Non-existence of localized travelling waves with non-zero speed in single reaction-diffusion equations

Yong Jung Kim *
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejon 305-701, Republic of Korea

Wei-Ming Ni †
Center for Partial Differential Equations
East China Normal University
Minhang, Shanghai, 200241, P.R. China
and
Department of Mathematics, University of Minnesota
127 Vincent Hall, 206 Church St. S.E.,
Minneapolis, MN 55455 USA

Masaharu Taniguchi ‡
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
O-okayama 2-12-1-W8-38, Meguro-ku, Tokyo 152-8552 Japan

Abstract

Assume a single reaction-diffusion equation has zero as an asymptotically stable stationary point. Then we prove that there exist no localized travelling waves with non-zero speed. That is, if $[\liminf_{|x|\to\infty} u(x), \limsup_{|x|\to\infty} u(x)]$ is included in an open interval of zero that does not include other stationary points, then the speed has to be zero or the travelling profile u has to be identically zero.

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*yongkim@kaist.edu
†ni@math.umn.edu
‡masaharu.taniguchi@is.titech.ac.jp, This work was supported by Grant-in-Aid for Scientific Research (C), 23540235, Japan Society for the Promotion of Science, 2011-2013.

1 Introduction

In this paper we consider the following equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) \qquad && \text{in } \mathbb{R}^n,\ t > 0, \\ u\big|_{t=0} &= u_0 && \text{in } \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here $n \geq 1$ and u_0 is a given bounded uniformly continuous function. The Laplacian Δ stands for $\Delta \overset{\text{def}}{=} \sum_{j=1}^n \partial^2/\partial x_j^2$. Here $f \in C^1(\boldsymbol{R})$ satisfies $f(0) = 0$. In addition, we assume either

$$f'(0) < 0 \tag{1.2}$$

or

$$0 < \liminf_{u \to 0} \frac{f(u)}{-u|u|^p} \leq \limsup_{u \to 0} \frac{f(u)}{-u|u|^p} < \infty \tag{1.3}$$

for a positive constant p with

$$n - 1 - \frac{2}{p} < 0.$$

Typical examples are

(a) $f(u) = -u + |u|^{q-1}u$ with $q > 1$;

(b) $f(u) = -u(u - b_0)(u - 1)$ with $0 < b_0 < 1$;

(c) $f(u) = u|u|(u - 1)$ with $n = 1, 2$.

We study a travelling wave solution $v(x_1, \ldots, x_n - ct)$ with speed c. We write $y = x_n - ct$ and $v(x_1, \ldots, x_{n-1}, y)$ satisfies

$$\left(\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2} \right) v + c\frac{\partial v}{\partial y} + f(v) = 0 \qquad (x_1, \ldots, x_{n-1}, y) \in \boldsymbol{R}^n.$$

This equation is the profile equation for a travelling wave solution $v(x_1, \ldots x_{n-1}, x_n - ct)$.

We write $v(x_1, \ldots, x_{n-1}, y)$ by $u(x_1, \ldots, x_n)$. Then we have

$$\Delta u + c\frac{\partial u}{\partial x_n} + f(u) = 0 \qquad \text{in } \boldsymbol{R}^n. \tag{1.4}$$

In this paper we study this equation with

$$[\liminf_{|x| \to \infty} u(x), \limsup_{|x| \to \infty} u(x)] \subset (-a_0, b_0). \tag{1.5}$$

Here we put

$$\begin{aligned} b_0 &\overset{\text{def}}{=} \sup\{b \in (0, \infty) \mid f(s) < 0 \quad \text{for all } s \in (0, b)\} \in (0, \infty], \\ a_0 &\overset{\text{def}}{=} \sup\{a' \in (0, \infty) \mid f(s) > 0 \quad \text{for all } s \in (-a', 0)\} \in (0, \infty], \end{aligned}$$

and have $-a_0 < 0 < b_0$. If u satisfies $\lim_{|x|\to\infty} u(x) = 0$, we have (1.5). If u satisfies (1.4) and $\lim_{|x|\to\infty} u(x) = 0$, we call u a localized travelling wave solution.

If $c = 0$, a localized travelling wave is a stationary solution. Many works have been studied for the existence, uniqueness and the spherical symmetry of stationary solutions. For the existence and the uniqueness of stationary solutions, see [14] , [3] , [4], [5], [16], [12] and [19] for instance. For the spherical symmetry of positive solutions, see [7], [8] and [13] for instance.

Localized travelling waves for systems of reaction-diffusion equations are studied in [15], [10] and [6] for examples, and they are sometimes called travelling spots or travelling spikes.

