COMPARISON PRINCIPLES AND ASYMPTOTICS IN A GENERAL DIFFUSION EQUATION.

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Abstract. A potential comparison technique has been developed for solutions to a nonlinear diffusion equation and the optimal asymptotic convergence order $O(1/t)$ has been obtained for a fast diffusion case [9]. This method has been applied to a conservation law without convexity after a suitable modification and the same asymptotic convergence order has been shown [6, 8]. In this paper we show that the potential comparison property collaborates with the steepness of a fundamental solution to produce the convergence order. It seems that these two properties are the essential features that gives the asymptotic convergence order. In this paper we obtain optimal contraction orders using these properties and show these properties in various cases.

1. Introduction

Let $u(x,t)$ be a positive solution to

$$u_t = \frac{\partial}{\partial x}\sigma(u,u_x), \quad u(x,0)=u_0(x)\ge 0, \quad t>0,\, x\in\mathbf{R}, \tag{1.1}$$

where the initial value is assumed to be positive $u_0\ge 0$, compactly supported and integrable. We simply assume that

$$\operatorname{spt}(u_0)\subset[0,L] \quad\text{and}\quad \int u_0(x)dx = M>0. \tag{1.2}$$

2. Comparison Principles

A primitive of a solution $u(x,t)$ is considered as a potential, which is denoted by

$$U(x,t)=\int_{-\infty}^x u(y,t)dy. \tag{2.1}$$

Then integrating the equation (1.1) over $(-\infty,x)\times(t_0,t)$ gives

$$U(x,t)=U(x,t_0)-\int_{t_0}^t \sigma(u(x,s),u_x(x,s))ds. \tag{2.2}$$

Let $v(x,t)$ be another positive solution with a different initial value $v_0(x)\ge 0$ that satisfies the same restrictions as in (1.2). Similarly, let $V(x,t)$ be its potential. We consider two kinds of comparison principles related to the solutions of (1.1). The first comparison property is between two potentials:

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Hypothesis 2.1 (Potential comparison principle)**.** *If there exist* $t_1, t_2 \geq 0$ *such that* $U(x,t_1) \leq V(x,t_2)$ *for all* $x \in \mathbf{R}$*, then for all* $t > 0$ *and* $x \in \mathbf{R}$ *we have*

$$U(x, t_1 + t) \leq V(x, t_2 + t). \tag{2.3}$$

The second comparison principle is about the number of sign changes of the difference between two solutions $e(x,t) := u(x,t) - v(x,t)$. The number of intersection points (or the lap number) is defined as

$$z(t; u, v) := \sup_{\Lambda}(\#P) - 1, \tag{2.4}$$

where $\#P$ is the number of elements in the partition $P := \{x_1 < \cdots < x_n\}$ that satisfies

$$e(x_i, t)e(x_{i+1}, t) < 0, \quad i = 1, \cdots, n-1,$$

and Λ is the collection of all possible such partitions. The second comparison property is about the monotonicity of the number of intersection points:

Hypothesis 2.2 (Monotonicity of the lap number)**.** *The number of intersection points between two solutions* u *and* v *decreases in time, i.e.,*

$$z(t_1; u, v) \geq z(t_2; u, v) \quad \textit{if} \quad t_1 < t_2. \tag{2.5}$$

We will show asymptotic convergence orders of a solution to a self-similar source-type solution of the problem. These two hypotheses are key players in doing that. To complete the theory one should classify the flux $\sigma = \sigma(u, u_x)$ and obtain these two hypotheses.

3. Steepness of a source-type solution

A general solution to the initial value problem (1.1) is compared to a fundamental or a source-type solution. Let $\rho(x,t) \geq 0$ be a source-type solution to (1.1), which satisfies

$$\rho_t = \frac{\partial}{\partial x}\sigma(\rho, \rho_x), \quad \rho(x,0) = q\delta(x), \quad x \in \mathbf{R},\ t > 0 \tag{3.1}$$

with a fixed $q > 0$. Comparison of a general solution to a source-type solution is convenient since in many cases they are given explicitly. Even if one can not expect such a luxury in more general cases a comparison with a source-type solution is convenient since it is the steepest one.

Theorem 3.1 (Number of intersections is at most two.)**.** *Let* ρ *be the source-type solution satisfying (3.1) and* ρ_c *be its space shift* $\rho_c(x,t) = \rho(x-c,t)$*. Then,*

$$z(t; u, \rho_c) \leq 2 \quad \textit{for all} \quad c \in \mathbf{R},\ t > 0. \tag{3.2}$$

Furthermore, if $z(t; u, \rho_c) = 2$*, then there exists* $a(t)$ *and* $b(t)$ *such that*

$$\rho_c(x,t) \geq u(x,t) \ \textit{if} \ \ a(t) \leq x \leq b(t), \ \ \textit{and} \quad \rho_c(x,t) \leq u(x,t) \ \ \textit{otherwise.} \tag{3.3}$$

Proof. Since $u(x,0) = u_0(x) \geq 0$ and $\rho_c(x,0) = 0$ for all $x \neq c$, we have $z(0; u, \rho_c) \leq 2$. Therefore, the monotonicity of the lap number gives (3.2). In fact one can easily see that $\min \operatorname{spt}(u_0) < c < \max \operatorname{spt}(u_0)$ if and only if $z(0; u, \rho_c) = 2$. Hence there exist $a(t) < b(t)$ that satisfies (3.4) for small $0 < t \ll 1$. This ordering is preserved as long as $z(t; u, \rho_c) = 2$ and hence (3.4) holds. $\square$

Remark 3.2. The proof of Theorem 3.1 is a little bit sloppy. It should be properly handled when Hypotheses 2.1 and 2.2 are considered. Except these three the rest of claims in this paper should be clean.

