An explicit solution of Burgers equation with stationary point source

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Abstract

Existence, uniqueness and regularity of the global weak solution to the Burgers equation with a reaction term is shown when the reaction term is given as a time independent point source and produces heat constantly. An explicit solution is obtained and used to show the long time asymptotic convergence of the solution to a steady state. For the heat equation case without any convection the solution diverges everywhere as time increases and hence it is the first order convection term that gives the compactness of the solution trajectory of the Burgers equation with reaction.

Key words: Cole-Hopf transformation, heat equation, long time asymptotics

1 Introduction

In many occasions a solution to an elliptic equation is understood as the longtime asymptotic limit of a solution to a parabolic equation. A study of such a convergence provides a good chance to understand the connection between

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the two groups of partial differential equations. However, there are subtle issues in studying such longtime asymptotics, which are overlooked in many cases. The purpose of this paper is to develop an explicit example to understand such subtle issues related to the roles of advection and diffusion. Consider the heat equation with a positive heat source:

$$\begin{aligned} &u_t - u_{xx} = \delta, \qquad && x \in \mathbf{R},\ 0 < t, \\ &u(x,0) = u_0(x), \qquad && x \in \mathbf{R}, \end{aligned} \tag{1.1}$$

where the heat source $\delta = \delta(x)$ is the time independent Dirac delta measure and the initial data u_0 is in $L^1(\mathbf{R})$. Notice that it is not the initial value, but the source that is remembered by the elliptic limit of the parabolic problem. Formally, one can see that the total heat energy increases constantly, i.e.,

$$\frac{d}{dt}\int u(x,t)dx = \int \delta(x)dx = 1.$$

Therefore, the solution does not converge in $L^1(\mathbf{R})$ as $t \to \infty$. Then, can we expect a pointwise convergence? [1]

Intuitively one might guess that as $t \to \infty$ the solution u would approach one of the steady states. Here, a steady state solution, say ω, of course satisfies

$$-\omega_{xx} = \delta$$

so that it can be written as a sum of the fundamental solution of Laplace's equation and a harmonic function $h(x)$:

$$\omega(x) = -|x|/2 + h(x). \tag{1.2}$$

However, this guess for the asymptotic behavior as $t \to \infty$ is wrong. Observe the solution u, which is explicitly given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbf{R}} e^{-y^2/4t}u_0(x-y)\,dy + \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-x^2/4(t-\tau)}\,d\tau.$$

The first term is from the initial heat distribution and vanishes as $t \to \infty$. The second term is from the inhomogeneous heat source and equals $\sqrt{t/\pi}$ at $x = 0$, which diverges to $+\infty$ with order $O(\sqrt{t}\,)$ as $t \to \infty$. If $x \neq 0$, by introducing $\xi = x^2/4(t-\tau)$, the second term is written as

$$\frac{|x|}{4\sqrt{\pi}}\int_{x^2/4t}^{\infty} \xi^{-3/2}e^{-\xi}\,d\xi.$$

[1] The answer depends on the space dimensions. For example, the answer is affirmative if the space dimension is $n \geq 3$. See [16] for more discussions.

Then, for a fixed x, the lower limit of integration $\frac{x^2}{4t} \to 0$ as $t \to \infty$ so that again the second term diverges to $+\infty$ as $t \to \infty$ since the integral $\int_0^1 \xi^{-3/2} d\xi$ diverges. Therefore the solution u diverges everywhere as $t \to \infty$, and the solution trajectory cannot be compact in any L^p space even on compact sets. A common belief is that diffusion is an elementary process that carries the effect of a heat source away from its point of application. The example given here shows that this is not true. The diffusion alone does not carry our constantly produced heat away from the heat source and hence the temperature blows up.

One might ask that, if a first order convection term is added, will it make the solution trajectory compact? Again intuitively one might guess "no" since only a lower order term is added to the second order equation and the higher order term usually decides intrinsic properties. In the aspect of regularity, the second order diffusion term 'u_{xx}' plays the main role so that one might expect the compactness of the solution trajectory would also be decided by the second order diffusion term. However, the guess is wrong again and this paper is devoted to this issue. The role of diffusion and convection has been intensively studied without an inhomogeneous source term but is less understood with one. Notice that it is not the initial distribution but the inhomogeneous source term that decides the asymptotic behavior and connects an elliptic problem to a parabolic one.

The purpose of this paper is to provide a clear view on the role of diffusion and convection when a non-autonomous reaction term produces heat constantly. Specifically we will consider the viscous Burgers equation with a positive heat source:

$$\begin{aligned} &u_t + uu_x - u_{xx} = \delta, \qquad x \in \mathbf{R},\ 0 < t, \\ &u(x,0) = u_0(x), \qquad\quad\ \ x \in \mathbf{R}, \end{aligned} \tag{1.3}$$

where the initial data u_0 is in $C^1(\mathbf{R}) \cap L^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$. The first main result of this paper is proof of existence, uniqueness and regularity of the solution to (1.3), Theorem 3.2, which will be obtained by constructing the solution explicitly. The explicit solution of the homogeneous Burgers equation has served as the foundation of the nonlinear theory since the pioneering work of E. Hopf [14]. Similarly, the explicit formula for the solution of this inhomogeneous case can be used to understand the dynamics of diffusion and convection in the presence of a heat source.

One may easily check that

$$w(x) = \begin{cases} 2/(\sqrt{2} - x), & x \le 0, \\ \sqrt{2}, & 0 < x \end{cases} \tag{1.4}$$

is a continuous steady state solution of (1.3), which is strictly positive everywhere. Note that the steady state solution (1.2) of the heat equation (1.1) cannot be nonnegative for any harmonic function $h(x)$. Hence this steady state solution with a convection term makes more sense in the presence of a nonnegative heat source. The second main result of this paper is a proof of asymptotic convergence of solutions to the steady state (1.4) as $t \to \infty$ in every L^p norm on compact sets, Theorem 3.5. Hence it is not the higher order diffusion, but the lower order convection that makes the solution trajectory compact. This result shows that our intuition, which usually based on the familiarity with problems without sources, may fail in the presence of a nonnegative source.

The role of diffusion and convection has been intensively studied *without* a source term. For example, consider a convection-diffusion equation

$$u_t + u^q u_x - (u^m)_{xx} = 0, \quad m, q > 0.$$

Since there is no heat source in this example, the solution vanishes as $t \to \infty$ and the scale of decay is decided by the dominating factor:

$$\|u(t)\|_\infty = O(t^{\frac{-1}{\alpha+1}}) \quad \text{as} \quad t \to \infty \;\; \text{with} \;\; \alpha = \min\{m, q\}. \tag{1.5}$$

See [5,9–11,22] and references therein for further discussions. In particular, for the Burgers equation case, $m = q = 1$, higher order asymptotics has been obtained using the Cole-Hopf transformation [6,17–19].

The decay rate in (1.5) indicates which one is the dominant factor of the evolution. For example, if $m < q$, the dynamics is dominated by the diffusion and, if $q < m$, then by the convection. Hence, the Burgers equation is the case under which the two of them are balanced. However, it is only for the homogeneous case; when there is a constant heat source, compactness of the solution trajectory is given by the convection, not by the diffusion, for the 'balanced' Burgers equation case. Hence, a better understanding seems needed for the dynamics of diffusion and reaction in the presence of a heat source.

Parabolic equations with a source term have been studied when the source term is *mild* enough so that it is integrable in some sense with respect to time and space. Hence the total heat is finite and the solution may decay asymptotically with the same rate as the homogeneous case. For diffusion equations with a mild source term, one can find an asymptotic profile in a similar way to the case without a source term [8,21]. For convection-diffusion equations, Schonbek [20] proved existence of a unique mild solution and its decay in L^p-norm. However, little is known when the source term is strong enough to change the asymptotics of the solution. Using a contraction mapping argument, Dix [7] proved local well-posedness of inhomogeneous viscous Burgers equation. For example, his result yields that our problem (1.3) is locally well-posed in L^2-norm. But as far as the authors know, asymptotic behavior of inhomogeneous viscous Burgers equation has not been studied.