For a single reaction-diffusion one might suppose that (1.1) is a gradient system for an energy functional and thus a localized initial state with finite energy has to be a stationary state or cannot keep its shape as time goes on. This intuitive idea suggests that there exist no localized travelling wave solutions with speed $c \neq 0$. The aim of this paper is to give a simple proof of this non-existence.

The main assertion of this paper is as follows.

Theorem 1. *Assume $f \in C^1(\boldsymbol{R})$ satisfies $f(0) = 0$. In addition assume $f'(0) < 0$ or (1.3). If $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies (1.4) and (1.5), one has $c = 0$ or $u \equiv 0$.*

Remark 1. If one assumes either

$$[\liminf_{|x|\to\infty} u(x), \limsup_{|x|\to\infty} u(x)] \subset [-a_0, b_0) \quad \text{or} \quad [\liminf_{|x|\to\infty} u(x), \limsup_{|x|\to\infty} u(x)] \subset (-a_0, b_0]$$

in stead of (1.5), Theorem 1 does not hold true. Indeed, we put $n = 1$, $a_0 = 1$, $b_0 = 1$, and choose f with $f'(-1) > 0$,

$$f(u) > 0, \quad f'(u) < f'(-1) \qquad \text{for all } u \in (-1, 0),$$

and have a monotone decreasing solution connecting 0 and -1 with any speed $c \geq 2\sqrt{f'(-1)}$ to (1.4) by [11, 1, 2]. Thus the interval $(-a_0, b_0)$ in the condition (1.5) is maximal in Theorem 1.

2 Proof of Theorem 1

We put

$$F(u) \overset{\text{def}}{=} \int_0^u f(s)\, ds.$$

For any $R > 0$ we use $B(0; R) \overset{\text{def}}{=} \{x \in \boldsymbol{R}^n \,|\, |x| \leq R\}$ and $\partial B(0; R) \overset{\text{def}}{=} \{x \in \boldsymbol{R}^n \,|\, |x| = R\}$. We write the outward unit vector $\nu = (\nu_1, \ldots, \nu_n)$ at $x \in \partial B(0; R)$ as

$$\nu = \frac{x}{|x|} = \frac{x}{R} \qquad \text{for } x \in \partial B(0; R).$$

We denote $\partial/\partial x_j$ by D_j and $\partial^2/\partial x_i \partial x_j$ by D_{ij} for $1 \le i, j \le n$. Under the assumption of Theorem 1 we put

$$
\begin{aligned}
A &\stackrel{\text{def}}{=} 1 + \sup_{x \in \boldsymbol{R}^n} |u(x)|, \qquad M \stackrel{\text{def}}{=} \max\left\{1, \max_{-A \le s \le A} |f'(s)|\right\}, \\
B_1 &\stackrel{\text{def}}{=} \liminf_{|x| \to \infty} u(x), \qquad B_2 \stackrel{\text{def}}{=} \limsup_{|x| \to \infty} u(x),
\end{aligned}
$$

and have $-A \le B_1 \le B_2 \le A$.

Lemma 1. *If $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies (1.4), one has*

$$
c \int_{B(0;R)} (D_n u)^2 = \int_{\partial B(0;R)} \left(\frac{1}{2} |\nabla u|^2 - F(u) \right) \nu_n - \int_{\partial B(0;R)} D_n u (\nabla u, \nu). \tag{2.6}
$$

Proof. Using

$$
\operatorname{div}(D_n u \nabla u) = D_n u \Delta u + \sum_{j=1}^n D_{jn} u D_j u
$$

and

$$
\frac{1}{2} D_n \left(|\nabla u|^2 \right) = \sum_{j=1}^n D_j u D_{jn} u,
$$

we get

$$
\operatorname{div}(D_n u \nabla u) = \frac{1}{2} D_n \left(|\nabla u|^2 \right) + D_n u \Delta u.
$$

Multiplying (1.4) by $D_n u$, we have

$$
0 = \operatorname{div}(D_n u \nabla u) - \frac{1}{2} D_n \left(|\nabla u|^2 \right) + c (D_n u)^2 + D_n (F(u)).
$$

Applying the Gauss divergence formula $\int_{B(0;R)} \operatorname{div} F = \int_{\partial B(0;R)} (F, \nu)$, we obtain

$$
0 = \int_{\partial B(0;R)} D_n u (\nabla u, \nu) - \frac{1}{2} \int_{\partial B(0;R)} |\nabla u|^2 \nu_n + c \int_{B(0;R)} (D_n u)^2 + \int_{\partial B(0;R)} F(u) \nu_n.
$$