Theorem 3.1 is about the number of intersection points between a source-type solution and a general one. One can easily conclude that source-type solution is the steepest one using this relation.

Theorem 3.3 (Source-type solution is the steepest one.)**.** *Let ρ be the source-type solution satisfying (3.1) and ρ_c be its space shift $\rho_c(x,t)=\rho(x-c,t)$. Then,*

$$z(t;u,\rho_c)\le 2,\qquad t>0. \tag{3.2}$$

Furthermore, if $z(t;u,\rho_c)=2$, then there exists $a(t)$ and $b(t)$ such that

$$\rho_c(x,t)\ge u(x,t)\ \textit{if}\ \ a(t)\le b(t),\ \ \textit{and}\quad \rho_c(x,t)\le u(x,t)\ \textit{otherwise}. \tag{3.4}$$

4. L^1 CONVERGENCE ORDER OF THE MAGNITUDE OF SOLUTIONS

In this section we show that the general solution converges to a source-type solution in the L^1 norm with convergence order which is same as the uniform norm of the source-type solution.

It is well known that the difference between a general solution $u(x,t)$ and the source-type solution $\rho(x,t)$ decays to zero in L^1 sense,

$$\|u(t)-\rho(t)\|_1\to 0\quad \text{as}\quad t\to\infty\quad \text{if}\ \ M=q. \tag{4.1}$$

Here and in the remainder of this paper, we use the notation $\|\cdot\|_1$ to denote the L^1-norm and $u(t)$ to denote the function on the real line given by $u(t)|_x=u(x,t)$. Since the L^1-norm of a solution is preserved, it is a natural way to measure the distance between two solutions. Note that one may obtain asymptotic orders in L^p-norm, $p\ge 1$ via classical interpolation arguments. Under the previous hypotheses one can obtain the correct convergence order.

Theorem 4.1. *Let $u(x,t)$ be the solution to (1.1)-(1.2) and $\rho(x,t)$ be the source-type solution (3.1). If Hypotheses 2.1 and 2.2 hold, then*

$$\|u(t)-\rho(t)\|_1\le 2L\max_x \rho(x,t). \tag{4.2}$$

Proof. Since $\lim_{t\downarrow 0}\rho(x,t)=\delta(x)$, the Dirac-delta measure, one may easily see that $z(t;\rho,u)=1$ for $t\ll 1$ small since the measure is placed at an end point of the support of the initial value u_0. Therefore, the monotonicity of the lap number implies that there is a unique sign-changing point $x=\xi(t)$ such that $\rho(x,t)\le u(x,t)$ for all $x\le\xi(t)$ and $\rho(x,t)\ge u(x,t)$ for all $x\ge\xi(t)$. Then, clearly,

$$\begin{aligned}\|u(t)-\rho(t)\|_1 &= \int|u(x,t)-\rho(x,t)|dx = 2\int_{-\infty}^{\xi(t)}[\rho(x,t)-u(x,t)]dx\\ &= 2\|U(t)-R(t)\|_\infty.\end{aligned}$$

Furthermore, since the initial value $u_0(x)$ satisfies (1.2) one can easily see that $R(x-L,0)\le U(x,0)\le R(x,0)$. Therefore, the potential comparison principle implies that

$$R(x-L,t)\le U(x,t)\le R(x,t),\qquad \text{for all } t\ge 0. \tag{4.3}$$

Now the potential difference is easily estimated by

$$|U(x,t)-R(x,t)|\le|R(x-L,t)-R(x,t)|=\int_{x-L}^{x}\rho(y,t)dy\le L\max_x\rho(x,t). \tag{4.4}$$

Since the estimate is independent of the point x, the estimate is uniform and the inequality in (6.5) is obtained. □

5. L^1 CONVERGENCE ORDER OF $O(1/t)$

The estimate (4.3) was obtained simply due to the structure (1.2) of the initial value. It is natural to ask if $U(x,t)$ can be similarly enveloped with R and its time shift this time. This is not trivial and not always possible. Certain cases one should choose a different potential to do that. However, if that is possible, one may obtain better convergence order which corresponds to a convergence order of $O(1/t)$.

Theorem 5.1. *Let $u(x,t)$ be the solution of (1.1)-(1.2) and $\rho(x,t)$ be a source-type solution (3.1). If there exist $T,S>0$ and $c\in\mathbf{R}$ such that*

$$R^c(x,T)\le U(x,S)\le R^c(x,0), \tag{5.1}$$

then

$$\|\rho^c(t)-u(t)\|_1\le 4T\max_x\big(\sigma(u,u_x)\big), \tag{5.2}$$

where we denote $\rho^c(x,t)=\rho(x-c,t)$ for the space shift.