The blow up of the heat equation (1.1) is closely related to the nonexistence of a positive steady state. Nonexistence of a positive solution has been intensively studied for nonlinear elliptic problems such as

$$-\Delta u = u^p, \quad -\Delta u = u^p + f(x), \quad \text{or} \quad -\Delta u = -V(x)u + u^p,$$

where the range of p depends on the dimension n. For $n \leq 2$, the nonexistence was shown for all $p < \infty$ and, for $n \geq 3$, the range of p depends on the dimension n, the potential V and the source f. For detailed discussions readers are referred to [1–3,12]. In the aspect of our result, a natural extension of these nonlinear elliptic problems is finding a positive solution of them after adding an advection term. Such a study may connect the theory of nonlinear elliptic problems to parabolic ones with reaction.

The paper is organized as follows. First a natural candidate for the solution of (1.3) is constructed in an explicit way via the Cole-Hopf transformation in Section 2. Then, in Section 3, this candidate solution is shown to be the unique weak solution. It is also shown in the section that the weak solution converges pointwise to the steady state solution (1.4). In appendix, the explicit solution with the zero initial value is given numerically to demonstrate how the solution converges to the steady state.

2 Cole-Hopf transformation for the Burgers equation with source

Consider the Cole-Hopf transformation

$$\Theta(x,t) := \exp\Big\{-\frac{1}{2}\int_{-\infty}^{x} u(y,t)\,dy\Big\}, \quad \Theta_0(x) := \exp\Big\{-\frac{1}{2}\int_{-\infty}^{x} u_0(y)\,dy\Big\}.$$

If u satisfies (1.3) with $u_0 \in C^1(\mathbf{R}) \cap L^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$, then $\Theta_0 \in C^2(\mathbf{R}) \cap W^{2,\infty}(\mathbf{R})$ and the transformation Θ satisfies

$$\begin{cases} \Theta_t - \Theta_{xx} = -\frac{1}{2}H(x)\,\Theta, & x \in \mathbf{R},\ 0 < t, \\ \Theta(x,0) \quad = \Theta_0(x), & x \in \mathbf{R}, \end{cases} \tag{2.1}$$

where $H(x)$ is the Heaviside function. The solution of this transformed problem is constructed by combining two solutions on domains $\{0 < x\}$ and $\{x < 0\}$.

Let R be the solution on the right side domain $\{0 < x\}$, which satisfies the following initial-boundary value problem:

$$\begin{cases} R_t - R_{xx} &= -\frac{1}{2}R, & 0 < x,\ 0 < t, \\ R(x,0) &= \Theta_0(x), & 0 \le x, \\ R(0,t) &= g(t), & 0 < t. \end{cases}$$

The boundary condition $g(t)$ will be decided later and we assume it is continuously differentiable for now. The inhomogeneous right side is due to the point source and we rescale R. Define

$$\widetilde{R}(x,t) := e^{t/2}\, R(x,t),$$

which satisfies

$$\begin{cases} \widetilde{R}_t - \widetilde{R}_{xx} &= 0, & 0 < x,\ 0 < t, \\ \widetilde{R}(x,0) &= \Theta_0(x), & 0 \le x, \\ \widetilde{R}(0,t) &= e^{t/2} g(t), & 0 < t. \end{cases}$$

One may find the solution to this problem in [15, p18 and p22]). If one returns back to R, then, for $x > 0$,

$$\begin{aligned} R(x,t) = e^{-t/2}\Big[&\tfrac{1}{2\sqrt{\pi t}} \textstyle\int_0^\infty \Theta_0(\xi)\Big(e^{-(\xi-x)^2/4t} - e^{-(\xi+x)^2/4t}\Big)\, d\xi \\ &+ \textstyle\int_0^t \Big(e^{\tau/2} g(\tau)\Big)' \operatorname{erfc}\Big(\tfrac{x}{2\sqrt{t-\tau}}\Big)\, d\tau + g(0)\operatorname{erfc}\Big(\tfrac{x}{2\sqrt{t}}\Big)\Big]. \end{aligned} \tag{2.2}$$

Similarly, let L be the solution on the left side domain $\{x < 0\}$, which satisfies

$$\begin{cases} L_t - L_{xx} &= 0, & x < 0,\ 0 < t, \\ L(x,0) &= \Theta_0(x), & x \le 0, \\ L(0,t) &= g(t), & 0 < t. \end{cases}$$

Then, for $x < 0$,

$$\begin{aligned} L(x,t) = &\tfrac{-1}{2\sqrt{\pi t}} \textstyle\int_0^\infty \Theta_0(-\xi)\Big(e^{-(\xi-x)^2/4t} - e^{-(\xi+x)^2/4t}\Big)\, d\xi \\ &+ \textstyle\int_0^t g'(\tau) \operatorname{erfc}\Big(\tfrac{-x}{2\sqrt{t-\tau}}\Big)\, d\tau + g(0)\operatorname{erfc}\Big(\tfrac{-x}{2\sqrt{t}}\Big). \end{aligned} \tag{2.3}$$

Now we decide the boundary condition $g(t)$. Since Θ is continuously differentiable, we impose a condition

$$\Theta_x(0+,t) = \Theta_x(0-,t) \quad \text{for } 0 < t,$$

where $0+$ and $0-$ denote the right and left side limits, respectively. Then we have $R_x(0+,t) = L_x(0-,t)$ for $0 < t$. A direct computation shows

$$\begin{aligned} R_x(x,t) = & \tfrac{e^{-t/2}}{4\sqrt{\pi}\,t^{3/2}} \int_0^\infty \Theta_0(\xi)\Big((\xi - x)e^{-(\xi-x)^2/4t} + (\xi + x)e^{-(\xi+x)^2/4t}\Big)\,d\xi \\ & -\tfrac{e^{-t/2}}{\sqrt{\pi}} \int_0^t \Big(e^{\tau/2}g(\tau)\Big)'\, \tfrac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}\,d\tau - \tfrac{e^{-t/2}g(0)}{\sqrt{\pi}}\,\tfrac{e^{-x^2/4t}}{\sqrt{t}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_x(x,t) = & \tfrac{-1}{4\sqrt{\pi}\,t^{3/2}} \int_0^\infty \Theta_0(-\xi)\Big((\xi - x)e^{-(\xi-x)^2/4t} + (\xi + x)e^{-(\xi+x)^2/4t}\Big)\,d\xi \\ & +\tfrac{1}{\sqrt{\pi}} \int_0^t g'(\tau)\, \tfrac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}\,d\tau + \tfrac{g(0)}{\sqrt{\pi}}\,\tfrac{e^{-x^2/4t}}{\sqrt{t}}. \end{aligned} \tag{2.5}$$

Hence the continuous differentiability of Θ implies that, for every $0 < t$,

$$\begin{aligned} & \tfrac{1}{2}\int_0^\infty \xi\Big(\Theta_0(\xi)e^{-t/2} + \Theta_0(-\xi)\Big)\tfrac{e^{-\xi^2/4t}}{t^{3/2}}\,d\xi - g(0)\,\tfrac{e^{-t/2}+1}{\sqrt{t}} \\ & \quad = \int_0^t \Big(g(\tau)/2 + g'(\tau)\Big)\,\tfrac{e^{-(t-\tau)/2}}{\sqrt{t-\tau}}\,d\tau + \int_0^t g'(\tau)\,\tfrac{1}{\sqrt{t-\tau}}\,d\tau. \end{aligned} \tag{2.6}$$

This integral equation decides a unique boundary condition $g(t)$.

Proposition 2.1 *There is a unique C^1 function $g(t)$ satisfying (2.6) for a given $\Theta_0 \in C^2(\mathbf{R}) \cap W^{2,\infty}(\mathbf{R})$ such that $g(0) = \Theta_0(0)$. Furthermore, there exist positive constants A and B such that*

$$|g(t)| \le Ae^{Bt} \quad \textit{for all } 0 \le t.$$

PROOF. Integration of the right hand side of (2.6) by parts gives

$$\begin{aligned} & \tfrac{1}{2}\int_0^\infty \xi\Big(\Theta_0(\xi)e^{-t/2} + \Theta_0(-\xi)\Big)\tfrac{e^{-\xi^2/4t}}{t^{3/2}}\,d\xi - g(0)\,\tfrac{e^{-t/2}+1}{\sqrt{t}} \\ & \quad = \sqrt{2\pi}\,g(0)\,\mathrm{erf}\Big(\sqrt{\tfrac{t}{2}}\Big) + \int_0^t \tfrac{g'(\tau)}{\sqrt{t-\tau}}\Big(\sqrt{2\pi(t-\tau)}\,\mathrm{erf}\Big(\sqrt{\tfrac{t-\tau}{2}}\Big) + e^{\frac{-(t-\tau)}{2}} + 1\Big)\,d\tau \end{aligned}$$

for all $0 < t$. Let $h(\tau) := g'(\tau)$. Then the above equation can be written as an *Abel integral equation*:

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{K(t,\tau)\,h(\tau)}{\sqrt{t-\tau}}\,d\tau = f(t) \quad \text{for all } 0 < t, \tag{2.7}$$

where

$$\begin{aligned} K(t,\tau) = & \frac{1}{2}\Big(\sqrt{2\pi(t-\tau)}\,\mathrm{erf}\Big(\sqrt{(t-\tau)/2}\Big) + e^{-(t-\tau)/2} + 1\Big), \\ f(t) = & \frac{1}{2\sqrt{\pi}}\int_0^\infty \xi(\Theta_0(\xi)e^{-t/2} + \Theta_0(-\xi))\frac{e^{-\xi^2/4t}}{t^{3/2}}\,d\xi \\ & -g(0)\Big(\frac{e^{-t/2}+1}{\sqrt{\pi t}} + \sqrt{2}\,\mathrm{erf}\Big(\sqrt{t/2}\,\Big)\Big). \end{aligned}$$

Then $K(t,\tau)$ is continuous, $K(t,t)=1$ and $\dfrac{\partial K}{\partial t}(t,\tau)$ is bounded for all t and τ such that $0\le\tau\le t<\infty$.

Integration by parts and the choice of $g(0)=\Theta_0(0)$ yield that

$$\begin{aligned} f(t) &= \frac{2}{\sqrt{\pi}}\int_0^\infty \Big(\Theta_0'(2\sqrt{t}\,\eta)e^{-t/2} - \Theta_0'(-2\sqrt{t}\,\eta)\Big)e^{-\eta^2}\,d\eta - \sqrt{2}\,\Theta_0(0)\,\mathrm{erf}\Big(\sqrt{t/2}\Big),\\ f'(t) &= \frac{2}{\sqrt{\pi t}}\int_0^\infty \Big(\Theta_0''(2\sqrt{t}\,\eta)\eta e^{-t/2} + \Theta_0''(-2\sqrt{t}\,\eta)\eta - \Theta_0'(2\sqrt{t}\,\eta)/2\Big)e^{-\eta^2}\,d\eta\\ &\quad - \frac{\Theta_0(0)}{\sqrt{\pi t}}e^{-t/2}. \end{aligned}$$

Hence $f(0)=0$ and $|f'(t)|\le C/\sqrt{t}$ for all $0<t$, where the constant C depends only on Θ_0. Therefore, we have

$$\begin{aligned} (Tf)(t) &:= \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\\ &= \frac{2}{\sqrt{\pi}}\frac{d}{dt}\int_0^t f'(\tau)\sqrt{t-\tau}\,d\tau = \frac{1}{\sqrt{\pi}}\int_0^t \frac{f'(\tau)}{\sqrt{t-\tau}}\,d\tau \end{aligned}$$

so that

$$|(Tf)(t)| \le \frac{C}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}\,\sqrt{t-\tau}}\,d\tau = C\sqrt{\pi},$$

i.e., $(Tf)(t)$ is continous and bounded for $t\ge 0$. Hence by [13, Theorem 5.1.3 and Theorem 5.1.4], there is a unique continuous solution $h(t)$ of the Abel integral equation (2.7) satisfying

$$|h(t)| \le e^{2Mt}\,\|Tf\|_{L^\infty([0,\infty))} \le C\sqrt{\pi}\,e^{2Mt}$$

where $M := \sup_{\tau\le t}\left|\frac{\partial K}{\partial t}(t,\tau)\right|$. Finally if we let $g(t) := \Theta_0(0)+\int_0^t h(\tau)\,d\tau$ then we can easily verify that g satisfies all the properties in the statement. □

Therefore, with this g, the solution of (2.1) is well defined and given by

$$\Theta(x,t) = \begin{cases} R(x,t), & 0\le x,\\ L(x,t), & x<0, \end{cases} \tag{2.8}$$

where the functions R and L are given in (2.2) and (2.3) respectively.

Example 2.2 (Explicit boundary condition for $u_0\equiv 0$) *In certain cases the boundary condition $g(t)$ satisfying (2.6) is given explicitly. Notice that the right hand side of (2.6) is a sum of two convolutions:*

$$\Big\{\Big(g(\tau)/2+g'(\tau)\Big) * \frac{e^{-\tau/2}}{\sqrt{\tau}}\Big\}(t) + \Big\{g'(\tau) * \frac{1}{\sqrt{\tau}}\Big\}(t).$$

The Laplace transform with respect to t on the both sides of (2.6) *yields that*

$$\begin{aligned} &\textstyle\int_0^\infty \Theta_0(\xi)e^{-\xi\sqrt{s+1/2}} + \Theta_0(-\xi)e^{-\xi\sqrt{s}}\, d\xi - g(0)\Big(\frac{1}{\sqrt{s+1/2}} + \frac{1}{\sqrt{s}}\Big) \\ &\quad = \left(G(s)/2 + sG(s) - g(0)\right)\frac{1}{\sqrt{s+1/2}} + \left(sG(s) - g(0)\right)\frac{1}{\sqrt{s}}\,, \end{aligned}$$

where $G(s)$ is the Laplace transform of $g(t)$. The solution of this algebraic equation is given by

$$G(s) = \frac{\sqrt{s+1/2}}{s+1/2+\sqrt{s(s+1/2)}}\int_0^\infty \Theta_0(\xi)e^{-\xi\sqrt{s+1/2}} + \Theta_0(-\xi)e^{-\xi\sqrt{s}}\, d\xi. \quad (2.9)$$

Now Proposition 2.1 and [4, Theorem 6.27] allows us to take the inverse Laplace transform to recover the function $g(t)$. For instance, if $u_0 \equiv 0$, then $\Theta_0 \equiv 1$ and $G(s) = 1/\sqrt{(s+1/4)^2 - (1/4)^2}$. Hence the inverse Laplace transform is

$$g(t) = e^{-t/4}\, I_0(t/4), \quad (2.10)$$

where I_0 is the modified Bessel function of the first kind. □

By definition, the transformed initial value satisfies $\Theta_0(-\infty) = 1$ formally. If we impose a stronger condition, that is,

$$x\,(\Theta_0(x) - 1) \to C \quad \text{as } x \to -\infty \quad (2.11)$$

for some real constant C, then the boundary condition $g(t)$ satisfies a common asymptotic behavior as the following lemma shows. In terms of the original initial value u_0, the condition (2.11) holds, for example, if

$$|u_0(x)| = O(|x|^{-2-\epsilon}) \quad \text{as } x \to -\infty$$

for some $\epsilon > 0$. In this case the constant C is zero:

$$\begin{aligned} |x|\,|\Theta_0(x) - 1| &\le |x|\left(\exp\left\{\frac{1}{2}\int_{-\infty}^x |u_0(y)|\, dy\right\} - 1\right) \simeq |x|\int_{-\infty}^x |u_0(y)|\, dy \\ &\lesssim |x|\int_{-\infty}^x |y|^{-2-\epsilon}\, dy \simeq |x|^{-\epsilon} \to 0 \quad \text{as } x \to -\infty. \end{aligned}$$

The conclusion in the following lemma is essential to obtain the asymptotics of u in the following section.

