m > 1$. Then the relative potential E is also compactly supported for all $t > 0$. Hence for a fixed time $T > 0$, there exists a constant $C_T > 0$ such that $E(\mathbf{x},t) = 0$ for all $|\mathbf{x}| \ge C_T$ and $t_0 \le t \le T$. Therefore, the maximum principle and the assumption $E(\mathbf{x},t_0) \ge 0$ for all $|\mathbf{x}| > 0$ imply that $E(\mathbf{x},t) \ge 0$ for all $t_0 \le t \le T$ and $|\mathbf{x}| < C_T$. Since we can take T and C_T arbitrarily large, the proof is done for the PME case. For the FDE case, $(d-2)_+/d < m < 1$, the solutions u_1 and u_2 become strictly positive for all $t > 0$ and the equation (28) becomes uniformly parabolic. Hence the maximum principle on the unbounded domain $\mathbf{R}^d \times [t_0, T]$ concludes the proposition. □

Let $d \ge 3$. Then, the Newtonian potentials U_1 and U_2 of two solutions u_1 and u_2 given by (16) obviously satisfy

$$E(r,t;u_1,u_2) = U_1(r,t) - U_2(r,t).$$

Hence, the proposition implies that $U_1(r,t) \ge U_2(r,t)$ for all $r > 0$ and $t \ge t_0$ if $U_1(r,t_0) \ge U_2(r,t_0)$ for all $r > 0$. Roughly speaking, the next step is to sandwich the potential $U(r,t)$ of the solution $u(x,t)$ by proving

$$R(r,t) \le U(r,t) \le R(r,T+t), \tag{29}$$

where R is the potential of the Barenblatt solution $\rho(x,t)$. In the following lemma we show this estimate in terms of relative potentials for all dimensions $d \ge 1$.

Lemma 3. *Let $u(r,t)$ be the solution of (22) with compactly supported initial value and mass $M > 0$. Let $\rho(r,t)$ be the Barenblatt solution with the same mass. Then,*
(i) The relative Newtonian potential satisfies

$$E(r,t;\rho,u) \le 0, \quad r,t \ge 0.$$

(ii) There exists $T > 0$ such that

$$E(r,t;\rho^T,u) \ge 0, \quad r,t \ge 0,$$

where $\rho^T(r,t) := \rho(r,t+T)$.

Proof. (i) This estimate, which corresponds to the lower estimate in (29), comes from Theorem 1 and the comparison principle.

(ii) We claim that there exist $T > 0$ such that for all $r \geq 0$,

$$-E(r, 0; \rho^T, u) = \int_r^\infty \Big(x^{1-d} \int_0^x y^{d-1} (\rho(y,T) - u_0(y))\, dy \Big) dx \leq 0.$$

Let $\mathrm{supp}(u_0) \subset [0, L]$. Then, for $l \geq r \geq L$, it holds that

$$\begin{aligned} \int_r^l \Big(x^{1-d} \int_0^x y^{d-1} \rho(y,t)\, dy \Big) dx &\leq \int_r^l x^{1-d} \frac{M}{\omega_d}\, dx \\ &= \int_r^l \Big(x^{1-d} \int_0^x y^{d-1} u_0(y)\, dy \Big) dx. \end{aligned}$$

Hence the above claim holds for $r \geq L$ with any $T > 0$. Now consider the case $0 < r < L$ with $2L \leq l$. Let

$$\epsilon_0 := \int_L^{2L} \Big(x^{1-d} \int_0^x y^{d-1} u_0(y)\, dy \Big) dx = \frac{M}{\omega_d} \int_L^{2L} x^{1-d} dx.$$

Since the Barenblatt solution $\rho(y,t)$ converges to zero uniformly, there exists a large time $T > 0$ such that $\rho(y,t) \leq (d\epsilon_0)/(2L^2)$ for all $t \geq T$ and $y > 0$. Then, for $t \geq T$, $0 < r < L$ and $l \geq 2L$,