Proof. This time the source-type solution ρ is place at $c\in\mathbf{R}$ which is not an end point of the support of initial value u_0 in general. Therefore, there can be two the sign-changing points and let $\xi_1(t)\le\xi_2(t)$ be those points. Then,

$$\begin{aligned}\|\rho^c(t)-u(t+S)\|_1 &= \Big(\int_{-\infty}^{\xi_1(t)}+\int_{\xi_1(t)}^{\xi_2(t)}+\int_{\xi_2(t)}^{\infty}\Big)\big|\rho(x-c,t)-u(x,t+S)\big|dx\\ &\le 4\|U(t+S)-R^c(t)\|_\infty.\end{aligned}$$

Using the comparison inequality (5.1) and the evolution equation for potentials (2.2), we obtain

$$\begin{aligned}|U(x,t+S)-R(x-c,t)|&\le|R(x-c,t+T)-R(x-c,t)|\\ &=\int_t^{t+T}\sigma\big(\rho(x-c,s),\rho_x(x-c,s)\big)ds\le T\|\sigma\big(\rho(t),\rho_x(t)\big)\|_\infty.\end{aligned}$$

Since the right hand side does not depend on x, it is a uniform estimate and hence (5.2) is obtained. □

The potential estimate (4.3) was immediate due to the structure of the initial value (1.2). However, the corresponding estimate (5.1) is not. In fact in many cases such an estimate is not possible and one should take a different kind of potential. For a general conservation law without the convexity assumption, such a potential comparison and hence the corresponding convergence order has been obtained using the primitive as its potential (see [6, **?**]). However, for the nonlinear diffusion equations, such a comparison is not possible using the primitive as the potential and hence one should use a different one. In [9] Newtonian potential has been employed to obtain potential comparison in (5.1).

6. Potential Comparison Principle

In this section we show Theorem 4.1. For the convection equation case, $\sigma=\sigma_c$, the entropy solutions are considered and the convergence order has been shown in [**?**], Theorem 1 (13), by trapping a solution between two rarefaction waves. In this chapter we consider the integrals of the solution $u(x,t)$ and the source-type one $\rho(x,t)$ as their potentials,

$$U(x,t)=\int_{-\infty}^x u(y,t)dy,\qquad R(x,t)=\int_{-\infty}^x\rho(y,t)dy. \tag{6.1}$$

Integrating (1.1) over $(-\infty, x)$ yields

$$U_t = \sigma(u, u_x) = \sigma(U_x, U_{xx}). \tag{6.2}$$

To justify this Hamilton-Jacobi type equation we should have

$$\lim_{x\to-\infty} \sigma(u(x,t), u_x(x,t)) = 0 \text{ and } \frac{\partial}{\partial t}\int_{-\infty}^{x} u(y,t)dy = \int_{-\infty}^{x} \frac{\partial}{\partial t} u(y,t)dy,$$

which can be easily verified.

It is well known that, if $u(x,0) \geq \tilde{u}(x,0)$, then the solutions $u, \tilde{u}$ to (1.1) satisfies $u(x,t) \geq \tilde{u}(x,t)$ for all $t > 0$. However, one can not expect this relation between u and ρ even after a time and space translation since ρ and u share the same total mass. In the following proposition we show that a similar comparison principle holds between potentials.

Proposition 6.1 (Comparison Principle). *Let σ be one of the 7 cases in Theorem 4.1, u and $\tilde{u}$ be solutions of (1.1) and U and $\tilde{U}$ be their potentials given by their integrals as in (6.1), respectively. Then, if $U(x,0) \geq \tilde{U}(x,0)$, then $U(x,t) \geq \tilde{U}(x,t)$ for all $t \geq 0$.*

Proof. Let $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$ where $c_i = 0$ or 1. One can easily check that the potential difference $E = U - \tilde{U}$ satisfies

$$\mathsf{L}[E] := (c_1 a(x,t) + c_2 b(x,t))E_{xx} + (c_3 c(x,t))E_x - E_t = 0, \; E(x,0) \geq 0,$$

where

$$a := \frac{\phi'(u)u_x - \phi'(\tilde{u})\tilde{u}_x}{u_x - \tilde{u}_x}, \quad b := \frac{\psi(u_x) - \psi(\tilde{u}_x)}{u_x - \tilde{u}_x}, \quad c := \frac{f(u) - f(\tilde{u})}{u - \tilde{u}}.$$

Notice that $b(x,t) \geq 0$ and $c(x,t) \geq 0$ since $f(u)$ and $\mathrm{sign}(u_x)\psi(|u_x|)$ are increasing functions with respect to $u \geq 0$ and u_x, respectively. However, $a(x,t)$ may change its sign. In the followings we check if the maximum principle type arguments hold. First observe the uniform decay of E, i.e.,

$$|E(x,t)| \leq \int_{-\infty}^{x} |u(y,t) - \tilde{u}(y,t)|dy \leq \|u(x,t) - \tilde{u}(x,t)\|_1 \to 0$$

as $t \to \infty$. Suppose that there exists $t \geq 0$ and $x \in \mathbf{R}$ such that $E(x,t) < 0$. Then the uniform convergence and the initial condition $E(x,0) \geq 0$ imply that there exists a global minimum point (x_0, t_0).