Lemma 2.3 *Let $g(t)$ be the boundary condition obtained in Proposition 2.1. If Θ_0 satisfies the condition* (2.11), *then*

$$\lim_{t\to\infty} \sqrt{t}\, g(t) = \sqrt{2/\pi} \quad \text{and} \quad \lim_{t\to\infty} t^{3/2}\, g'(t) = -1/\sqrt{2\pi}.$$

PROOF. (Limit of $\sqrt{t}\, g(t)$) Let

$$G_K(s) := \frac{\sqrt{s+1/2}}{s+1/2+\sqrt{s(s+1/2)}}\,,$$
$$G_{[f]}(s) := \int_0^\infty \left(f(\xi)\, e^{-\xi\sqrt{s+1/2}} + f(-\xi)\, e^{-\xi\sqrt{s}} \right) d\xi.$$

Then, by (2.9), the Laplace transform of $g(t)$ is

$$G(s) = G_K(s)\, G_{[\Theta_0]}(s).$$

Also let

$$G_1(s) := G_K(s)\, G_{[1]}(s) = \frac{1}{\sqrt{(s+1/4)^2-(1/4)^2}}$$

then its inverse Laplace transform,

$$g_1(t) := \mathcal{L}^{-1}\{G_1(s)\} = e^{-t/4}\, I_0(t/4),$$

satisfies the conclusion. Now consider the remaining part

$$G_2(s) := G(s) - G_1(s) = G_K(s)\, G_{[\Theta_0-1]}(s).$$

Then we can easily verify that $G_2(s)$ decays algebraically as $s \to \infty$ so that by [4, Theorem 6.30], it has the inverse Laplace transform $g_2(t)$. We claim that $\lim\limits_{t\to\infty} t\, g_2(t) = -C/\sqrt{2}$. Because

$$\lim_{t\to\infty} t\, g_2(t) = \lim_{s\to 0} s\, \mathcal{L}\{t\, g_2(t)\} = -\lim_{s\to 0} s\, G_2'(s),$$

it suffices to show $\lim_{s\to 0} s\, G_2'(s) = C/\sqrt{2}$. Changing the variable $\eta = \xi\sqrt{s}$, we have

$$G_{[f]}(s) = \int_0^\infty f(\xi)\, e^{-\xi\sqrt{s+1/2}} + \frac{1}{\sqrt{s}} \int_0^\infty f\Big(-\frac{\eta}{\sqrt{s}}\Big)\, e^{-\eta}\, d\eta,$$
$$G_{[f]}'(s) = -\frac{1}{\sqrt{4s+2}} \int_0^\infty f(\xi)\, \xi e^{-\xi\sqrt{s+1/2}}\, d\xi - \frac{1}{2s} \int_0^\infty \frac{\eta}{\sqrt{s}}\, f\Big(-\frac{\eta}{\sqrt{s}}\Big)\, e^{-\eta}\, d\eta.$$

Hence by the Dominated Convergence Theorem, the assumption (2.11) yields that

$$\lim_{s\to 0} s\, G_{[\Theta_0-1]}'(s) = C/2.$$

Using two easy observations $\lim\limits_{s\to 0} \sqrt{s}\, G_K'(s) = -1$ and $\lim\limits_{s\to 0} G_K(s) = \sqrt{2}$, finally we have

$$\begin{aligned} s\, G_2'(s) &= \left(\sqrt{s}\, G_K'\right)\left(\sqrt{s}\, G_{[\Theta_0-1]}\right) + G_K \left(s\, G_{[\Theta_0-1]}'\right) \\ &\to (-1)\cdot 0 + \sqrt{2}\cdot (C/2) = C/\sqrt{2} \quad \text{as } s \to 0. \end{aligned}$$

Therefore $\lim_{t\to\infty} t\, g_2(t) = -C/\sqrt{2}$, which in turn implies $\lim_{t\to\infty} \sqrt{t}\, g_2(t) = 0$. Now the first statement follows:

$$\lim_{t\to\infty} \sqrt{t}\, g(t) = \lim_{t\to\infty} \sqrt{t}\, \Big(g_1(t) + g_2(t)\Big) = \lim_{t\to\infty} \sqrt{t}\, g_1(t) = \sqrt{\frac{2}{\pi}}.$$

To prove the second statement, we claim that $\lim_{t\to\infty} t^2\, g_2'(t) = C/\sqrt{2}$. Because

$$\begin{aligned}\lim_{t\to\infty} t^2\, g_2'(t) &= \lim_{s\to 0} s\, \mathcal{L}\{t^2\, g_2'(t)\} = \lim_{s\to 0} s\, \frac{d^2}{ds^2}\mathcal{L}\{g_2'(t)\} \\ &= \lim_{s\to 0} s \frac{d^2}{ds^2}\Big(s\, G_2(s) - g_2(0)\Big) = \lim_{s\to 0} 2sG_2'(s) + s^2 G_2''(s) \\ &= \sqrt{2}C + \lim_{s\to 0} s^2\, G_2''(s),\end{aligned}$$

it suffices to show

$$\lim_{s\to 0} s^2 G_2''(s) = \lim_{s\to 0} s^2 \Big(G_K'' G_{[\Theta_0-1]} + 2G_K' G_{[\Theta_0-1]}' + G_K G_{[\Theta_0-1]}''\Big) \tag{2.12}$$

is equal to $-C/\sqrt{2}$. The first two terms in (2.12) vanish because

$$\begin{aligned} s^2\, G_K'' G_{[\Theta_0-1]} &= \Big(s^{3/2} G_K''\Big)\Big(\sqrt{s} G_{[\Theta_0-1]}\Big) \to \frac{1}{2}\cdot 0 = 0 \quad \text{as } s \to 0, \\ s^2\, G_K' G_{[\Theta_0-1]}' &= \Big(sG_K'\Big)\Big(sG_{[\Theta_0-1]}'\Big) \to 0\cdot (C/2) = 0 \quad \text{as } s \to 0.\end{aligned}$$

To compute the third term in (2.12), observe that

$$\begin{aligned} G_{[f]}''(s) &= \frac{1}{4(s+\frac{1}{2})^{\frac{3}{2}}} \int_0^\infty f(\xi)\, \xi\, e^{-\xi\sqrt{s+\frac{1}{2}}}\, d\xi + \frac{1}{4(s+\frac{1}{2})} \int_0^\infty f(\xi)\, \xi\, e^{-\xi\sqrt{s+\frac{1}{2}}}\, d\xi \\ &\quad + \frac{1}{4s^2}\left(\int_0^\infty \frac{\eta}{\sqrt{s}}\, f\Big(-\frac{\eta}{\sqrt{s}}\Big)\, e^{-\eta}\, d\eta + \int_0^\infty \frac{\eta}{\sqrt{s}}\, f\Big(-\frac{\eta}{\sqrt{s}}\Big)\, \eta\, e^{-\eta}\, d\eta\right).\end{aligned}$$

Then by the Dominated Convergence Theorem and the assumption (2.11),

$$s^2 G_K G_{[\Theta_0-1]}'' = G_K\Big(s^2\, G_{[\Theta_0-1]}''\Big) \to \sqrt{2}\cdot(-C/2) = -C/\sqrt{2}.$$

Therefore $\lim_{t\to\infty} t^2\, g_2'(t) = C/\sqrt{2}$, which in turn implies that $\lim_{t\to\infty} t^{3/2}\, g_2'(t) = 0$. Hence the second statement follows:

$$\lim_{t\to\infty} t^{3/2}\, g'(t) = \lim_{t\to\infty} t^{3/2}\, \Big(g_1'(t) + g_2'(t)\Big) = \lim_{t\to\infty} t^{3/2}\, g_1'(t) = -\frac{1}{\sqrt{2\pi}}.$$

□

Lastly we prove strict positivity of Θ, which is need to validate the inverse Cole-Hopf transformation in the following section.

Proposition 2.4 *Let $\Theta_0 \in C^2(\mathbf{R}) \cap W^{2,\infty}(\mathbf{R})$. The solution Θ in* (2.8) *is strictly positive, i.e.,*

$$\Theta(x,t) > 0 \quad \textit{for all } x \in \mathbf{R} \textit{ and } 0 \le t.$$

PROOF. First we show that the boundary condition $\Theta(0,t) = g(t)$ is strictly positive for all $0 \le t$. Assume the contrary. Then, because $g(0) = \Theta_0(0) > 0$, there exists a point $a > 0$ such that $g(a) = 0$ and $g(\tau) \ge 0$ for all $\tau \le a$. Then for all $t > a$, integration by parts yields that

$$\int_0^a g'(\tau)\, \frac{e^{-(t-\tau)/2}}{\sqrt{t-\tau}}\, d\tau = -\int_0^a \frac{g(\tau)}{2} \Big(\frac{e^{-(t-\tau)/2}}{\sqrt{t-\tau}} + \frac{e^{-(t-\tau)/2}}{(t-\tau)^{3/2}} \Big)\, d\tau - g(0)\, \frac{e^{-t/2}}{\sqrt{t}}$$

and

$$\int_0^a g'(\tau)\, \frac{1}{\sqrt{t-\tau}}\, d\tau = -\int_0^a \frac{g(\tau)}{2}\, \frac{1}{(t-\tau)^{3/2}}\, d\tau - \frac{g(0)}{\sqrt{t}}.$$

Applying the identities into (2.6), we have, for $t > a$,

$$\begin{aligned} &\frac{1}{2}\int_0^\infty \xi \Big(\Theta_0(\xi) e^{-t/2} + \Theta_0(-\xi) \Big) \frac{e^{-\xi^2/4t}}{t^{3/2}}\, d\xi + \int_0^a \frac{g(\tau)}{2}\, \frac{e^{-(t-\tau)/2} + 1}{(t-\tau)^{3/2}}\, d\tau \\ &\quad = \int_a^t \Big(g(\tau)/2 + g'(\tau) \Big) \frac{e^{-(t-\tau)/2}}{\sqrt{t-\tau}}\, d\tau + \int_a^t g'(\tau)\, \frac{1}{\sqrt{t-\tau}}\, d\tau. \end{aligned}$$

Note that the left hand side has a uniform, positive lower bound for all $t \ge a$. However, if we take the limit as $t \downarrow a$, the right hand side vanishes, which is a contradiction. Therefore $\Theta(0,t) = g(t) > 0$ for all $0 \le t$.