This completes the proof. $\square$

We take $\delta_0 > 0$ small enough such that we have

$$
s f(s) < 0 \qquad \text{if } s \in [-\delta_0, \delta_0] \backslash \{0\}
$$

and

$$
\begin{aligned}
-\frac{1}{2} f'(0) \le \inf_{s \in [-\delta_0, \delta_0] \backslash \{0\}} \frac{|f(s)|}{|s|} \le \sup_{s \in [-\delta_0, \delta_0] \backslash \{0\}} \frac{|f(s)|}{|s|} \le -2 f'(0) \qquad & \text{if } f'(0) < 0, \\
a \le \inf_{s \in [-\delta_0, \delta_0] \backslash \{0\}} \frac{|f(s)|}{|s|^{1+p}} \le \sup_{s \in [-\delta_0, \delta_0] \backslash \{0\}} \frac{|f(s)|}{|s|^{1+p}} < +\infty \qquad & \text{if } f \text{ satisfies (1.3).}
\end{aligned}
$$

Here $a \in (0,\infty)$ is a constant. We put

$$\alpha \stackrel{\text{def}}{=} -\frac{1}{2}f'(0) > 0$$

if $f'(0) < 0$.

For any $x_0 \in [-\delta_0, \delta_0]$ we define $T(t;x_0)$ by

$$\begin{aligned} T'(t;x_0) &= f(T(t;x_0)) \qquad t > 0, \\ T(0;x_0) &= x_0. \end{aligned}$$

Here $'$ means d/dt. If $x_0 \neq 0$, we have

$$t = \int_{x_0}^{T(t;x_0)} \frac{ds}{f(s)} = \int_{T(t;x_0)}^{x_0} \frac{ds}{-f(s)}.$$

We put

$$\varepsilon_0 \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}\min\{1, a_0 + B_1, b_0 - B_2\} & \text{if } a_0 < \infty,\ b_0 < \infty, \\ \frac{1}{2}\min\{1, b_0 - B_2\} & \text{if } a_0 = \infty,\ b_0 < \infty, \\ \frac{1}{2}\min\{1, a_0 + B_1\} & \text{if } a_0 < \infty,\ b_0 = \infty, \\ \frac{1}{2} & \text{if } a_0 = \infty,\ b_0 = \infty, \end{cases}$$

and have $0 < \varepsilon_0 \leq 1/2$ and $[B_1 - \varepsilon, B_2 + \varepsilon] \subset (-a_0, b_0)$ for all $\varepsilon \in (0, \varepsilon_0)$. For all $\varepsilon \in (0, \varepsilon_0)$ there exists $r_\varepsilon > 0$ such that

$$B_1 - \varepsilon \leq u(x) \leq B_2 + \varepsilon \qquad \text{if } |x| \geq r_\varepsilon. \tag{2.7}$$

Lemma 2. *Suppose $f'(0) < 0$. For $x_0 \in [-\delta_0, \delta_0]$ one has*

$$|T(t;x_0)| \leq |x_0| e^{-\alpha t} \qquad \text{for all } t \geq 0.$$

One has

$$\sup_{t \geq 1} e^{\alpha t} \max\{|T(t; B_1 - \varepsilon)|, |T(t; B_2 + \varepsilon)|\} < +\infty.$$

Proof. The assertion holds true for the case $x_0 = 0$. For the former statement we give a proof for the case $x_0 > 0$, since the case $x_0 < 0$ can be proved similarly. Using

$$\frac{1}{-f(s)} \leq \frac{1}{\alpha s}$$

for all $s \in (0, \delta_0]$, we get

$$\int_{T(t;x_0)}^{x_0} \frac{ds}{-f(s)} = t \leq \int_{T(t;x_0)}^{x_0} \frac{ds}{\alpha s}.$$

This gives

$$\log T(t;x_0) \leq -\alpha t + \log x_0$$

for all $t \geq 0$. The latter statement follows from the former statement. $\square$

Lemma 3. *Suppose f satisfies (1.3). For $x_0 \in [-\delta_0, 0) \cup (0, \delta_0]$ one has*

$$|T(t; x_0)| \le \left(apt + |x_0|^{-p}\right)^{-\frac{1}{p}} \qquad \text{for all } t \ge 0.$$

One has

$$\sup_{t\ge 1}(apt)^{\frac{1}{p}} \max\left\{|T(t; B_1 - \varepsilon)|, |T(t; B_2 + \varepsilon)|\right\} < +\infty.$$

Proof. For the former statement we give a proof for the case $x_0 > 0$, since the case $x_0 < 0$ can be proved similarly. We have

$$t \le \int_{T(t;x_0)}^{x_0} \frac{ds}{as^{1+p}},$$

which gives

$$t \le \frac{1}{ap}\left(T(t; x_0)^{-p} - x_0^{-p}\right).$$

The latter statement follows from the former statement. □

We define a constant $k > 0$ as

$$k \overset{\text{def}}{=} \begin{cases} 1 + \sup_{t\ge 1} e^{\alpha t} \max\left\{|T(t; B_1 - \varepsilon)|, |T(t; B_2 + \varepsilon)|\right\} & \text{if } f \text{ satisfies } f'(0) < 0, \\ 1 + \sup_{t\ge 1}(apt)^{\frac{1}{p}} \max\left\{|T(t; B_1 - \varepsilon)|, |T(t; B_2 + \varepsilon)|\right\} & \text{if } f \text{ satisfies (1.3).} \end{cases}$$