First consider the case $c_1 = 1$. Since the solutions $u, \tilde{u}$ are continuous, E is differentiable and hence

$$E_x(x_0, t_0) = 0.$$

Therefore, $u(x_0,t_0) = \tilde{u}(x_0,t_0)$ and $a(x_0,t_0) = \phi'(u(x_0,t_0))$. The point x_0 should be an interior point of the support of $u(\cdot,t_0)$ or $\tilde{u}(\cdot,t_0)$ (otherwise $E(x_0,t_0) = 0$) and hence $a(x_0,t_0) > 0$ and the equation is uniformly parabolic in a small neighborhood of (x_0,t_0). The maximum principle now easily derives a contradiction.

Now consider the case $c_1 = 0$ and $c_2 = 1$. (In this case the coefficient b may diverges and hence the maximum principle is not obvious.) Since $E(x,t_0)$ is not a constant function and x_0 is a minimum point, there exists $\delta > 0$ such that $E(x,t_0) > E(x_0,t_0)$ for $x \in (x_0 - \delta, x_0)$ after retaking x_0 if needed. By taking smaller $\delta > 0$ if needed, we may assume $E_x(\cdot,t_0) < 0$ and $E_{xx}(\cdot,t_0) > 0$ on $(x_0 - \delta, x_0)$. If

$b(x_0,t_0)>0$, the problem is uniformly parabolic in neighborhood of (x_0,t_0) again. Therefore, consider the case when $b(x_0,t_0)=0$. Let $W=E+\epsilon z$, where

$$z(x)=1-e^{x_0-x}\quad\text{and } 0<\epsilon<\frac{E(x_0,t_0)-E(x_0-\delta,t_0)}{z(x_0-\delta)}\ .$$

Then, since $z(x_0)=0$ and $z_x(x_0)=1$, we get $W(x_0,t_0)=E(x_0,t_0)$, $W_x(x_0,t_0)=\epsilon>0$ and $W(x_0-\delta,t_0)>E(x_0,t_0)$. Therefore, $W(\cdot,t_0)$ has an interior minimum point in $(x_0-\delta,x_0)$. Let (y_0,t_0) be the minimum point. Since $E(x_0,t_0)$ is the global minimum of E and the auxiliary function z depends on x variable only, this minimum point is also a minimum of W on $Q:=(x_0-\delta,x_0)\times(0,T)$ for any $T>0$. Notice that, since $E_x(\cdot,t_0)\neq 0\neq E_{xx}(\cdot,t_0)$ in the interval $(x_0-\delta,x_0)$, both of $b(x,t)$ and $c(x,t)$ are strictly positive. Therefore, L is uniformly parabolic in a neighborhood of (y_0,t_0) and

$$\mathsf{L}[W]=\mathsf{L}[E]+\epsilon\mathsf{L}[z]=\epsilon(bz_{xx}-c_3cz_x)=-\epsilon(b+c_3c)\le 0$$

on it. The assumption that (y_0,t_0) is a interior minimum point of W contradicts to the maximum principle. The case $c_1=c_2=0$ and $c_3=1$ is done in [**?**] and hence omitted. ☐

Notice that this comparison principle holds for potentials given by integrals of solutions (6.1) and that it seems to hold under a more general operator $\sigma=\sigma(u,u_x)$. The next step is to place the potential U between two potentials of canonical solutions.

Corollary 6.2 (Trapped!)**.** *Let u be a solution of (1.1)-(1.2) and ρ be the canonical solution of (3.1). Then their potentials U,R given by (6.1) satisfies*

$$R(x-L,t)\le U(x,t)\le R(x,t)\ \textit{for all}\ t>0,\ x\in\mathbf{R}. \tag{6.3}$$

Proof. Since $\rho(x,0)=\delta(x)$, $R(x,0)$ is the heavy side function

$$R(x,0)=\begin{cases}0, & \text{if } x<0,\\ 1, & \text{if } x\ge 0.\end{cases}$$

Therefore the restriction (1.2) on the initial value u_0 implies that

$$R(x-L,0)\le U(x,0)\le R(x,0)$$

(**see Figure 1**). Proposition 6.1 implies that (6.3) holds for all $t>0$. ☐

The next step is to compute the convergence rate of potentials using (6.3) which comes easily:

Lemma 6.3 (Potential Convergence Rate)**.** *Under the same conditions as in Corollary 6.2,*

$$\|U(x,t)-R(x,t)\|_\infty\le L\max_{x\in\mathbf{R}}\rho(x,t). \tag{6.4}$$

Proof. Using the comparison inequality (6.3) and the Mean Value Theorem, we obtain

$$\begin{aligned}|U(x,t)-R(x,t)| &\le |R(x-L,t)-R(x,t)|\\ &= LR_x(c,t)=L\rho(c,t)\le L\max_{x\in\mathbf{R}}\rho(x,t),\end{aligned}$$

where $c\in[x-L,x]$. Since the estimate is independent of the point x, the estimate is uniform as in (6.4). ☐

Now the last step is to transfer this potential convergence rate to the desired solution convergence rate. Following the approach in [9], one may employ the classical regularity theory in [11] to derive the estimate of $U_x - R_x (\equiv u - \tilde{u})$ from $U - R$. It seems that there is a simpler approach under a conjecture that ρ is steeper than any other solution u.