The strict positivity of Θ follows from the maximum principle because $\Theta_0(x) \ge e^{-\|u_0\|_1/2} > 0$ and the boundary condition is strictly positive. □

3 Global weak solution and its asymptotic behavior

The weak solution of the Burgers equation with a point source (1.3) is constructed in this section using the solution Θ of the transformed problem (2.1), which has been derived in the previous section. We start with the definition of a weak solution of (1.3).

Definition 3.1 *A function $u(x,t)$ defined in $\mathbf{R} \times (0,\infty)$ is said to be a* (global) weak solution *of* (1.3) *if $u \in C([0,\infty); H^1(\mathbf{R}) \cap L^\infty(\mathbf{R}))$ and u satisfies* (1.3) *in the sense of distributions, i.e.,*

$$\int_0^\infty \int_{\mathbf{R}} u\, \phi_t + \frac{1}{2} u^2\, \phi_x - u_x\, \phi_x\, dx\, dt + \int_0^\infty \phi(0,t)\, dt - \int_{\mathbf{R}} u_0(x)\, \phi(x,0)\, dx = 0$$

for all test functions $\phi \in C_0^\infty(\mathbf{R} \times [0, \infty))$.

We now show that the inverse Cole-Hopf transform of Θ is the unique weak solution of (1.3).

Theorem 3.2 *If the initial value* u_0 *is in* $C^1(\mathbf{R}) \cap L^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$, *there exists a unique weak solution of* (1.3) *and the solution is given by*

$$u(x,t) = -2\frac{\Theta_x(x,t)}{\Theta(x,t)}, \tag{3.1}$$

where the function Θ *is given in* (2.8). *The solution is in* $C^\infty(\mathbf{R}\setminus\{0\}\times(0,\infty))$.

PROOF. (Regularity) We have verified that Θ is well defined and strictly positive under the conditions on u_0; hence $u := -2\Theta_x/\Theta$ is well defined. Since Θ is a solution of the homogeneous heat equation for $x < 0$ and $x > 0$, it is in $C^\infty(\mathbf{R} \setminus \{0\} \times (0, \infty))$. Hence, it is obvious that $u \in C^\infty(\mathbf{R} \setminus \{0\} \times (0, \infty))$.

(Existence) Now we show that u is a weak solution of (1.3). By the regularity of u,

$$-\int_0^\infty \int_{\mathbf{R}} u_x\, \phi_x\, dx\, dt = \int_0^\infty \int_{\mathbf{R}} u_{xx}\, \phi\, dx\, dt + \int_0^\infty (u_x\phi)(0+,t) - (u_x\phi)(0-,t)\, dt.$$

From the definition of u and Θ, we also have

$$\begin{aligned}
u_x(0+,t) - u_x(0-,t) &= -2\frac{\Theta_{xx}\Theta - (\Theta_x)^2}{\Theta^2}\Big|_{x=0-}^{x=0+} \\
&= -\frac{2}{\Theta(0,t)}\Big(\Theta_t(0+,t) - \Theta_t(0-,t)\Big) - 1 \\
&= -2\partial_t\Big(\ln\Theta(0+,t) - \ln\Theta(0-,t)\Big) - 1.
\end{aligned}$$

On the other hand, by a simple computation, we can verify that that u is a classical solution of the viscous Burgers equation (without a source term) if $x \neq 0$ and $0 < t$. Hence we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}} \Big(u\,\phi_t + \frac{1}{2}u^2\,\phi_x - u_x\,\phi_x\Big)\, dx\, dt + \int_0^\infty \phi(0,t)\, dt - \int_{\mathbf{R}} u_0(x)\,\phi(x,0)\, dx \\
&\quad = \int_0^\infty \Big((u_x\phi)(0+,t) - (u_x\phi)(0-,t) + \phi(0,t)\Big)\, dt \\
&\quad = -2\int_0^\infty \frac{\partial}{\partial t}\Big(\ln\Theta(0+,t) - \ln\Theta(0-,t)\Big)\,\phi(0,t)\, dt \\
&\quad = 2\int_0^\infty \Big(\ln\Theta(0+,t) - \ln\Theta(0-,t)\Big)\,\phi_t(0,t)\, dt = 0.
\end{aligned}$$

Therefore u is a weak solution of (1.3).

(Uniqueness) Let u, v be two weak solutions with the same initial data u_0. Then $e := u - v$ satisfies

$$\int_0^\infty \int_{\mathbf{R}} e\,\phi_t + \frac{1}{2}(u+v)e\,\phi_x - e_x\,\phi_x\,dx\,dt = 0$$

for all test functions $\phi \in C_0^\infty(\mathbf{R} \times [0,\infty))$. Let $\phi(x,t) = e(x,T)\,H(T-t)$ for a fixed $T > 0$. Since this function is not a test function, we cannot directly apply it. However, using an usual approximation procedure, we may have

$$\begin{aligned}
\|e(T)\|_2^2 + \int_0^T \|e_x(t)\|_2^2\,dt &= -\frac{1}{2}\int_0^T \int_{\mathbf{R}} (u+v)\,e\,e_x\,dx\,dt \\
&\le \frac{1}{2}\int_0^T \|(u+v)(t)\|_\infty\,\|e(t)\|_2\,\|e_x(t)\|_2\,dt \\
&\le \frac{1}{4}\int_0^T \|(u+v)(t)\|_\infty^2\,\|e(t)\|_2^2\,dt + \frac{1}{4}\int_0^T \|e_x(t)\|_2^2\,dt.
\end{aligned}$$

Since $T > 0$ is arbitrary, we have $\|e(t)\|_2 = 0$ for all $0 < t$ by Gronwall's inequality. □

Next we consider the asymptotics of Θ, which consists of two lemmas for R and L, respectively.

Lemma 3.3 *If Θ_0 satisfies* (2.11)*, then $\sqrt{t}\,R$ and $\sqrt{t}\,R_x$ converge* uniformly *on compact sets and the limits are*

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{t}\,R(x,t) &= \frac{x}{2\sqrt{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{\tau}{2}}\,e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\,d\tau = \sqrt{\frac{2}{\pi}}\,e^{-\frac{x}{\sqrt{2}}}, \\
\lim_{t\to\infty} \sqrt{t}\,R_x(x,t) &= -\frac{1}{2\sqrt{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{\tau}{2}}\,e^{-\frac{x^2}{4\tau}}}{\tau^{1/2}}\,d\tau = -\frac{1}{\sqrt{\pi}}\,e^{-\frac{x}{\sqrt{2}}}.
\end{aligned}$$

PROOF. Assume $0 < x \le A$ for some $A > 0$. First we prove the uniform convergence of $\sqrt{t}\,R$. Integration by parts gives that, from (2.2),

$$\begin{aligned}
&\sqrt{t}\,R(x,t) \\
&= \frac{\sqrt{t}\,e^{-\frac{t}{2}}}{\sqrt{\pi}}\left(\int_{-\frac{x}{2\sqrt{t}}}^\infty \Theta_0(2\sqrt{t}\,\eta + x)\,e^{-\eta^2}\,d\eta - \int_{\frac{x}{2\sqrt{t}}}^\infty \Theta_0(2\sqrt{t}\,\eta - x)\,e^{-\eta^2}\,d\eta\right) \\
&\quad + \frac{x}{2\sqrt{\pi}}\int_0^t \sqrt{t}\,g(t-\tau)\,\frac{e^{-\frac{\tau}{2}}\,e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\,d\tau.
\end{aligned}$$

