ADDENDUM TO "RELATIVE NEWTONIAN POTENTIALS OF RADIAL FUNCTIONS AND ASYMPTOTICS OF NONLINEAR DIFFUSION"*

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Abstract. Theorem 9.7 of the Lieb and Loss [2] is an extended version of Newton's theorem and was cited in the authors' previously published paper [1]. However, the statement of this theorem is incorrect for dimensions $d \le 2$. A couple of comments and, in particular, Figure 1 in the authors' paper [1] are based on this theorem and are incorrect because of this reason. In this note we show another extended version of Newton's theorem and provide corrected figures. This correction makes the arguments in the original paper more general.

Key words. Newtonian potential, Newton's theorem, radial symmetry, relative potential

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The fundamental solution of Laplace's equation in $\mathbf{R}^d$ is

$$\Phi(\mathbf{x}) := \begin{cases} \dfrac{-1}{(d-2)\omega_d}|\mathbf{x}|^{2-d}, & d \ge 3, \\ \dfrac{1}{\omega_d}\ln|\mathbf{x}|, & d = 2, \\ \dfrac{1}{2}|\mathbf{x}|, & d = 1, \end{cases} \tag{1}$$

where $\omega_d := 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbf{R}^d$. The Newtonian potential of a Radon measure $v(\mathbf{x})$ is given by the convolution

$$V(\mathbf{x}) := (\Phi * v)(\mathbf{x}),$$

which solves Poisson's equation $\Delta V = v$. The fundamental solution Φ also satisfies $\Delta\Phi = \delta$, where δ is the Dirac delta distribution. The following theorem is an extended version of Newton's theorem on radial potentials.

THEOREM 1. *Let $d \ge 1$ and $v = v^+ - v^-$ be a radial Radon measure in $\mathbf{R}^d$ with $v^\pm \ge 0$. Let $M^\pm = \int v^\pm(\mathbf{x})d\mathbf{x}$, $M = \int v(\mathbf{x})d\mathbf{x} = M^+ - M^-$, and*

$$|v(\mathbf{x})| = O(|\mathbf{x}|^{-\gamma}),\ \gamma > 2, \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{2}$$

(i) Let $B_L(\mathbf{x})$ be the ball of radius $L > 0$ centered at $\mathbf{x}$. If $\operatorname{supp}(v) \subset B_L(0)$,

$$M\Phi(\mathbf{x}) = V(\mathbf{x}) \quad \text{for all} \quad |\mathbf{x}| \ge L. \tag{3}$$

(ii) If $v \ge 0$ (i.e., $v^- = 0$, but not necessarily compactly supported), then

$$M\Phi(\mathbf{x}) \le V(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbf{R}^d. \tag{4}$$

(iii) If $d \ge 3$ and v is not necessarily nonnegative, then

$$M^+\Phi(\mathbf{x}) \le V(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbf{R}^d. \tag{5}$$

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Proof. The proof for the first part (i) is a direct computation. Let $\mathrm{supp}(v) \subset B_L(0)$ for $L > 0$. Then, since v is radial and Φ is harmonic for $\mathbf{x} \neq 0$, we have, for $|\mathbf{x}| > L$,

$$\begin{aligned} V(\mathbf{x}) &= \int_{B_L(\mathbf{x})} \Phi(\mathbf{y}) v(\mathbf{x}-\mathbf{y}) d\mathbf{y} = \int_0^L \Big(\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{y}) v(\mathbf{x}-\mathbf{y}) dS(\mathbf{y}) \Big) dr \\ &= \Phi(\mathbf{x}) \int_0^L \Big(\int_{\partial B_r(\mathbf{x})} v(\mathbf{x}-\mathbf{y}) dS(\mathbf{y}) \Big) dr = \Phi(\mathbf{x}) \int_{B_L(0)} v(\mathbf{y}) d\mathbf{y} = M\Phi(\mathbf{x}). \end{aligned}$$

Here, the first equality depends on the assumption $\mathrm{supp}(v) \subset B_L(0)$ and the third one on three facts that Φ is harmonic on $B_L(\mathbf{x})$, $v(\mathbf{x} - \cdot)$ is constant on $\partial B_r(\mathbf{x})$ and the mean value property. Therefore, the first part (i) holds.