Proposition 6.4. *Let $u(x,t)$ and $\rho(x,t)$ be solutions of (1.1)-(1.2) and (3.1), respectively. Then,*

$$\|\rho(t) - u(t)\|_1 = 2\|R(t) - U(t)\|_\infty \le 2L \max_x \rho(x,t). \tag{6.5}$$

Proof. Let $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$ where $c_i = 0$ or 1. One can easily check that the potential difference $E = R - U$ satisfies

$$\mathsf{L}[E] := (c_1 a(x,t) + c_2 b(x,t))E_{xx} + (c_3 c(x,t))E_x - E_t = 0,\ E(x,0) \ge 0,$$

where

$$\begin{aligned} a &:= \frac{\phi'(\rho)\rho_x - \phi'(u)u_x}{\rho_x - u_x}, \\ b &:= \frac{\operatorname{sign}(\rho_x)\psi(|\rho_x|) - \operatorname{sign}(u_x)\psi(|u_x|)}{\rho_x - u_x}, \\ c &:= \frac{f(\rho) - f(u)}{\rho - u}. \end{aligned}$$

Since $\operatorname{sign}(x)\psi(x)$ is an odd function with $\psi'(x) \ge 0$ for $x > 0$, the coefficient b is non-negative. Therefore, if $c_1 = 0$, we may apply the property of the nonincrease of the lap number [15]. If $c_1 = 1$, then the coefficient of the leading order term $a + c_2 b$ can be negative. However, the lap number changes at the points $\rho = u$ and $a = \phi'(u) > 0$ near such a point. Therefore, we may apply the argument similarly and show the nonincrease of the lap number. We may conclude that the number of sign changes is at most once since $\lim_{t\downarrow 0} \rho(x,t) = \delta(x)$, the Dirac-delta measure. Let $x(t)$ be the sign-changing point of $e = \rho - u$. Then, clearly,

$$\begin{aligned} \|u(t) - \rho(t)\|_1 &= \int |u(x,t) - \rho(x,t)|dx = 2\int_{-\infty}^{x} [\rho(x,t) - u(x,t)]dx \\ &= 2\|U(t) - R(t)\|_\infty. \end{aligned}$$

Furthermore, using the comparison inequality (6.3), we obtain

$$|U(x,t) - R(x,t)| \le |R(x-L,t) - R(x,t)| = \int_{x-L}^{x} \rho(y,t)dy \le L \max_x \rho(x,t). \tag{6.6}$$

Since the estimate is independent of the point x, the estimate is uniform and the inequality in (6.5) is obtained. ☐

7. Porous medium type nonlinear diffusion

In this section we show Theorem 5.1(i). For the nonlinear diffusion case, $\sigma = \sigma_d$, the equation (1.1) is written as

$$u_t = \phi(u)_{xx}, \quad \lim_{t\downarrow 0} u(x,t) = u_0(x) \ge 0, \quad x \in \mathbf{R},\ t > 0. \tag{7.1}$$

The source solution ρ satisfies

$$\rho_t = \phi(\rho)_{xx}, \quad \lim_{t\downarrow 0} \rho(x,t) = \delta(x), \quad x \in \mathbf{R},\ t > 0. \tag{7.2}$$

Under the power law, $\phi(u) = u^m$, $m > 0$, the canonical solution $\rho(x,t)$ is explicitly given by

$$\rho^{m-1}(x,t) = \max\left\{0, At^{\frac{1-m}{m+1}} + \frac{1-m}{2m(m+1)}|x|^2 t^{-1}\right\}, \tag{7.3}$$

where the constant σ is decided by the relation $\int \rho(x,t)dx = 1$. One can easily check that $\rho(x,t)$ has its maximum at $x = 0$ and

$$\max_{x\in\mathbf{R}} \rho(x,t) = A^{\frac{1}{m-1}} t^{\frac{-1}{m+1}} = O(t^{\frac{-1}{m+1}}). \tag{7.4}$$

For the fast diffusion range, $0 < m < 1$, the convergence order $1/t$ has been shown in [9].

To obtain a convergence order $1/t$ we have trapped the initial potential $U(x,0)$ between $R(x,t)$ and $R(x,t+T)$ in the convection case. However, the primitive of a solution does not decouple the equation enough. Therefore, we introduce Newtonian potentials:

$$\mathsf{U}(x,t) = \int_{-\infty}^{x} U(y,t)dy, \qquad \mathsf{R}(x,t) = \int_{-\infty}^{x} R(y,t)dy. \tag{7.5}$$

Let $\varphi(x) = \max\{0, x\}$. Then, one may check that $\Delta\varphi(x) = \delta(x)$ and $\mathsf{U} = \varphi * u$. Therefore, it is natural to call U a Newtonian potential. A typical way is to take $\varphi(x) = |x|/2$ as a kernel. Here, to show the connection to Section 6 more closely, the Newtonian potential is given by (7.5).