We define a positive constant β as

$$\beta \overset{\text{def}}{=} \begin{cases} \dfrac{1}{4}\min\left\{\dfrac{1}{\max\{1, |c|\}}, \dfrac{1}{\sqrt{\alpha + M}}\right\} & \text{if } f \text{ satisfies (1.2),} \\ \dfrac{1}{4}\min\left\{\dfrac{1}{\max\{1, |c|\}}, \dfrac{1}{\sqrt{1 + M}}\right\} & \text{if } f \text{ satisfies (1.3).} \end{cases}$$

Using a method in [18], we will prove the following a priori estimate.

Proposition 1. *For any $\varepsilon \in (0, \varepsilon_0)$ let $r_\varepsilon > 0$ be as in (2.7). Assume that $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies (1.4) and (1.5). In the case of (1.2) one has*

$$|u(x)| \le 2ke^{-\alpha\beta|x|} \tag{2.8}$$

if

$$|x| \ge \max\left\{1, \frac{1}{\beta}, \frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\alpha\beta^2} + \frac{1}{\alpha\beta}\log\left(\frac{2A(r_\varepsilon)^n}{k(\pi\beta)^{\frac{n}{2}}}\right)\right\}.$$

In the case of (1.3) one has

$$|u(x)| \le 2k\left(ap\beta|x|\right)^{-\frac{1}{p}} \tag{2.9}$$

if

$$|x| \ge \max\left\{1, \frac{1}{\beta}, \frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta^2} + \frac{1}{\beta}\log\left(\frac{2A(r_\varepsilon)^n a^{\frac{1}{p}}}{k(\pi\beta)^{\frac{n}{2}} e^{\frac{1}{p}}}\right)\right\}.$$

Proof. We only give a proof for the upper estimate on u, since the lower estimate on u can be proved similarly.

We put $w(x,t) \overset{\text{def}}{=} u(x) - T(t; B_2 + \varepsilon)$ for $x \in \boldsymbol{R}^n$ and $t > 0$. Then we have

$$w_t - \Delta w - cD_n w = f(u(x)) - f(T(t; B_2 + \varepsilon))$$

for $x \in \boldsymbol{R}^n$ and $t > 0$. Thus we have

$$\begin{aligned} w_t - \Delta w - cD_n w - w \int_0^1 f'(\theta u(x) + (1-\theta)T(t; B_2 + \varepsilon))\, d\theta = 0 \qquad & \text{for all } x \in \boldsymbol{R}^n,\ t > 0, \\ w(x,0) = u(x) - B_2 - \varepsilon \qquad & \text{for all } x \in \boldsymbol{R}^n. \end{aligned}$$

We introduce $\widetilde{w}(x,t)$ by

$$\begin{aligned} \widetilde{w}_t - \Delta \widetilde{w} - cD_n \widetilde{w} - \widetilde{w} \int_0^1 f'(\theta u(x) + (1-\theta)T(t; B_2 + \varepsilon))\, d\theta = 0 \qquad & \text{for all } x \in \boldsymbol{R}^n,\ t > 0, \\ \widetilde{w}(x,0) = \begin{cases} 2A & \text{if } |x| \le r_\varepsilon, \\ 0 & \text{if } |x| > r_\varepsilon. \end{cases} & \end{aligned}$$

By the maximum principle, we obtain

$$\begin{aligned} 0 \le \widetilde{w}(x,t) \qquad & \text{for all } x \in \boldsymbol{R}^n,\ t > 0, \\ w(x,t) \le \widetilde{w}(x,t) \qquad & \text{for all } x \in \boldsymbol{R}^n,\ t > 0. \end{aligned}$$

For the maximum principles see [17] or [9] for instance. We introduce $W(x,t)$ by

$$\begin{aligned} W_t - \Delta W - cD_n W - MW = 0 \qquad & \text{for all } x \in \boldsymbol{R}^n,\ t > 0, \\ W(x,0) = \begin{cases} 2A & \text{if } |x| \le r_\varepsilon, \\ 0 & \text{if } |x| > r_\varepsilon. \end{cases} & \end{aligned}$$