$$\begin{aligned} &\int_r^l \Big(x^{1-d} \int_0^x y^{d-1} \rho(y,t)\, dy \Big) dx \\ &\leq \int_0^{2L} \Big(x^{1-d} \int_0^x y^{d-1} \rho(y,t)\, dy \Big) dx + \int_{2L}^l \Big(x^{1-d} \int_0^x y^{d-1} \rho(y,t)\, dy \Big) dx \\ &\leq \epsilon_0 + \int_{2L}^l x^{1-d} \frac{M}{\omega_d}\, dx \quad \Big(\text{because } \rho(y,t) \leq \frac{d\epsilon_0}{2L^2} \Big) \\ &= \int_L^{2L} \Big(x^{1-d} \int_0^x y^{d-1} u_0(y)\, dy \Big) dx + \int_{2L}^l \Big(x^{1-d} \int_0^x y^{d-1} u_0(y)\, dy \Big) dx \\ &\leq \int_r^l \Big(x^{1-d} \int_0^x y^{d-1} u_0(y)\, dy \Big) dx. \end{aligned}$$

Hence the claim is proved. Finally the potential comparison principle gives $E(r, t; \rho^T, u) \geq 0$ for all $r, t \geq 0$. $\square$

Lemma 4. *Let $\rho(\mathbf{x}, t)$ be the Barenblatt solution given in (11). Then, there exists a constant $C > 0$ such that, for any given $t, T > 0$,*

$$\|\rho(t) - \rho(t+T)\|_1 \leq \frac{CT}{t}.$$

Proof. Since the Barenblatt solution is explicit, one may explicitly compute this contraction order. However, in the following, we show the lemma in a relatively general way. The Barenblatt solution in the radial variable is in the form of $\rho(r,t) = t^{-d\alpha} f(rt^{-\alpha})$ for $\alpha > 0$. then,

$$\rho_t(r,t) = -t^{-1}\Big(d\alpha t^{-d\alpha} f(rt^{-\alpha}) + \alpha t^{-d\alpha} f'(rt^{-\alpha}) rt^{-\alpha}\Big).$$

Therefore, using the similarity variable $\zeta = rt^{-\alpha}$, one can see that

$$\int_0^\infty r^{d-1}|\rho_t(r,t)|dr \le \frac{\alpha}{t}\int_0^\infty \zeta^{d-1}(df(\zeta) + \zeta|f'(\zeta)|)d\zeta = \frac{C}{\omega_d t},$$

where $C = \omega_d \alpha \int_0^\infty \zeta^{d-1}(df(\zeta) + \zeta|f'(\zeta)|)d\zeta$. Similarly, for all $s > t$, $\int_0^\infty r^{d-1}|\rho_t(r,s)|dr \le \frac{C}{\omega_d s} \le \frac{C}{\omega_d t}$. Finally, for any $T > 0$,

$$\begin{aligned}
\frac{1}{\omega_d}\|\rho(t) - \rho(t+T)\|_1 &= \int_0^\infty r^{d-1}\,|\rho(r,t) - \rho(r,t+T)|\,dr \\
&= \int_0^\infty r^{d-1}\left|\int_t^{t+T} \rho_t(r,s)\,ds\right| dr \\
&\le \int_t^{t+T}\int_0^\infty r^{d-1}|\rho_t(r,s)|\,dr\,ds \\
&\le \int_t^{t+T} \frac{C}{\omega_d t}\,ds = \frac{CT}{\omega_d t},
\end{aligned}$$

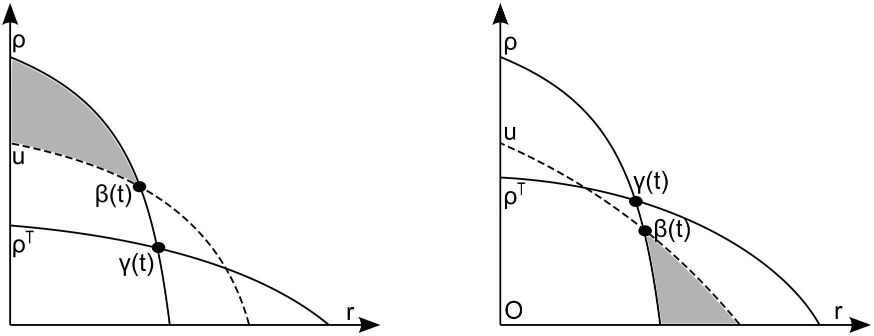
which completes the proof. □

The proof of Theorem 2 employs the well known zero-set theory (see [3,25]). For example, Theorem B in [3] says that the number of zeros of the solution e to