Because the first term vanishes uniformly, we may ignore it. For the second term, if we can take the limit $t \to \infty$ inside the integral, Lemma 2.3 gives the limit in the statement. But it needs some analysis to validate the limit process. Observe that

$$
\begin{aligned}
&\left|\sqrt{t}\, R(x,t) - \sqrt{\frac{2}{\pi}}\, e^{-\frac{x}{\sqrt{2}}}\right| \\
&\quad = \frac{x}{2\sqrt{\pi}} \left| \int_0^t \sqrt{t}\, g(t-\tau)\, \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau - \int_0^\infty \sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau \right| \\
&\quad \le \frac{x}{2\sqrt{\pi}} \int_0^t \left| \sqrt{t}\, g(t-\tau) - \sqrt{\frac{2}{\pi}} \right| \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau + \frac{x}{2\sqrt{\pi}} \int_t^\infty \sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau.
\end{aligned}
$$

Changing the variable $\eta = \frac{x}{2\sqrt{\tau}}$, we can rewrite the second term and find its uniform bound:

$$
2^{nd} \text{ term} = \frac{2\sqrt{2}}{\pi} \int_0^{\frac{x}{2\sqrt{t}}} e^{-\frac{x^2}{8\eta^2}}\, e^{-\eta^2}\, d\eta \le \frac{2\sqrt{2}}{\pi} \int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2}\, d\eta = \sqrt{\frac{2}{\pi}}\, \mathrm{erf}\Big(\frac{A}{2\sqrt{t}}\Big),
$$

which vanishes uniformly. The first term needs more care to analyze. Because

$$
\sqrt{t} - \sqrt{t-\tau} \le \frac{\tau}{\sqrt{t}} \quad \text{for all } t \ge \tau > 0,
$$

the first term is bounded by

$$
\frac{x}{2\sqrt{\pi}} \int_0^t \left| \sqrt{t-\tau}\, g(t-\tau) - \sqrt{\frac{2}{\pi}} \right| \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau + \frac{x}{2\sqrt{\pi}\sqrt{t}} \int_0^t g(t-\tau)\, \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{1/2}}\, d\tau. \tag{3.2}
$$

Again by changing the variable $\eta = \frac{x}{2\sqrt{\tau}}$, the first term in (3.2) becomes

$$
\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^\infty \left| \sqrt{t-\tau}\, g(t-\tau) - \sqrt{\frac{2}{\pi}} \right| e^{-\frac{x^2}{8\eta^2}}\, e^{-\eta^2}\, d\eta. \tag{3.3}
$$

By Lemma 2.3, for any fixed $\epsilon > 0$, there exists $T_1 > 0$ such that $|\sqrt{t}\, g(t) - \sqrt{\frac{2}{\pi}}| \le \epsilon$ for all $t \ge T_1$. Also choose $T_2 > 0$ such that $\mathrm{erf}\big(\frac{A}{2\sqrt{T_2}}\big) \le \epsilon$. Assume $t \ge T_1 + T_2$. Now split the integral of (3.3) into two parts: $\int_{\frac{A}{2\sqrt{t-T_1}}}^\infty + \int_{\frac{x}{2\sqrt{t}}}^{\frac{A}{2\sqrt{t-T_1}}} d\eta$. Then in the first part, $\eta \ge \frac{A}{2\sqrt{t-T_1}} \ge \frac{x}{2\sqrt{t-T_1}}$ so that $t - \tau = t - \frac{x^2}{4\eta^2} \ge T_1$. Hence the first part is bounded by

$$
\frac{2}{\sqrt{\pi}} \int_{\frac{A}{2\sqrt{t-T_1}}}^\infty \epsilon\, e^{-\frac{x^2}{8\eta^2}}\, e^{-\eta^2}\, d\eta \le \frac{2}{\sqrt{\pi}} \int_0^\infty \epsilon\, e^{-\eta^2}\, d\eta = \epsilon.
$$

Because the first term in the integrand of (3.3) is bounded by $\left\| \sqrt{t}\, g(t) - \sqrt{\frac{2}{\pi}} \right\|_\infty$, ignoring it, the second part is bounded by

$$
\frac{2}{\sqrt{\pi}} \int_0^{\frac{A}{2\sqrt{t-T_1}}} e^{-\eta^2}\, d\eta = \mathrm{erf}\Big(\frac{A}{2\sqrt{t-T_1}}\Big) \le \mathrm{erf}\Big(\frac{A}{2\sqrt{T_2}}\Big) = \epsilon.
$$

Therefore the first term in (3.2) vanishes uniformly. Lastly, the second term in (3.2) has a uniform bound

$$\frac{A}{2\sqrt{\pi}\sqrt{t}} \int_0^\infty \|g\|_\infty \frac{e^{-\tau/2}}{\tau^{1/2}}\, d\tau = \frac{A\,\|g\|_\infty}{\sqrt{2}\sqrt{t}}.$$

Now we consider the uniform convergence of $\sqrt{t}\, R_x$. From R_x in (2.4), we have

$$\begin{aligned}
&\sqrt{t}\, R_x(x,t) \\
&= \frac{e^{-\frac{t}{2}}}{\sqrt{\pi}} \left(\int_{-\frac{x}{2\sqrt{t}}}^\infty \Theta_0(2\sqrt{t}\,\eta + x)\, \eta e^{-\eta^2}\, d\eta + \int_{\frac{x}{2\sqrt{t}}}^\infty \Theta_0(2\sqrt{t}\,\eta - x)\, \eta e^{-\eta^2}\, d\eta \right) \\
&\quad - \frac{1}{2\sqrt{\pi}} \int_0^t \left(2\sqrt{t}\, g'(t-\tau) + \sqrt{t}\, g(t-\tau) \right) \frac{e^{-\frac{\tau}{2}}\, e^{-\frac{x^2}{4\tau}}}{\tau^{1/2}}\, d\tau - \frac{g(0)}{\sqrt{\pi}}\, e^{-\frac{t}{2}}\, e^{-\frac{x^2}{4t}}.
\end{aligned}$$

The first and the third term vanish uniformly so we may ignore them. For the second term, if we can take the limit as $t \to \infty$ inside the integral, by Lemma 2.3, we have the required limit. The limit process, which is in fact uniform on compact sets, can be justified in a similar fashion as before. $\square$

Lemma 3.4 *If Θ_0 satisfies* (2.11)*, then $\sqrt{t}\, L$ and $\sqrt{t}\, L_x$ converge* uniformly *on compact sets and the limits are*

$$\lim_{t\to\infty} \sqrt{t}\, L(x,t) = \frac{\sqrt{2} - x}{\sqrt{\pi}} \quad \text{and} \quad \lim_{t\to\infty} \sqrt{t}\, L_x(x,t) = -\frac{1}{\sqrt{\pi}}.$$

PROOF. (Uniform Convergence of $\sqrt{t}\, L$) Assume $-A \le x < 0$ for some $A > 0$. First we prove the uniform convergence of $\sqrt{t}\, L$. Integrating L in (2.3) by parts, we have

$$\begin{aligned}
\sqrt{t}\, L(x,t) &= \frac{-1}{\sqrt{\pi}} \int_0^\infty \Theta_0(-2\sqrt{t}\,\eta)\, e^{-\eta^2 - \frac{x^2}{4t}}\, \sqrt{t}\left(e^{\frac{\eta x}{\sqrt{t}}} - e^{-\frac{\eta x}{\sqrt{t}}} \right) d\eta \\
&\quad + \frac{-x}{2\sqrt{\pi}} \int_0^t \sqrt{t}\, g(t-\tau)\, \frac{e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau.
\end{aligned}$$

As we did in the proof of Lemma 3.3, we can show that the second term converges uniformly to $\sqrt{2/\pi}$. The only difficulty rises when we replace the integrand $\sqrt{t}\, g(t-\tau)$ by $\sqrt{t-\tau}\, g(t-\tau)$, in which case we have an additional term

$$\frac{-x}{2\sqrt{\pi}} \int_0^t \frac{\tau}{\sqrt{t}}\, g(t-\tau)\, \frac{e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}}\, d\tau = \frac{1}{2\sqrt{\pi}} \frac{-x}{\sqrt{t}} \int_0^t g(t-\tau)\, \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{\tau}}\, d\tau.$$

However, because $\sqrt{t}\, g(t)$ is bounded, the above term is bounded by a constant multiple of

$$\frac{A}{\sqrt{t}} \int_0^t \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{t-\tau}\,\sqrt{\tau}}\, d\tau \le \frac{A}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{t-\tau}\,\sqrt{\tau}}\, d\tau = \frac{A\,\pi}{\sqrt{t}},$$

which vanishes uniformly. For the first term, we claim that it converges uniformly to

$$\frac{-1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2}\, 2\eta x\, d\eta = -\frac{x}{\sqrt{\pi}}.$$