We have

$$\widetilde{w}(x,t) \le W(x,t) \qquad \text{for all } x \in \mathbb{R}^n,\ t > 0,$$

and

$$u(x) \le T(t; B_2 + \varepsilon) + W(x,t) \qquad \text{for all } x \in \mathbb{R}^n,\ t > 0. \tag{2.10}$$

Using

$$K(x,t) \overset{\text{def}}{=} e^{Mt} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(- \frac{\sum_{j=1}^{n-1} x_j^2 + (x_n + ct)^2}{4t} \right),$$

we have

$$0 < W(x,t) = 2A \int_{B(0; r_\varepsilon)} K(x - y, t)\, dy.$$

Using $B(0; r_\varepsilon) \subset [-r_\varepsilon, r_\varepsilon]^n$ we have

$$0 < W(x,t) < 2A \int_{[-r_\varepsilon, r_\varepsilon]^n} K(x - y, t)\, dy,$$

and thus

$$0 < W(x,t) < 2Ae^{Mt} \int_{\frac{x_n - r_\varepsilon + ct}{\sqrt{4t}}}^{\frac{x_n + r_\varepsilon + ct}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \prod_{j=1}^{n-1} \int_{\frac{x_j - r_\varepsilon}{\sqrt{4t}}}^{\frac{x_j + r_\varepsilon}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \tag{2.11}$$

for all $x \in \boldsymbol{R}^n$ and all $t > 0$.

Then (2.10) and (2.11) give

$$u(x) \le T(t; B_2 + \varepsilon) + 2Ag(x,t) \tag{2.12}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$, where

$$g(x,t) \stackrel{\text{def}}{=} e^{Mt} \int_{\frac{x_n - r_\varepsilon + ct}{\sqrt{4t}}}^{\frac{x_n + r_\varepsilon + ct}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \prod_{j=1}^{n-1} \int_{\frac{x_j - r_\varepsilon}{\sqrt{4t}}}^{\frac{x_j + r_\varepsilon}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz.$$

Now we study g. Using

$$\min\{(x_j + r_\varepsilon)^2, (x_j - r_\varepsilon)^2\} \ge |x_j|^2 - 2r_\varepsilon |x_j| + r_\varepsilon^2 = (|x_j| - r_\varepsilon)^2,$$

we have

$$0 < \int_{\frac{x_j - r_\varepsilon}{\sqrt{4t}}}^{\frac{x_j + r_\varepsilon}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \exp\left(-\frac{(|x_j| - r_\varepsilon)^2}{4t} \right) \qquad \text{if } |x_j| > r_\varepsilon,$$

and

$$0 < \int_{\frac{x_j - r_\varepsilon}{\sqrt{4t}}}^{\frac{x_j + r_\varepsilon}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \qquad \text{if } |x_j| \le r_\varepsilon.$$

Combining these estimates together, we obtain

$$0 < \int_{\frac{x_j - r_\varepsilon}{\sqrt{4t}}}^{\frac{x_j + r_\varepsilon}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \exp\left(\frac{{r_\varepsilon}^2}{4t} \right) \exp\left(-\frac{(|x_j| - r_\varepsilon)^2}{4t} \right)$$

for all $x_j \in \mathbb{R}$ and $t > 0$. Using

$$\min\left\{(x_n + ct + r_\varepsilon)^2, (x_n + ct - r_\varepsilon)^2\right\} \ge (|x_n + ct| - r_\varepsilon)^2,$$

we obtain

$$\int_{\frac{x_n - r_\varepsilon + ct}{\sqrt{4t}}}^{\frac{x_n + r_\varepsilon + ct}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \exp\left(-\frac{(|x_n + ct| - r_\varepsilon)^2}{4t} \right) \quad \text{if } |x_n + ct| > r_\varepsilon,$$

and

$$\int_{\frac{x_n - r_\varepsilon + ct}{\sqrt{4t}}}^{\frac{x_n + r_\varepsilon + ct}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \qquad \text{if } |x_n + ct| \le r_\varepsilon.$$

Combining these two estimates together, we obtain

$$0 < \int_{\frac{x_n - r_\varepsilon + ct}{\sqrt{4t}}}^{\frac{x_n + r_\varepsilon + ct}{\sqrt{4t}}} \frac{e^{-z^2}}{\sqrt{\pi}}\, dz \le \frac{r_\varepsilon}{\sqrt{\pi t}} \exp\left(\frac{r_\varepsilon^2}{4t}\right) \exp\left(-\frac{(|x_n + ct| - r_\varepsilon)^2}{4t}\right)$$

for all $x_n \in \mathbb{R}$ and $t > 0$. Now we obtain

$$0 < g(x,t) \le e^{Mt} \frac{(r_\varepsilon)^n}{(\pi t)^{\frac{n}{2}}} \exp\left(\frac{n(r_\varepsilon)^2}{4t}\right) \exp\left(-\frac{J(x,t)}{4t}\right),$$

where

$$J(x,t) \stackrel{\text{def}}{=} (|x_n + ct| - r_\varepsilon)^2 + \sum_{j=1}^{n-1} (|x_j| - r_\varepsilon)^2 .$$