$$e_t = a(x,t)e_{xx} + b(x,t)e_x + c(x,t)e, \quad e(x,0) = e_0(x)$$

decreases in t if $a > 0$ and $a, a^{-1}, a_t, a_x, a_{xx}, b, b_x, c$ are bounded. Let u and v be radial solutions of a parabolic problem with radial symmetry. Then, one may view a intersection point between u and v as a zero point of $e = u - v$. In many cases the difference e or its regularized version can be written in the above form and one may obtain the decrease of number of intersection points. (For more details readers are referred to [18].)This property holds for the nonlinear diffusion equations such as the PME or PLE and it has been applied to obtain intermediate asymptotics (see Corollary 15.10 in [29]). Since the Barenblatt solution is a delta sequence as $t \to 0$, there exists exactly one intersection point between $\rho(r,t)$ and $u(r,t)$ for $r > 0$. In other words, there exists a unique point $r = \beta(t)$ such that

$$(0, \beta(t)) = \{r > 0 : \rho(r,t) > u(r,t)\}.$$



a) $\beta(t) < \gamma(t)$ b) $\gamma(t) < \beta(t)$

Fig. 2. These diagrams show the relation among ρ, u and ρ^T. For the FDE case the supports are unbounded.

We reserve the notation $\beta(t)$ for this unique intersection point between ρ and u.

First proof of Theorem 2 for PME and FDE. Let $T > 0$ be the one given in Lemma 3(ii). It is well-known that there exists a finite time $t_0 > 0$ such that the pressure $p(u(t_0)) = \frac{m}{m-1} u^{m-1}(t_0)$ becomes concave (see [4,21–23]). Hence, by taking larger $T > 0$ if needed, there exists a unique intersection point between $\rho(r, t_0 + T)$ and $u(r, t_0)$. Hence the zero set theory implies that $\rho(r, t)$, $u(r, t)$ and $\rho(r, t + T)$ intersect each other exactly once for all $t > t_0$. Let $\gamma(t)$ be the intersection point between $\rho(\cdot, t)$ and $\rho(\cdot, t+T)$. Now we show that, for $t > t_0$,

$$\rho(0, t + T) < u(0, t) < \rho(0, t). \tag{30}$$

First we show the first inequality $\rho(0, t+T) < u(0, t)$. If $\rho(0, t+T) = u(0, t)$ for for a time $t > t_0$, then $r = 0$ is the only intersection point between $u(t)$ and $\rho(t + T)$. Since $\rho(t + T)$ and $u(t)$ are non-negative functions that share the same mass, $\rho(t+T)$ and $u(t)$ should be identical, which is a contradiction. Suppose that $\rho(0, t + T) > u(0, t)$ for a time $t > t_0$ and $\alpha(t)$ is the unique intersection point. Then, after the intersection point, $x > \alpha(t)$, the order is reversed, i.e., $\rho(x, t + T) < u(x, t)$. Therefore, for $r > \alpha(t)$,

$$\begin{aligned} E(r, t; \rho^T, u) &= -\int_r^\infty \Big(x^{1-d} \int_0^x y^{d-1} (\rho(y, t + T) - u(y, t))\, dy \Big) dx \\ &= \int_r^\infty \Big(x^{1-d} \int_x^\infty y^{d-1} (\rho(y, t + T) - u(y, t))\, dy \Big) dx < 0, \end{aligned}$$

which contradicts Lemma 3(ii). Therefore, we have $\rho(0, t + T) < u(0, t)$. The other inequality in (30) is also similarly obtained.

Now suppose that $\beta(t) \le \gamma(t)$. One may see that (c.f., Figure 2a)

$$\int_0^{\beta(t)} r^{d-1}|\rho(r,t)-u(r,t)|dr \le \int_0^{\gamma(t)} r^{d-1}|\rho(r,t)-\rho(r,t+T)|dr.$$

Since u, ρ and ρ^T share the same mass, the positive and the negative mass of their differences should be identical. Furthermore, since they intersect to each other at a single point, we have

$$\begin{aligned} \|u(t)-\rho(t)\|_1 &= 2\omega_d \int_0^{\beta(t)} r^{d-1}|u(r,t)-\rho(r,t)|dr, \\ \|\rho(t)-\rho^T(t)\|_1 &= 2\omega_d \int_0^{\gamma(t)} r^{d-1}|\rho(r,t)-\rho(r,t+T)|dr. \end{aligned}$$