Proposition 7.1. *Let u be the solution of (7.1) and U be its Newtonian potential. Then*

$$\mathsf{U}_t(x,t) = \phi(u(x,t)) > 0 \qquad \text{for each } x \in \mathbf{R},\ t > 0. \tag{7.6}$$

Proof. Basically, we need to check if

$$\lim_{h\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{x} \frac{u(y,t+h)-u(y,t)}{h} dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{x} \lim_{h\to\infty} \frac{u(y,t+h)-u(y,t)}{h} dydx.$$

Under the power law $\phi(u) = u^m$, $m > 0$, it is clear for $m > 1$ since the integrands are compactly supported. For the fast diffusion case $0 < m \le 1$ we simply refer [9]. Therefore, we need to check for the case of non-power law. □

Proposition 7.2 (Potential comparison). *Let $\mathsf{U}(x,t)$ and $\tilde{\mathsf{U}}(x,t)$ be the Newtonian potentials of two bounded solutions u and $\tilde{u}$ of (7.1), respectively. Then $\mathsf{U}, \tilde{\mathsf{U}}$ are continuous on the closure of $\mathbf{R} \times]0,\infty[$. If $\mathsf{U}(x,0) \le \tilde{\mathsf{U}}(x,0)$ for all $x \in \mathbf{R}$, then $\mathsf{U}(x,t) \le \tilde{\mathsf{U}}(x,t)$ for all $t > 0$.*

Proof. Let $\mathsf{V} = \mathsf{U} - \tilde{\mathsf{U}}$. Then V satisfies

$$\mathsf{V}_t = \mathsf{U}_t - \tilde{\mathsf{U}}_t = \phi(u) - \phi(\tilde{u}) = \frac{\phi(u)-\phi(\tilde{u})}{u-\tilde{u}}\mathsf{V}_{xx}\big(\equiv a(x,t)\mathsf{V}_{xx}\big). \tag{7.7}$$

The coefficient $a(x,t)$ is non-negative since ϕ is an increasing function and hence the maximum principle implies that $\mathsf{V}(x,t)$ is nonnegative if the initial value $\mathsf{V}(x,0)$ is nonnegative. □

Lemma 7.3. *Let c be the center of mass $c = \int xu_0(x)dx$. Then, there exists $T > 0$ such that*

$$R(x,t) \le \mathsf{U}(x+c,t) \le R(x,t+T), \quad t > 0. \tag{7.8}$$

Proof. First we can easily check that $\partial_x \mathsf{U}(x,t) = U(x,t) \le 1$, $\partial_x \mathsf{U}(x,0) = 1$ for $x > L$ and

$$\mathsf{U}(L,0) = \int_0^L \int_0^x u_0(y)dydx = \Big[x \int_0^x u_0(y)dy\Big]_{-\infty}^L - \int_0^L x u_0(x)dx = L - c.$$

Therefore, $\mathsf{U}(x,0) = x - c$ for all $x \ge L$ and $\mathsf{U}(x,0) \ge x - c$. Clearly $\mathsf{U}(x,0) \ge 0$ and $\mathsf{U}(x,0) = 0$ for all $x < 0$. $\mathsf{R}(x,0)$ is easy to compute that we have

$$\mathsf{U}(x+c,0) \ge \mathsf{R}(x,0) = \begin{cases} 0, & x < 0, \\ x, & x > 0. \end{cases}$$

Considering the explicit formula (7.3) for the canonical solution $\rho(x,t)$ we can easily check that $\mathsf{R}(x,t) \to \infty$ as $t \to \infty$ uniformly on any compact interval. Therefore, there exists $T > 0$ such that

$$\mathsf{R}(x,0) \le \mathsf{U}(x+c,0) \le \mathsf{R}(x,T)$$

(**see Figure 3**). Now the comparison principle, Proposition 7.2, completes the proof. ☐

The convergence order of potentials is easily computed like followings:

Lemma 7.4 (Potential convergence order)**.** *Let U, R be Newtonian potentials of u, ρ, respectively, and $c = \int x u_0(x)dx$. Then, there exists $T > 0$ depending on the initial value u_0 such that*

$$\|\mathsf{R}(t) - \mathsf{U}_c(t)\|_\infty = T \max_x \big(\phi(\rho(x,t))\big). \tag{7.9}$$

Proof. Let $T > 0$ be the constant that satisfies (7.8). Then,

$$\begin{aligned} |\mathsf{R}(x,t) - \mathsf{U}(x+c,t)| &\le |\mathsf{R}(x,t) - \mathsf{R}(x,t+T)| \\ &\le T\|\mathsf{R}_t\|_\infty = T \max_x \big(\phi(\rho(x,t))\big). \end{aligned}$$

Since the estimate is independent of x, it is a uniform estimate. ☐

Now we transfer this convergence order to solutions. Since the coefficient a in (7.7) is non-negative, the lap number decreases and hence there is at most two sign-changing points and hence $U(x,t) - R(x,t)$ may change its sign once. Therefore, by the same arguments in the proof of Proposition 6.4, we have

$$\|R(t) - U_c(t)\|_1 = 2\|\mathsf{R}(t) - \mathsf{U}_c(t)\|_\infty \le 2T \max_x \big(\phi(\rho(x,t))\big).$$

We need to convert this L^1 convergence rate to a uniform one to use the arguments again. This comes from the $L^1 - L^\infty$ interpolation and

$$\|R(t) - U_c(t)\|_\infty \le 2T \max_x \big(\phi(\rho(x,t))\big) \max_x \big(\rho(x,t)\big). \tag{7.10}$$

Since $\rho - u$ may changes its sign at most twice, the estimate corresponding to the first step is

$$\|\rho(t) - u_c(t)\|_1 \le 4\|R(t) - U_c(t)\|_\infty \le 8T \max_x \big(\phi(\rho(x,t))\big) \max_x \big(\rho(x,t)\big).$$

This completes the proof of Theorem 5.1(i).