Now we get

$$J(x,t) = |x|^2 + 2ctx_n + c^2t^2 - 2r_\varepsilon |x_n + ct| - 2r_\varepsilon \sum_{j=1}^{n-1} |x_j| + n(r_\varepsilon)^2.$$

Now we set $t = \beta |x|$ and have

$$J(x, \beta|x|) = |x|^2 + 2\beta c x_n |x| + \beta^2 c^2 |x|^2 - 2r_\varepsilon \sum_{j=1}^{n-1} |x_j| - 2r_\varepsilon |x_n + ct| + nr_\varepsilon^2.$$

Using

$$\sum_{j=1}^{n} |x_j| \le \sqrt{n}|x|,$$

we get

$$J(x, \beta|x|) \ge (1 + \beta^2 c^2 - 2\beta|c|)|x|^2 - 2r_\varepsilon \sqrt{n}|x| - 2r_\varepsilon \beta |c||x| + nr_\varepsilon^2,$$

and thus

$$J(x, \beta|x|) \ge (1 - \beta|c|)^2 |x|^2 - 2r_\varepsilon(\sqrt{n} + \beta|c|)|x| + nr_\varepsilon^2.$$

From the definition of β we have

$$(1 - \beta|c|)^2 \ge \frac{1}{2}$$

and

$$J(x, \beta|x|) \ge \frac{1}{2}|x|^2 - 2r_\varepsilon(\beta|c| + \sqrt{n})|x| + nr_\varepsilon^2.$$

Thus we get

$$0 < g(x, \beta|x|) \le \frac{(r_\varepsilon)^n}{(\pi\beta|x|)^{\frac{n}{2}}} \exp\left(\frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta}\right) \exp\left(-\left(\frac{1}{8\beta} - M\beta\right)|x|\right).$$

From the definition of β we have

$$2\alpha\beta < \frac{1}{8\beta} - M\beta \qquad \text{if } f'(0) < 0,$$
$$2\beta < \frac{1}{8\beta} - M\beta \qquad \text{if } f \text{ satisfies (1.3)}.$$

In the case of $f'(0) < 0$ we have

$$u(x) \leq 2ke^{-\alpha\beta|x|}$$

if

$$e^{\alpha\beta|x|} \geq \frac{2A(r_\varepsilon)^n}{k(\pi\beta|x|)^{\frac{n}{2}}} \exp\left(\frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta}\right).$$

This gives (2.8).

In the case of (1.3) we have

$$\begin{aligned} u(x) &\leq k\,(ap\beta|x|)^{-\frac{1}{p}} + \frac{2A(r_\varepsilon)^n}{(\pi\beta|x|)^{\frac{n}{2}}} \exp\left(\frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta}\right) e^{-2\beta|x|} \\ &\leq k\,(ap\beta|x|)^{-\frac{1}{p}} \left(1 + \frac{2A(r_\varepsilon)^n}{k(\pi\beta)^{\frac{n}{2}}} \exp\left(\frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta}\right) (ap\beta|x|)^{\frac{1}{p}} e^{-\beta|x|} e^{-\beta|x|}\right) \\ &\leq k\,(ap\beta|x|)^{-\frac{1}{p}} \left(1 + \frac{2A(r_\varepsilon)^n}{k(\pi\beta)^{\frac{n}{2}}} \exp\left(\frac{r_\varepsilon(\beta|c| + \sqrt{n})}{2\beta}\right) \left(\frac{a}{e}\right)^{\frac{1}{p}} e^{-\beta|x|}\right) \end{aligned}$$

if $|x| \geq 1$. Here we used

$$\sup_{r>0} (ap\beta r)^{\frac{1}{p}} e^{-\beta r} = \left(\frac{a}{e}\right)^{\frac{1}{p}}.$$

This gives (2.9) and completes the proof. $\square$

Let $s \in (n, +\infty)$ be arbitrarily given. If $w \in W^{2,s}_{\mathrm{loc}}(B(0;2)) \cap L^s(B(0;2))$ and $h \in L^s(B(0;2))$ satisfy

$$(-\Delta - cD_n)w = h \qquad \text{in } B(0;2),$$

one has

$$\|w\|_{W^{2,s}(B(0;1))} \leq K_0 \|h\|_{L^s(B(0;2))}$$

by the Schauder interior estimate, where K_0 is a positive constant. See [9, Theorem 9.11] for a general theory. Then $W^{2,s}(B(0;1)) \subset C^1(\overline{B(0;1)})$ and $L^\infty(B(0;2)) \subset L^s(B(0;2))$ yield

$$\|w\|_{C^1(\overline{B(0;1)})} \leq K\|h\|_{L^\infty(B(0;2))},$$

where K is a positive constant. See also [9] for the Sobolev imbedding theorems.