Therefore, by Lemma 4,

$$\|u(t)-\rho(t)\|_1 \le \|\rho(t)-\rho(t+T)\|_1 = O(t^{-1}) \quad \text{as} \quad t \to \infty. \tag{31}$$

Now suppose that $\gamma(t) \le \beta(t)$. One may similarly see that (c.f., Figure 2b)

$$\int_{\beta(t)}^{\infty} r^{d-1}|\rho(r,t)-u(r,t)|dr \le \int_{\gamma(t)}^{\infty} r^{d-1}|\rho(r,t)-\rho(r,t+T)|dr.$$

Since

$$\begin{aligned} \|u(t)-\rho(t)\|_1 &= 2\omega_d \int_{\beta(t)}^{\infty} r^{d-1}|u(r,t)-\rho(r,t)|dr, \\ \|\rho(t)-\rho^T(t)\|_1 &= 2\omega_d \int_{\gamma(t)}^{\infty} r^{d-1}|\rho(r,t)-\rho(r,t+T)|dr, \end{aligned}$$

one obtains (31) again. □

In the following we provide another proof which is based on a scaling method that has been used to find intermediate asymptotics for various cases (see sections 3 and 4 of [31]). This method has been used for the FDE case in [19]. Hence we consider the PME case in the following. From the explicit formula of the Barenblatt solution in (11), one may easily observe the following.

Lemma 5. *The Barenblatt solution ρ and the intersection point $\beta(t)$ between ρ and a L^1-solution u satisfy that*

$$\begin{aligned} &\|\rho^m(\cdot,t)\|_\infty = O\left(t^{-dm\alpha}\right) \ \textit{as}\ t\to\infty, \\ &\beta(t) \le \zeta(t) = O\left(t^{\alpha}\right) \ \textit{as}\ t\to\infty, \end{aligned}$$

where $\alpha := \frac{1}{d(m-1)+2}$ and $\zeta(t)>0$ denotes the positive interface of the Barenblatt solution, that is, $\zeta(t) := \sup\{r>0 : \rho(r,t)>0\}$.

Proposition 2 (L^∞-distance between potentials). *Let $u(r,t)$ be the solution of (22) with a compactly supported initial value and a finite total mass $M>0$. Let $\rho(r,t)$ be the Barenblatt solution of the same mass. Then, the relative potential has the order*

$$\|E(\cdot,t;u,\rho)\|_\infty = O\left(t^{-dm\alpha}\right) \quad \textit{as } t\to\infty.$$

Proof. Let $T>0$ be the one in Lemma 3(ii). For any $r,t\ge 0$,

$$E(r,t;\rho^T,\rho) = E(r,t;\rho^T,u) + E(r,t;u,\rho) \ge E(r,t;u,\rho) \ge 0.$$

Therefore, we have

$$\begin{aligned}
0 &\le E(r,t;u,\rho) \le E(r,t;\rho^T,\rho)\\
&= \left| \int_r^\infty \left(x^{1-d} \int_0^x y^{d-1}(\rho(y,t)-\rho(y,t+T))\,dy\right) dx \right|\\
&= \left| \int_r^\infty \left(x^{1-d} \int_0^x y^{d-1} T\rho_t(y,\tau(y))\,dy\right) dx \right|\\
&\le \sup_{\tau\in(t,t+T)} T|\rho^m(r,\tau)| \le T\|\rho^m(t)\|_\infty,
\end{aligned}$$

where $\tau=\tau(y)\in(t,t+T)$. Using Lemma 5, we conclude that

$$\|E(\cdot,t;u,\rho)\|_\infty \le T\|\rho^m(t)\|_\infty = O\left(t^{-dm\alpha}\right) \quad \text{as } t\to\infty.$$

□

Now we are ready to provide the second proof of Theorem 2 for the solutions of PME by translating the above uniform estimate of the relative potential to the required L^1-convergence order. The same proof was given for the FDE case in [19]. We now apply the technique to the PME case. To complete the mission we need an additional information on the intersection point $\beta(t)$, which is

$$\frac{\beta(t)}{t^\alpha} \to \sqrt{2C_M md} \quad \text{as } t\to\infty.$$

This kind of estimate of intersection hypersurface is of independent interest (see the first author's Ph.D. thesis [10]). Note that the tail analysis was enough for the FDE case in [19] since the solution is positive everywhere.