8. p-LAPLACIAN TYPE NONLINEAR DIFFUSION

In this section we show Theorem 5.1(ii). Consider the case $\sigma = \sigma_p$, i.e., u is the solution of

$$u_t = \psi(u_x)_x, \quad \lim_{t \downarrow 0} u(x,t) = u_0(x), \quad x \in \mathbf{R},\ t > 0, \tag{8.1}$$

where ψ is an odd function satisfying (**??**) and the initial value u_0 is symmetric and continuous, and satisfies

$$\operatorname{spt}(u_0) = [-L, L], \quad u_0(x) > 0 \text{ for } x \in (-L, L) \text{ and } \int u_0(x)dx = 1. \tag{8.2}$$

The source solution ρ satisfies

$$\rho_t = \psi(\rho_x)_x, \quad \lim_{t \downarrow 0} \rho(x,t) = \delta(x), \quad x \in \mathbf{R},\ t > 0. \tag{8.3}$$

Let

$$v = u_x, \qquad \varrho = \rho_x.$$

Then after differentiating (8.1) one can easily check that v and ϱ are solutions to the porous medium type equations

$$v_t = (\psi(v))_{xx}, \qquad \varrho_t = (\psi(\varrho))_{xx}. \tag{8.4}$$

The difference from the previous case is that v, ϱ are sign-changing solution with zero total mass, i.e.,

$$\int_{-\infty}^{\infty} v(x,t)dx = \int_{-\infty}^{\infty} \varrho(x,t)dx = 0.$$

Let $u, \tilde{u}$ be solutions to (8.1) and $v, \tilde{v}$ be their derivatives, respectively. We take the integrals $U, \tilde{U}$ of $u, \tilde{u}$ as their potentials. Then, $U, \tilde{U}$ are Newtonian potentials of $v, \tilde{v}$. We already saw in Proposition 6.1 that $E := U - \tilde{U} \geq 0$ for all $t \geq 0$ if $E(x,0) \geq 0$. In the proof we basically used the maximum principle to the relation

$$E_t = b(x,t)E_{xx}, \quad b(x,t) := \frac{\psi(v) - \psi(\tilde{v})}{v - \tilde{v}} \geq 0.$$

Now we construct an estimate related to the time translation. Since $\rho(x,t) \geq 0$ decays to zero as $t \to \infty$, preserves its initial mass, and is symmetric with respect to $x = 0$, we have, for any $t \geq 0$,

$$R(x,t) \to 1 \text{ as } x \to \infty,\ R(x,t) \to 0 \text{ as } x \to -\infty,$$
$$R_x(x,t) \to 0 \text{ as } t \to \infty,\ R(0,t) = 1/2 \text{ for } t \geq 0.$$

Similarly, since $u(x,t)$ is symmetric, the potential U of the solution u satisfies

$$U(x,t) \to 1 \text{ as } x \to \infty,\ U(x,t) \to 0 \text{ as } x \to -\infty,$$
$$U_x(x,t) \to 0 \text{ as } t \to \infty,\ U(0,t) = 1/2 \text{ for } t \geq 0.$$

The initial potentials $U(x,0), R(x,0)$ satisfy

$$U(x,0) = 1 \text{ for all } x > L, \quad U(x,0) = 0 \text{ for all } x < -L,$$
$$R(x,0) = 1 \text{ for all } x > 0, \quad R(x,0) = 0 \text{ for all } x < 0.$$

Under the extra assumption that $U_x(x_0,0) \neq 0$ at the point such that $U(x_0,0) = 1/2$, one can easily see that there exist $T > 0, c \in \mathbf{R}$ such that

$$R(x-c,T) \leq U(x,0) \leq R(x-c,0), \quad x > 1/2a.$$
$$R(x-c,0) \leq U(x,0) \leq R(x-c,T), \quad x < 1/2a.$$

It is clear that such an estimates as (7.8) is not possible since $R(x,t)$ and $U(x,t)$ cross to each other at the point $x=0$, i..e, $U(0,t)=R(0,t)=1/2$ for all $t>0$. However we may apply the maximum principle arguments by restricting the domain to $x\geq 0$.