Proof of Theorem 1. For any $y \in \mathbb{R}^n$ we put $w(x) \stackrel{\text{def}}{=} u(x+y)$ and have

$$-\Delta w(x) - cD_n w(x) = f(w(x)) \qquad \text{for } x \in B(y;2).$$

The Schauder estimate gives

$$\|w\|_{C^1(\overline{B(0;1)})} \leq K\|f(w)\|_{L^\infty(B(0;2))},$$

which yields

$$|\nabla u(y)| \leq K \sup_{|x|\leq 2} |f(u(y+x))| \qquad \text{for all } y \in \mathbb{R}^n.$$

Especially we get

$$|\nabla u(x)| \leq KM \sup_{|y|\leq 2} |u(x+y)| \qquad \text{for all } x \in \mathbb{R}^n.$$

In the case of (1.2) we obtain

$$|\nabla u(x)| \leq 2kKMe^{-\alpha\beta(|x|-2)} \leq 2kKMe^{2\alpha\beta}e^{-\alpha\beta|x|}$$

if

$$|x| \geq \max\left\{3, \frac{1}{\beta}, \frac{r_\varepsilon(\beta|c|+\sqrt{n})}{2\alpha\beta^2} + \frac{1}{\alpha\beta}\log\left(\frac{2A(r_\varepsilon)^n}{k(\pi\beta)^{\frac{n}{2}}}\right)\right\}.$$

In the case of (1.3) we obtain

$$|\nabla u(x)| \leq 2kKM3^{\frac{1}{p}}(ap\beta|x|)^{-\frac{1}{p}}$$

if

$$|x| \geq \max\left\{3, \frac{1}{\beta}, \frac{r_\varepsilon(\beta|c|+\sqrt{n})}{2\beta^2} + \frac{1}{\beta}\log\left(\frac{2A(r_\varepsilon)^n a^{\frac{1}{p}}}{k(\pi\beta)^{\frac{n}{2}}e^{\frac{1}{p}}}\right)\right\}.$$

Using estimates on u and $|\nabla u|$, we get

$$\left|\int_{\partial B(0;R)} \left(\frac{1}{2}|\nabla u|^2 - F(u)\right)\nu_n - \int_{\partial B(0;R)} D_n u(\nabla u, \nu)\right| \leq \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}R^{n-1}\left(\frac{3}{2}\max_{\partial B(0;R)}|\nabla u|^2 + \frac{M}{2}\max_{\partial B(0;R)}|u|^2\right).$$

Here Γ is the Gamma function. Recall that $2\pi^{n/2}/\Gamma(n/2)$ is the area of a unit ball in $\mathbb{R}^n$. If $f'(0) < 0$, we have

$$\lim_{R\to\infty}\left|\int_{\partial B(0;R)} \left(\frac{1}{2}|\nabla u|^2 - F(u)\right)\nu_n - \int_{\partial B(0;R)} D_n u(\nabla u, \nu)\right| = 0$$

from Proposition 1.

If f satisfies (1.3), we have

$$\max_{\partial B(0;R)}|u| = O(R^{-\frac{1}{p}}), \quad \max_{\partial B(0;R)}|\nabla u| = O(R^{-\frac{1}{p}})$$

as $R \to \infty$ using Proposition 1. Thus we obtain

$$\left|\int_{\partial B(0;R)} \left(\frac{1}{2}|\nabla u|^2 - F(u)\right)\nu_n - \int_{\partial B(0;R)} D_n u(\nabla u, \nu)\right| = O(R^{n-1-\frac{2}{p}})$$

as $R \to \infty$. Using

$$n - 1 - \frac{2}{p} < 0$$

and sending $R \to \infty$ in (2.6) in Lemma 1, we obtain

$$c \int_{\mathbb{R}^n} (D_n u)^2 = 0$$

for both cases. This equality gives $c = 0$ or $D_n u \equiv 0$. If $D_n u \equiv 0$, the condition $\lim_{|x| \to \infty} u(x) = 0$ implies $u \equiv 0$. Thus we find that $c = 0$ or $u \equiv 0$. This completes the proof of Theorem 1. $\square$