Second proof of Theorem 2 for PME. First a family of rescaled solutions is introduced:

$$u^\lambda(r,t) := \lambda^{d\alpha} u(\lambda^\alpha r, \lambda t), \quad \lambda>0.$$

The Barenblatt solution is unchanged by this scaling, i.e., $\rho = \rho^\lambda$. Then changes of variables yield that

$$E(r,t;\rho^\lambda,u^\lambda) = \lambda^{(d-2)\alpha} E(\lambda^\alpha r, \lambda t; \rho, u).$$

Hence by Proposition 2,

$$\begin{aligned}\|E(r,t;\rho^\lambda,u^\lambda)\|_\infty &= O\left(\lambda^{(d-2)\alpha}\lambda^{-dm\alpha}t^{-dm\alpha}\right)\\ &= O(\lambda^{-1}t^{-dm\alpha}).\end{aligned}$$

This verifies that $\lambda E(r,t;\rho^\lambda,u^\lambda)$ is uniformly bounded on λ. Therefore, by virtue of an a priori estimate (minutely explained in [19]), their Laplacians are uniformly bounded in a region in which the Barenblatt solution ρ is strictly positive:

$$|\Delta\lambda E(r,t;\rho^\lambda,u^\lambda)| = |\lambda(u^\lambda-\rho)(r,t)| \le C_K, \quad \text{for } \lambda>0,\ |r|\le K.$$

When $t\approx 1$, K is a fixed constant which is strictly smaller than $\sqrt{C_M/k}$; this condition assures that the Barenblatt solution ρ is strictly positive in the region $|r|\le K$. Fixing $t=1$ and replacing λ by λt in the above inequality, we obtain

$$\begin{aligned}C_K &\ge \lambda t|(u^{\lambda t}-\rho)(r,1)|\\ &= \lambda t\cdot t^{d\alpha}|(u^\lambda-\rho)(t^\alpha r,t)| \quad \text{for all } \lambda>0 \text{ and } |r|\le K.\end{aligned}$$

On the other hand, since

$$\rho(t^\alpha K,t) = t^{-d\alpha}(C_M - kK^2)_+^{1/(m-1)},$$

we have

$$\begin{aligned}|(u^\lambda-\rho)(t^\alpha r,t)| &\le \frac{C_K}{\lambda t}t^{-d\alpha} \le \frac{\tilde C_K}{\lambda t}\rho(t^\alpha K,t)\\ &\le \frac{\tilde C_K}{\lambda t}\rho(t^\alpha r,t) \quad \text{for all } |r|\le K\end{aligned}$$

for some constant $\tilde C_K$ which depends only on K. Substituting $\lambda=1$, we can deduce the inequality

$$|(u-\rho)(r,t)| \le \frac{\tilde C_K}{t}\rho(r,t) \quad \text{for all } \frac{|r|}{t^\alpha}\le K.$$

Choose a constant K satisfying

$$\sqrt{2C_M md} < K < \sqrt{C_M/k} = \sqrt{2C_M m\frac{d(m-1)+2}{m-1}}.$$

Assume $\beta(t)/t^\alpha$ is strictly smaller than K in a finite time so that

$$|(u-\rho)(r,t)| \le \frac{\tilde{C}_K}{t}\rho(r,t) \quad \text{for all } |r| \le \beta(t).$$

Then integration of the above inequality gives us

$$\begin{aligned} \|\rho(\cdot,t) - u(\cdot,t)\|_1 &= 2\omega_d \int_0^{\beta(t)} r^{d-1}(\rho - u)(r,t)\,dr \\ &\le \frac{\tilde{C}_K}{t} \cdot 2\omega_d \int_0^{\beta(t)} r^{d-1}\rho(r,t)\,dr \\ &\le \frac{2\tilde{C}_K M}{t} = O(1/t) \quad \text{as} \quad t \to \infty, \end{aligned}$$

which completes the proof. $\square$

4. The p-Laplacian Equation

In this section we show Theorem 2 for the solutions to the PLE given in (12). Since the solution $u(\mathbf{x},t)$ and the initial value $u^0(\mathbf{x})$ are radially symmetric, we may rewrite the problem as

$$u_t = r^{1-d}(r^{d-1}|u_r|^{p-2}u_r)_r,\ u(r,0) = u^0(r) \ge 0,\ u_r(0,t) = 0,\ r,t > 0. \tag{32}$$