Now we show part (ii). To do this we will first introduce a relative Newtonian potential. For the notational convenience, we will abuse notations by writing $\delta = \delta(r)$ and $v = v(r)$ with $r = |\mathbf{x}|$. Since $M = \int v(\mathbf{x}) d\mathbf{x}$ and v decays with order (2), the relative potential

$$E(r; M\delta, v) := \int_\infty^r z^{1-d} f(z) dz, \quad f(z) := \int_0^z r^{d-1} \big(M\delta(r) - v(r) \big) dr \tag{6}$$

is well defined. Then, $E(r; M\delta, v) \to 0$ as $r \to \infty$ and

$$\Delta E = r^{1-d} (r^{d-1} E'(r))' = r^{1-d} (f(r))' = M\delta - v = \Delta(M\Phi - V).$$

Therefore, the difference $G := E - (M\Phi - V)$ satisfies $\Delta G = 0$ on $\mathbf{R}^d$, i.e., G is an entire harmonic function.

The positivity assumption $v \geq 0$ implies that $f(z)$ in (6) decreases as $z \to \infty$ since only the negative part of the domain is added as $z \to \infty$. It is also clear that $f(z)$ converges to 0 as $z \to \infty$ since $M = \int v(\mathbf{x}) d\mathbf{x}$. Therefore, $f \geq 0$ and hence $E \leq 0$. The next step is to show the following equality.

$$M\Phi(\mathbf{x}) - V(\mathbf{x}) = E(|\mathbf{x}|; M\delta, v), \quad \mathbf{x} \in \mathbf{R}^d. \tag{7}$$

Then, the previous estimate $E \leq 0$ implies that $M\Phi \leq V$ for all $\mathbf{x} \in \mathbf{R}^d$ and the proof for (ii) is completed. First consider the case that $\mathrm{supp}(v) \subset B_L(0)$. Then, by part (i), $M\Phi(\mathbf{x}) = V(\mathbf{x})$ for all $|\mathbf{x}| > L$. Since $E(\mathbf{x}) = 0$ for $|\mathbf{x}| > L$, the difference $G(\mathbf{x}) = E(\mathbf{x}) - (M\Phi(\mathbf{x}) - V(\mathbf{x})) = 0$ for $|\mathbf{x}| > L$. Therefore, the maximum principle implies that $G(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^d$. Therefore, (7) holds if v is compactly supported.

Now consider a case that v is not compactly supported. For $\epsilon > 0$ small there exists $L_\epsilon > 0$ such that $M - \epsilon = \int_{B_{L_\epsilon}(0)} v(\mathbf{y}) d\mathbf{y}$. Let $V_\epsilon := \Phi * (v \chi_{B_{L_\epsilon}(0)})$, where $\chi_{B_{L_\epsilon}(0)}$ is the usual characteristic function. Then, $V_\epsilon \to V$ monotonously as $\epsilon \to 0$ and we have already shown for compactly supported measures that, for all $\mathbf{x} \in \mathbf{R}^d$,

$$(M - \epsilon)\Phi(\mathbf{x}) \leq V_\epsilon(\mathbf{x}).$$

Therefore, by taking the limit, $\epsilon \to 0$, we have (4), which completes the part (ii).

Finally, the last part (iii) comes from the following computation,

$$V = \Phi * (v^+ - v^-) = \Phi * v^+ - \Phi * v^- \geq \Phi * v^+ \geq M^+ \Phi.$$

Here, the second equality is the linearity of the convolution and the last inequality is from the second part (ii), which hold for all dimensions. However, the other inequality holds for $d \geq 3$ due to the sign of Φ. ∎