The difference between the first term and $-x/\sqrt{\pi}$ is bounded by

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_0^\infty \Big|\Theta_0(-2\sqrt{t}\,\eta) - 1\Big| e^{-\eta^2 - \frac{x^2}{4t}}\, \sqrt{t}\left(e^{\frac{\eta|x|}{\sqrt{t}}} - e^{-\frac{\eta|x|}{\sqrt{t}}}\right) d\eta \\
&\qquad + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2}(1 - e^{-\frac{x^2}{4t}})\, \sqrt{t}\left(e^{\frac{\eta|x|}{\sqrt{t}}} - e^{-\frac{\eta|x|}{\sqrt{t}}}\right) d\eta \\
&\qquad + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2}\bigg|\sqrt{t}\left(e^{\frac{\eta x}{\sqrt{t}}} - e^{-\frac{\eta x}{\sqrt{t}}}\right) - 2\eta x\bigg|\, d\eta \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Fix an $\epsilon > 0$. Then there exists $T > 0$ such that

$$\Big|\Theta_0(-x) - 1\Big| \le \epsilon \quad \text{for all } x \ge T.$$

Now we split the integral of I_1 into two parts: $\int_0^{\frac{T}{2\sqrt{t}}} + \int_{\frac{T}{2\sqrt{t}}}^\infty d\eta =: I_1' + I_1''$. In I_1'', $2\sqrt{t}\,\eta \ge T$ so that

$$\begin{aligned}
I_1'' &\le \frac{1}{\sqrt{\pi}} \int_0^\infty \epsilon\, e^{-\eta^2}\, \sqrt{t}\left(e^{\frac{\eta|x|}{\sqrt{t}}} - e^{-\frac{\eta|x|}{\sqrt{t}}}\right) d\eta = \epsilon\, e^{\frac{x^2}{4t}}\, \sqrt{t}\, \mathrm{erf}\Big(\frac{|x|}{2\sqrt{t}}\Big) \\
&\le \epsilon\, e^{\frac{A^2}{4t}}\, \sqrt{t}\, \mathrm{erf}\Big(\frac{A}{2\sqrt{t}}\Big) \le \epsilon\, e^{A}\, \frac{A}{\sqrt{\pi}} \quad \text{if } t \ge A/4.
\end{aligned}$$

To estimate I_1', we notice that by Taylor's Theorem, there exists $h^* > 0$ such that $e^h - e^{-h} \le 3h$ for all $0 \le h \le h^*$. Assume $t \ge TA/(2h^*)$. Then in I_1',

$$\frac{\eta|x|}{\sqrt{t}} \le \frac{\eta A}{\sqrt{t}} \le \frac{TA}{2t} \le h^*$$

so that we have

$$\begin{aligned}
I_1' &\le \frac{1}{\sqrt{\pi}} \int_0^{\frac{T}{2\sqrt{t}}} \Big|\Theta_0(-2\sqrt{t}\,\eta) - 1\Big|\, e^{-\eta^2}\, \sqrt{t}\, \frac{3\eta|x|}{\sqrt{t}}\, d\eta \\
&\le \frac{3A}{\sqrt{\pi}} \int_0^{\frac{T}{2\sqrt{t}}} \Big\|\Theta_0(x) - 1\Big\|_{L^\infty((-\infty,0])}\, \eta e^{-\eta^2}\, d\eta \lesssim A(1 - e^{-\frac{T^2}{4t}}).
\end{aligned}$$

Therefore $I_1 = I_1' + I_1''$ vanishes uniformly. Estimation of I_2 is straightforward:

$$\begin{aligned} I_2 &\le \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2} (1 - e^{-\frac{A^2}{4t}}) \sqrt{t} \left(e^{\frac{\eta A}{\sqrt{t}}} - e^{-\frac{\eta A}{\sqrt{t}}} \right) d\eta \\ &= (1 - e^{-\frac{A^2}{4t}})\, e^{\frac{A^2}{4t}} \sqrt{t}\, \mathrm{erf}\Big(\frac{A}{2\sqrt{t}}\Big) \le (1 - e^{-\frac{A^2}{4t}})\, e^{\frac{A^2}{4t}} \frac{A}{\sqrt{\pi}}. \end{aligned}$$

Lastly, to estimate I_3, we have by Taylor's Theorem,

$$|e^h - e^{-h} - 2h| \le \frac{h^2}{2}(e^{|h|} - 1) \quad \text{for all } h \in \mathbf{R}.$$

Hence, if we put $h = \frac{\eta x}{\sqrt{t}}$,

$$\begin{aligned} I_3 &\le \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2} \left| \sqrt{t} \left(e^{\frac{\eta x}{\sqrt{t}}} - e^{-\frac{\eta x}{\sqrt{t}}} \right) - 2\eta x \right| d\eta \\ &\le \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2} \frac{\eta^2 A^2}{2\sqrt{t}} (e^{\frac{\eta A}{\sqrt{t}}} - 1)\, d\eta \\ &\lesssim \frac{A^2}{\sqrt{t}} \int_0^\infty \eta^2 e^{-\eta^2} (e^{\eta} - 1)\, d\eta \quad \text{if } t \ge A^2. \end{aligned}$$

(Uniform Convergence of $\sqrt{t}\, L_x$) From L_x in (2.5), we have

$$\begin{aligned} &\sqrt{\pi t}\, L_x(x,t) \\ &\quad = -\int_{\frac{|x|}{2\sqrt{t}}}^\infty \Theta_0(-2\sqrt{t}\,\eta + |x|)\eta e^{-\eta^2}\, d\eta - \int_{\frac{-|x|}{2\sqrt{t}}}^\infty \Theta_0(-2\sqrt{t}\,\eta - |x|)\eta e^{-\eta^2}\, d\eta \\ &\qquad + \int_0^t \sqrt{t}\, g'(t-\tau) \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{\tau}}\, d\tau + g(0)\, e^{-\frac{x^2}{4t}}. \end{aligned} \tag{3.4}$$

Assume $-A \le x < 0$ for some $A > 0$. First we claim that the second term in (3.4) vanishes uniformly. By Lemma 2.3, for any fixed $\epsilon > 0$, there exists $T > 0$ such that $|g(t)| \le \epsilon$ for all $t \ge T$. Split the integral of the second term in (3.4) into two parts: $\int_0^{t-T} + \int_{t-T}^t d\tau =: I_1 + I_2$ and let I_2 contain the explicit term $g(0)\, e^{-\frac{x^2}{4t}}$. Then integrating I_2 by parts, we have

$$I_2 = \int_{t-T}^t \sqrt{t}\, g(t-\tau) \frac{(2\tau - x^2)\, e^{-\frac{x^2}{4\tau}}}{4\tau^{5/2}}\, d\tau + g(T) \sqrt{\frac{t}{t-T}}\, e^{-\frac{x^2}{4(t-T)}}.$$

If we assume $t \ge 2T$,

$$\begin{aligned} |I_2| &\le \sqrt{t} \int_{t-T}^t \|\sqrt{t}\, g(t)\|_\infty \frac{(2\tau + A^2)}{4\sqrt{t-\tau}\, \tau^{5/2}}\, d\tau + \sqrt{2}\, \epsilon \\ &\lesssim \frac{\sqrt{t}}{\sqrt{t}\, \sqrt{t-T}} + \frac{(3t - 2T)\sqrt{t}}{t^{3/2}\, (t-T)^{3/2}} A^2 + \epsilon \lesssim \frac{1}{\sqrt{t}} + \frac{A^2}{t^{3/2}} + \epsilon. \end{aligned}$$

On the other hand, if we assume $T \geq 1/\epsilon^2$, it holds that in I_1

$$|I_1| \lesssim \sqrt{t} \int_0^{t-T} \frac{\|t^{3/2}\, g'(t)\|_\infty}{(t-\tau)^{3/2}\sqrt{\tau}}\, d\tau \lesssim \frac{\sqrt{t-T}}{\sqrt{T}\,\sqrt{t}} \leq \epsilon.$$