The initial value $u^0 \ge 0$ is assumed to be compactly supported. The proof is based on the potential comparison technique. First observe that the radial PLE (32) is easily transformed to the radial PME (22) for the case $d = 1$. Let $\nu := u_r$. Then, after differentiating (32) with respect to r once, one obtains

$$\nu_t = (\mathrm{sign}(\nu)|\nu|^{p-1})_{rr},\ \nu(x,0) = \partial_r(u^0(r)),\ \nu(0,t) = 0,\ r,t > 0. \tag{33}$$

In other words, the PME and PLE has an equivalence relation for the one space dimension given by

$$\nu = u_r, \quad m = p - 1. \tag{34}$$

It seems that this equivalence relation is of independent interest.

Note that, for the case of dimension one, the Newtonian potential of $\nu := u_r$ is simply the antiderivative of the solution u, which gives the mass concentration of the solution. (This antiderivative successfully played the role of a potential for a convection problem in [16, 17].) In fact, for all dimensions $d \ge 1$, we take the mass concentration,

$$\mathcal{U}(r,t) := \omega_d \int_0^r x^{d-1} u(x,t)\,dx, \quad r,t \ge 0,$$

in the place of the Newtonian potential. Since we are considering L^1-solutions, the concept of relative potential is not needed. However, for a situation without integrability, the relative mass in (18) can be useful. In any case, one can see that only the mass difference plays a role in the following asymptotic analysis.

The Barenblatt-type solution $\rho(r,t)$ of the PLE given in (13) can be written in the radial variable,

$$\rho(r,t) = t^{-d\alpha}(C_M - k(rt^{-\alpha})^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}, \tag{35}$$

where $\alpha = \frac{1}{d(p-2)+p}$, $k = \frac{p-2}{p}\alpha^{1/(p-1)}$. Let $\mathcal{R}(r,t)$ be the mass concentration of $\rho(r,t)$. Then the mass conservation gives

$$M = \lim_{r\to\infty}\mathcal{U}(r,t) = \lim_{r\to\infty}\mathcal{R}(r,t), \quad t>0. \tag{36}$$

Proposition 3 (Comparison Principle). *Let u_1 and u_2 be two bounded solutions of the radial p-Laplacian equation (32) and $\mathcal{U}_1$ and $\mathcal{U}_2$ be their mass concentrations, respectively. If there exists $t_0>0$ such that*

$$\mathcal{U}_1(r,t_0) - \mathcal{U}_2(r,t_0) \ge 0 \quad \text{for all } r>0,$$

then, if $t>t_0$,

$$\mathcal{U}_1(r,t) - \mathcal{U}_2(r,t) \ge 0 \quad \text{for all } r>0.$$

Proof. Let $\mathcal{E} := \mathcal{U}_1 - \mathcal{U}_2$ be the relative mass concentration. Then, the initial condition gives $\mathcal{E}(r,t_0)\ge 0$ for all $r>0$, and we will show $\mathcal{E}(r,t)\ge 0$ for all $r>0$ if $t>t_0$. Consider the relations

$$\begin{aligned}
\mathcal{U}_t &= \omega_d r^{d-1}|u_r|^{p-2}u_r,\\
\omega_d u_r &= \frac{1}{r^{d-1}}\mathcal{U}_{rr} - \frac{d-1}{r^d}\mathcal{U}_r,\\
\mathcal{E}_t &= a(r,t)\mathcal{E}_{rr} - \frac{(d-1)a(r,t)}{r}\mathcal{E}_r,
\end{aligned}$$

where

$$a(r,t) := \frac{|u_{1r}|^{p-2}u_{1r} - |u_{2r}|^{p-2}u_{2r}}{u_{1r}-u_{2r}} \ge 0 \ \text{ for } \ p>1.$$