3 Applications

Let $n \geq 3$, $1 < q < (n+2)/(n-2)$ and $f(u) = -u + |u|^{q-1}u$. Equation (1.4) for $c = 0$ is called the scalar field equation. It is well known that there exists a unique positive U of class $C^2[0, \infty)$ satisfying

$$\begin{aligned} U''(r) + \frac{n-1}{r} U'(r) + f(U(r)) = 0 \quad \text{for all } r > 0, \\ U'(0) = 0, \quad \lim_{r \to \infty} U(r) = 0. \end{aligned}$$

If $u > 0$ satisfies

$$\Delta u + f(u) = 0 \qquad x \in \mathbb{R}^n,$$

$u(x) = U(|x - y|)$ for some $y \in \mathbb{R}^n$ from [7, 8]. From Theorem 1 in this paper one has $c = 0$ if $u \not\equiv 0$ satisfies (1.4) and (1.5) for some $c \in \mathbb{R}$. Thus there exist no localized travelling waves with non-zero speed for $f(u) = -u + |u|^{q-1}u$.

We give another example of f that satisfies (1.3). Let $f(u) = u|u|(u-1)$. If $u > 0$ satisfies

$$\Delta u + f(u) = 0 \qquad x \in \mathbb{R}^n,$$

u has to be spherically symmetric for some point in $\mathbb{R}^2$ from [13]. For an example of spherically symmetric solution is as follows. For $n = 2$, the following equation

$$\begin{aligned} U''(r) + \tfrac{1}{r} U'(r) + f(U(r)) = 0 \quad \text{for all } r > 0, \\ \lim_{r \to \infty} U(r) = 0. \end{aligned} \tag{3.13}$$

has a solution

$$U(r) = \frac{4}{r^2 + 2}.$$

From Theorem 1 one has $c = 0$ if $n = 1, 2$ and $u \not\equiv 0$ satisfies (1.4) and (1.5) for some $c \in \mathbb{R}$. Thus there exist no localized travelling waves with non-zero speed for $f(u) = u|u|(u-1)$ when $n = 1, 2$. However our assumption does not hold true when $p = 1$ and $n \geq 3$. Thus it is an interesting open problem to prove the existence or non-existence of travelling waves for $f = u|u|(u-1)$ when $n \geq 3$ or more generally for f with (1.3) when $n \geq 1 + (2/p)$.

References

[1] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics*, Partial Differential Equations and Related Topics, ed. J. A. Goldstein, Lecture Notes in Mathematics, 446 (1975) 5–49.

[2] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math., 30, No.1 (1978), 33–76.

[3] H. Berestycki, P.-L. Lions, L. A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbf{R}^N$, Indiana Univ. Math. J. 30, No.1 (1981) 141–157.

[4] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., 82, No. 4 (1983) 313–345.

[5] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal., 82, No. 4, (1983) 347–375.

[6] S.-I. Ei, M. Mimura and M. Nagayama, *Interacting spots in reaction diffusion systems*, Discrete Contin. Dyn. Syst., 14, No. 1 (2006) 31–62.

[7] B. Gidas, W.-M. Ni and L.Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68, No. 3 (1979) 209–243.

[8] B. Gidas, and W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud., 7 (1981) 369–402, Academic Press, New York,

[9] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, 1983.

[10] S. Kawaguchi and M. Mimura, *Collision of travelling waves in a reaction-diffusion system with global coupling effect*, SIAM J. Appl. Math., 59, No. 3 (1999) 920–941.

[11] A. Kolmogoroff, I. Petrovsky and N. Piscounoff, *Study of the diffusion equation with growth of the quantity of matter and its application to a biology problem*, Moscow University Bulletin of Mathematics, 1, (1937) 1–25.

[12] M.-K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R}^n$, Arch. Rational Mech. Anal., 105, No. 3 (1989) 243–266.

[13] Y. Li and W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Comm. Partial Differential Equations, 18, (1993) 1043–1054

[14] Z. Nehari, *On a nonlinear differential equation arising in nuclear physics*, Proc. Roy. Irish Acad. Sect. A, 62 (1963) 117–135.

[15] Y. Nishiura, T. Teramoto and K.-I. Ueda, *Scattering of traveling spots in dissipative systems*, Chaos, 15, No. 4 (2005) 047509, 10.

[16] L. A. Peletier and J. Serrin, *Uniqueness of positive solutions of semilinear equations in* $\mathbf{R}^n$, Arch. Rational Mech. Anal., 81 (1983) 181–197.

[17] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations," Springer-Verlag, Berlin, 1984.

[18] M. Taniguchi, *Multi-dimensional traveling fronts in bistable reaction-diffusion equations equations*, Discrete Contin. Dyn. Syst., 32, No. 3 (2012) 1011–1046.

[19] E. Yanagida, *Uniqueness of positive radial solutions of* $\Delta u + f(u, |x|) = 0$, Nonlinear Anal., 19, No. 12 (1992) 1143–1154.