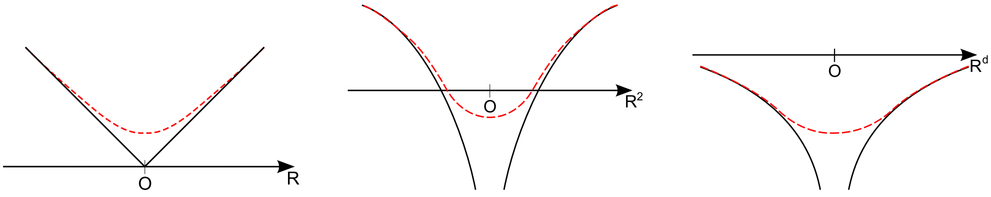

FIG. 1. *These diagrams show the relation between the fundamental solution $M\Phi$ (solid lines), and the Newtonian potential V of a radial Radon measure $v \geq 0$ with mass $M > 0$ (dashed lines) given by Theorem 1. One can see that $V > 0$ in dimension $d = 1$ and $V < 0$ in dimension $d \geq 3$. For dimension $d = 2$, $V \geq 0$ or V may have sign changes as in the figure.*

The original Newton's theorem is the relation (3) for the case that the measure v is radial symmetric, nonnegative, compactly supported, and in dimension $d \geq 3$. Theorem 1 extends it to measures with unbounded supports and sign changes, to a relation including the interior of the support, and to dimensions $d \geq 1$. However, the relation (5) holds for dimensions $d \geq 3$ if sign changing measures are included. For dimensions $d \leq 2$, such an estimate is not possible for sign changing measures. For example, let $\text{supp}(v) \subset B_L(0)$ and $M^+ = M^-$. Then, by (3),

$$V(\mathbf{x}) = M\Phi(\mathbf{x}) = 0 \quad \text{for all} \quad |\mathbf{x}| > L.$$

Since $\Phi(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ in dimensions $d \leq 2$, it is clear that a relation such as (5) is not possible for dimensions $d \leq 2$.

One can find a similar extended version of Newton's theorem from Lieb and Loss [2, Theorem 9.7]. It claims that, for $d \geq 1$,

$$|V(\mathbf{x})| \leq (M^+ + M^-)|\Phi(\mathbf{x})| \quad \text{for all} \quad \mathbf{x} \in \mathbf{R}^d. \tag{8}$$

For $d \geq 3$, the lower bound on V(x) given in the estimate (5) is finer than that of (8). However, for dimensions $d \leq 2$, this inequality does not hold. For example, if $d = 2$ and $|\mathbf{x}_0| = 1$, or if $d = 1$ and $|\mathbf{x}_0| = 0$, then, according to (8),

$$\Phi * v(\mathbf{x}_0) = V(\mathbf{x}_0) = 0 \tag{9}$$

for any integrable v. On the other hand, one can easily see that the equality (9) does not hold in general as is seen in the following example. Let $v^r := \frac{1}{|B(0,r)|}\chi_{B(0,r)}$ be a characteristic function. Then, $M = M^+ = 1$ and $M^- = 0$. Then $(\Phi * v^r)(\mathbf{x}_0)$ is just the average of Φ in the ball of radius r and centered at $\mathbf{x}_0$. If $|\mathbf{x}_0| = 0$ or 1 and $r > 2$, then $(\Phi * v^r)(\mathbf{x}_0)$ is strictly increasing as a function of r, $r > 2$, since the support of v^r includes more positive part of Φ as $r > 2$ increases. Therefore, $(\Phi * v^r)(\mathbf{x}_0)$ can not be zero for all $r > 0$, which contradicts (9).

The relation (4) gives three different scenarios due to the signs of the Newtonian potential V and the fundamental solution Φ, which are given in Figure 1. However, in the authors' previously published paper, similar figures, [1, Figure 1], were given according to the relation (8) which are incorrect for dimensions $d = 1, 2$. Hence those figures should be replaced by the ones in this note. The equality in (7) was claimed only for $d \geq 3$ in the paper (Theorem 1.1 (9)) due to the disagreement to Theorem 9.7 of [2], which was believed to be correct at the time. However, we have just shown that it holds for all dimensions $d \geq 1$. Notice that the positivity of v is related to the relation $E \leq 0$ and (4), but not to the equality in (7).

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