If $\frac{2d}{d+1} < p < 2$, the solutions are supported on the whole space and the diffusion is non-degenerate. Hence the maximum principle gives $\mathcal{E}(r,t)\ge 0$ for all $r>0$ if $t>t_0$. For $p>2$, the solutions are compactly supported and the diffusion is degenerate at the zero points. Then, for a fixed large time $T>t_0$, there exists a radius C_T such that $\mathcal{E}(r,t)=0$ for all $r\ge C_T$ and $t_0\le t\le T$. So we can apply the maximum principle in $[1/n, C_T]\times[t_0,T]$ for any $n>0$. Assume $\mathcal{E}$ has a negative value, say $-\epsilon<0$, in $\{x_0\}\times[t_0,T]$. Then the minimum in $\{1/n\} \times [t_0, T]$ for all $1/n < x_0$ must be less than or equal to $-\epsilon$ by the maximum principle. However, this contradicts the facts $\mathcal{E}(0,t) = 0$ for $t_0 \le t \le T$ and $\mathcal{E}$ is continuous. Therefore, $\mathcal{E}(r,t) \ge 0$ for all $r > 0$ if $t > t_0$. □

Now we sandwich the mass concentration of a solution between those of the Barenblatt solution and of its time delay.

Lemma 6 (Sandwiched). *There exist $t_0, T > 0$ such that*

$$\mathcal{R}(r, t+T) \le \mathcal{U}(r,t) \le \mathcal{R}(r,t), \quad r > 0,\ t > t_0. \tag{37}$$

Proof. We first check the second inequality. From the explicit formula, we can easily verify that $\rho(x,t) = t^{-d\alpha}\rho(xt^{-\alpha}, 1)$ for any $x \ge 0$ and $t \ge 0$. Hence for any $r \ge 0$, we have

$$\begin{aligned}\mathcal{R}(r,t) &= \omega_d \int_0^r x^{d-1}\rho(x,t)\,dx \\ &= \omega_d \int_0^r (xt^{-\alpha})^{d-1}\rho(xt^{-\alpha},1)\,t^{-\alpha}dx \\ &= \omega_d \int_0^{rt^{-\alpha}} y^{d-1}\rho(y,1)\,dy = \mathcal{R}(rt^{-\alpha},1).\end{aligned}$$

Therefore for any $r > 0$, it holds that

$$\lim_{t\to 0+} \mathcal{R}(r,t) = \lim_{t\to 0+} \mathcal{R}(rt^{-\alpha},1) = M \ge \mathcal{U}(r,0),$$

which implies $\mathcal{R}(r,0) \ge \mathcal{U}(r,0)$ for every $r \ge 0$. The second inequality follows from Proposition 3, the comparison principle.

Since $\|u(t) - \rho(t)\|_\infty \to 0$ as $t \to \infty$ and the solution u becomes strictly positive at the origin in a finite time, there exist positive numbers t_0, ϵ and δ such that

$$u(r,t_0) \ge \epsilon > 0 \quad \text{for all } |r| \le \delta.$$

Let $L(t)$ be the outside boundary of the support of the solution u at time t, that is,

$$L(t) := \sup(\mathrm{supp}(u(\cdot,t))).$$

Then obviously $L(t_0) \ge \delta$. Now pick a number $T > 0$ such that $\epsilon(\delta/L(t_0))^d \ge \rho(r, t_0+T)$ for all $r \in \mathbf{R}$. In this setting we consider three different cases according to the intervals.

1. If $|r| \le \delta$, $\mathcal{U}(r,t_0) \ge \omega_d \int_0^{|r|} x^{d-1}\epsilon(\delta/L(t_0))^d\,dx \ge \mathcal{R}(r, t_0+T)$.
2. If $\delta \le |r| \le L(t_0)$, then

$$\begin{aligned}\mathcal{U}(r,t_0) &\ge \mathcal{U}(\delta,t_0) \ge \omega_d \frac{\epsilon}{d}\delta^d \\ &= \omega_d \int_0^{L(t_0)} x^{d-1}\epsilon\left(\frac{\delta}{L(t_0)}\right)^d dx \ge \mathcal{R}(r,t_0+T)\end{aligned}$$

3. If $|r| \geq L(t_0)$, then $\mathcal{U}(r,t_0) = M \geq \mathcal{R}(r,t_0+T)$.

Therefore, $\mathcal{R}(r,t_0+T) \leq \mathcal{U}(r,t_0)$. Hence Proposition 3 gives the rest of the proof. □