Lemma 8.1. *There exists $T>0$ such that*

$$R(x,t+T)\leq U(x,t)\leq R(x,t),\quad t>0,\, x\geq 0. \tag{8.5}$$

Proof. The second inequality in (8.5) is already obtained and we show the first one. Since the initial value is continuous and $u_0(0)>0$, there exists $\epsilon>0,\delta>0$ such that

$$u_0(x)>\epsilon \qquad \text{for} \qquad 0<x<\delta.$$

Since $\rho(x,t)\to 0$ as $t\to\infty$, there exists $T>0$ such that $\max_x\rho(x,T)<\epsilon$. Therefore, $R(x,T)\leq U(x,0)$ for $0<x<\delta$. Since $R(x,t)\to 0$ uniformly as $t\to\infty$ on the domain $[\delta,L]$ and $U(x,0)>\epsilon\delta$ on $[\delta,L]$, we have $R(x,T)\leq U(x,0)$, $0<x<L$, after taking larger $T>0$ if needed. Since $U(x,0)=1$ for $x\geq L$ and $R(x,t)\leq 1$ for $x\geq L$, we have

$$R(x,T)\leq U(x,0),\qquad t>0,\, x\geq 0.$$

Now we may apply the maximum principle on the domain $\Omega:=\{(x,t):0\leq x<\infty,\, 0\leq t<\infty\}$. Since (8.5) hold on the boundary $\partial\Omega$, we may conclude that (8.5) holds on Ω. $\square$

Now the convergence order is similarly estimated as

$$\|u-\varrho\|_1\leq 2\|R(t)-R(T+t)\|_\infty\leq 2T\|R_t\|_\infty\leq 2T\max_x\psi(\varrho(x,t)))$$

and the proof for Theorem 5.1(ii) is complete.

In the diffusion case we obtained extra convergence order of similarity scale since the solution is the second order derivative of its potential. Here we could not do that. However, since ϱ has zero total mass, we may expect that it may decay faster than ρ.

9. Explicit computation of convergence orders

In this section we compute the convergence order in Theorems 4.1 and 5.1 explicitly under power laws:

$$f(u)=u^q/q,\ \phi(u)=u^m,\ \psi(v)=\mathrm{sign}(v)|v|^{p-1},\ m>0,q>1,p>1. \tag{9.1}$$

If $\sigma=f(u)$, $\phi(u)_x$ or $\psi(u_x)$, the the source-type solution ρ is given explicitly and it is well known that

$$\max_{x\in\mathbf{R}}\rho(x,t)=O(t^{-\alpha}),\quad \alpha=\begin{cases}1/q & \text{if } \sigma=f(u),\\ 1/(m+1) & \text{if } \sigma=\phi(u)_x,\\ 1/(2p-2) & \text{if } \sigma=\psi(u_x),\end{cases} \tag{9.2}$$

as $t\to\infty$. If σ is a sum of two or all of these three terms, the decay power α is simply the largest one among these three. For the general case $0\neq\sigma=c_1\sigma_c+c_2\sigma_d+c_3\sigma_p$, the decay rate of the solution ρ is

$$\max_{x\in\mathbf{R}}\rho(x,t)=O(t^{-\alpha}),\quad \alpha=\max\Big(\frac{c_1}{q},\frac{c_2}{m+1},\frac{c_3}{2p-2}\Big) \tag{9.3}$$

as $t\to\infty$. (This decay order has been shown for the convection-diffusion case in [5], i.e., $c_1=c_3=1,c_2=0$. I am not sure if there is a paper for (9.3) for the case when all of these three terms are together.)

If $\alpha = 1/q$, then we may say that the convection dominates the evolution. Similarly, if $\alpha = 1/(m+1)$ or $\alpha = 1/(2p-2)$, then the nonlinear diffusion operator σ_d or the p-Laplacian operator σ_p dominates the evolution, respectively. Combining (4.2) and (9.3) we obtain the convergence order

$$\|u(t)-\rho(t)\|_1 = O(t^{-\alpha}),\ \alpha = \max\left(\frac{c_1}{q}, \frac{c_2}{m+1}, \frac{c_3}{2p-2}\right),\ \text{as } t\to\infty \tag{9.4}$$

for the general case in (**??**).

For the case $\sigma = \phi(u)_x$ the order of ρ is $t^{-1/(m+1)}$ and hence the convergence order in (**??**) is

$$\|\rho(t) - u_c(t)\|_1 = O(t^{\frac{-1}{m+1}} t^{\frac{-m}{m+1}}) = O(t^{-1})$$

as $t \to \infty$.

Remark 9.1. If $1 < p < 3/2$, the decay order in (9.4) is higher than $1/t$. It seems that this case the diffusion speed is so fast that the the time translation has bigger impact than space translation. The similar phenomenon is observed for the nonlinear diffusion when the diffusion is so fast that the total mass is not preserved.

Remark 9.2. For the nonlinear diffusion in multi-dimensions, $\sigma = \Delta(u^m)$, the convergence order of magnitude of the source solution has been known by Dolbeault & del Pino [2] and Otto [16]. For p-Laplacian $\sigma = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with p in a certain range L^1 convergence order has been obtained by Dolbeault & del Pino [3] for dimension $n \geq 2$, which is lower than the magnitude of solutions. However, under the presence of several phenomena any convergence order is not known until a recent work by Carrillo & Fellner [1]. In [1] the same convergence order has been shown for the case $\sigma = \sigma_c + \sigma_d$ with $m+1 < q$ and $1 \leq m < 2$.

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