Lemma 7. *The fundamental solution ρ of PLE satisfies*

$$\|r^{d-1}|\rho_r|^{p-1}\|_\infty = O(t^{-1}) \quad \text{as} \quad t \to \infty.$$

Proof. The Barenblatt-type solution is

$$\rho(r,t) = t^{-d\alpha}(C_M - k(rt^{-\alpha})^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}},$$

where $\alpha = \frac{1}{d(p-2)+p}$, $k = \frac{p-2}{p}\alpha^{1/(p-1)}$, and C_M is the positive constant that sets the total mass to the solution to be $M > 0$. Consider the case $p > 2$. Then, $|r|^{d-1}|\rho_r|^{p-1}$ has the maximum at the interface of ρ and the interface is of order t^α. Therefore, it suffices to consider

$$\bar{\rho}(r,t) := t^{-d\alpha}(rt^{-\alpha})^{\frac{p}{p-2}}$$

to obtain the order of $\|r^{d-1}|\rho_r|^{p-1}\|_\infty$. Now compute

$$\begin{aligned} r^{d-1}|\bar{\rho}_r|^{p-1}\Big|_{r=t^\alpha} &= t^{(d-1)\alpha}\left(\frac{p}{p-2}t^{-d\alpha-\frac{p\alpha}{p-2}+\frac{2\alpha}{p-2}}\right)^{p-1} \\ &= \left(\frac{p}{p-2}\right)^{p-1} t^{(d-1)\alpha}\cdot t^{-(d+1)(p-1)\alpha} \\ &= O(t^{-1}) \quad \text{as} \quad t \to \infty. \end{aligned}$$

If $p < 2$, then the Barenblatt-type solution is differentiable everywhere and one may easily obtain the same result. □

Proof of Theorem 2 for PLE. We have obtained from Lemma 6 that there exist $T > t_0 > 0$ such that

$$\mathcal{R}(r,t+T) \leq \mathcal{U}(r,t) \leq \mathcal{R}(r,t) \quad \text{for all } r > 0,\ t \geq t_0.$$

Therefore for all $t \geq t_0$, we have

$$\begin{aligned} |\mathcal{R}(r,t) - \mathcal{U}(r,t)| &\leq |\mathcal{R}(r,t) - \mathcal{R}(r,t+T)| \\ &\leq \sup_{c\in(t,t+T)} T|\mathcal{R}_t(r,c)| \\ &\leq T\omega_d\|r^{d-1}|\rho_r(r,t)|^{p-1}\|_\infty. \end{aligned}$$

Hence $\|\mathcal{R}(\cdot,t) - \mathcal{U}(\cdot,t)\|_\infty \leq \omega_d T\|r^{d-1}|\rho_r|^{p-1}\|_\infty = O(1/t)$ as $t \to \infty$. Finally by virtue of the fact that the Barenblatt-type solution ρ and a solution u have the only one positive intersection point $\beta(t)$, we can translate this L^∞-distance between mass concentration to the required L^1-convergence order:

$$\begin{aligned}\frac{1}{\omega_d}\|\rho(t)-u(t)\|_1 &= \int_0^\infty r^{d-1}|\rho(r,t)-u(r,t)|\,dr\\ &= 2\int_0^{\beta(t)} r^{d-1}(\rho(r,t)-u(r,t))\,dr\\ &= \frac{2}{\omega_d}\|\mathcal{R}(\cdot,t)-\mathcal{U}(\cdot,t)\|_\infty\\ &= O(1/t) \quad \text{as} \quad t\to\infty,\end{aligned}$$

which completes the proof. □

Epilogue

YJK presented one dimensional results related to this paper in a BIRS workshop which was held April 15-20, 2006. J.L. Vázquez pointed out during the talk that an extension to a radial case would not be simple due to its geometric property. This paper is in fact about such an extension. Vázquez himself also subsequently submitted [14] a radial version of the equivalence relation (32)-(34), where this one dimensional version was also presented in the talk. YJK would like to thank Banff International Research Station for hosting the workshop and participants of the workshop for valuable comments. Authors would like to thank anonymous reviewers, whose comments improved this presentation considerably.

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