Now we claim that the first term in (3.4) converges uniformly to -1. Both integrals in the term converge to $-1/2$ and they can be dealt in a similar way. Here we will consider the second integral only and show that

$$J := \int_{\frac{-|x|}{2\sqrt{t}}}^{\infty} \Theta_0(-2\sqrt{t}\,\eta - |x|)\eta e^{-\eta^2}\, d\eta \to \int_0^\infty \eta e^{-\eta^2}\, d\eta = \frac{1}{2} \quad \text{uniformly as } t \to \infty.$$

Split the integral into two parts: $\int_{\frac{-|x|}{2\sqrt{t}}}^{0} + \int_0^\infty d\eta =: J_1 + J_2$. Then J_1 vanishes uniformly:

$$|J_1| \leq \int_{\frac{-|x|}{2\sqrt{t}}}^{0} \|\Theta_0\|_\infty\, |\eta| e^{-\eta^2}\, d\eta \lesssim \int_0^{\frac{A}{2\sqrt{t}}} \eta e^{-\eta^2}\, d\eta = \frac{1}{2}\Big(1 - e^{-\frac{A^2}{4t}}\Big).$$

On the other hand, from the assumption (2.11), J_2 converges to $1/2$ uniformly:

$$\begin{aligned} \left|J_2 - \frac{1}{2}\right| &\leq \int_0^\infty \Big|\Theta_0(-2\sqrt{t}\,\eta - |x|) - 1\Big|\, \eta e^{-\eta^2}\, d\eta \\ &\leq \int_0^\infty \frac{\Big\|x\Big(\Theta_0(x) - 1\Big)\Big\|_{L^\infty((-\infty,0])}}{2\sqrt{t}\,\eta + |x|}\, \eta e^{-\eta^2}\, d\eta \lesssim \frac{1}{\sqrt{t}}. \end{aligned}$$

Therefore the proof is complete. $\square$

Now we are ready to show that the weak solution of (1.3) converges to a steady state uniformly on compact sets. A steady state solution of (1.3) should satisfy

$$w w_x - w_{xx} = \delta \quad \text{in } \mathbf{R}.$$

Assuming the boundary conditions $w(-\infty) = w_x(-\infty) = 0$, integrating this equation from $-\infty$ to x yields that

$$\frac{1}{2} w^2(x) - w_x(x) = H(x),$$

where $H(x)$ is the Heaviside function. If $x < 0$, then $H(x) \equiv 0$ so that the solution is $w(x) = 2/(c - x)$ for some constant c. On the other hand, if $0 < x$, $H(x) \equiv 1$ so that the solution is $w(x) \equiv \sqrt{2}$. Therefore assuming continuity of the solution, we can find the steady state:

$$w(x) = \begin{cases} 2/(\sqrt{2} - x) & \text{if } x \leq 0, \\ \sqrt{2} & \text{if } 0 < x. \end{cases} \tag{1.4}$$

Theorem 3.5 *Let u be the weak solution of* (1.3) *with an initial value $u_0 \in C^1(\mathbf{R}) \cap L^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$. If Θ_0 satisfies* (2.11)*, then for any compact set $K \subset \mathbf{R}$,*

$$\|u(t) - w\|_{L^\infty(K)} \to 0 \quad \text{as } t \to \infty,$$

where the steady state $w(x)$ is given in (1.4)*. Furthermore, for any $1 \le p \le \infty$,*

$$\|u(t) - w\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty.$$

PROOF. By Theorem 3.2, the unique weak solution is given by $u = -2\Theta_x/\Theta$. Hence, for $0 < x$, Lemma 3.3 gives that

$$u(x,t) = -2\,\frac{\sqrt{t}\,R_x(x,t)}{\sqrt{t}\,R(x,t)} \to -2\,\frac{-\frac{1}{\sqrt{\pi}}e^{-\frac{x}{\sqrt{2}}}}{\sqrt{\frac{2}{\pi}}e^{-\frac{x}{\sqrt{2}}}} = \frac{1}{\sqrt{2}} \quad \text{as } t \to \infty.$$

For $x \le 0$, Lemma 3.4 also gives that

$$u(x,t) = -2\frac{\sqrt{t}\,L_x(x,t)}{\sqrt{t}\,L(x,t)} \to -2\,\frac{-\frac{1}{\sqrt{\pi}}}{\frac{1}{\sqrt{\pi}}(\sqrt{2}-x)} = \frac{2}{\sqrt{2}-x} \quad \text{as } t \to \infty.$$

The uniform convergence on compact sets is clear since the denominator converges to a positive function. The L^1-convergence on compact sets follows from the uniform convergence and the Dominated Convergence Theorem. Finally the L^p-convergence follows by interpolation. $\square$

As a corollary of the *pointwise* convergence, we can prove that the solution is uniformly bounded.

Corollary 3.6 *Let u be the weak solution of* (1.3) *with an initial value $u_0 \in C^1(\mathbf{R}) \cap L^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$. If Θ_0 satisfies* (2.11)*, then*

$$u \in L^\infty(\mathbf{R} \times [0,\infty)).$$

PROOF. From the hypothesis, $u_0 \in L^\infty(\mathbf{R})$. Now note that u satisfies the following initial-boundary value problem:

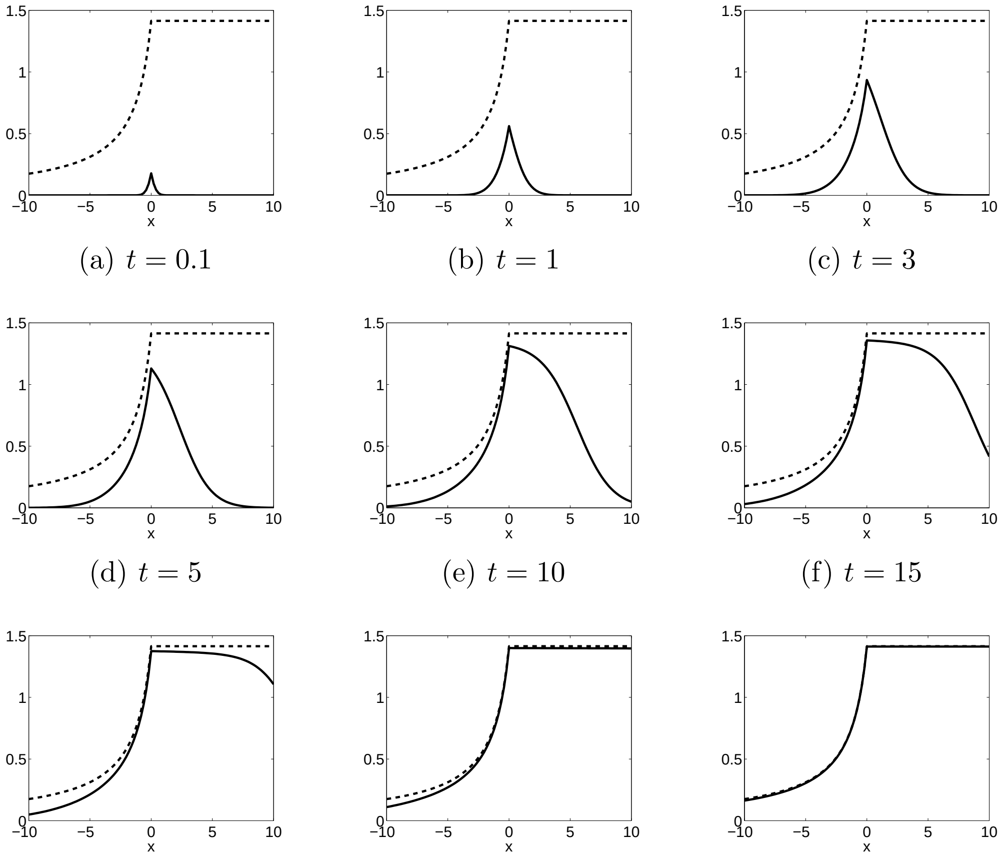
$$\begin{cases} v_t + vv_x - v_{xx} = 0, & 0 \ne x \in \mathbf{R},\ 0 < t, \\ v(x,0) = u_0(x), & x \in \mathbf{R}, \\ v(0,t) = u(0,t), & 0 < t. \end{cases}$$

By Theorem 3.5, $u(0,t) \to w(0) = \sqrt{2}$ as $t \to \infty$ so that $u(0,t)$ is bounded for $0 \le t$. Therefore by the maximum principle, we have

$$\|u\|_{L^\infty(\mathbf{R}\times[0,\infty))} \le \max\Big\{\|u_0\|_{L^\infty(\mathbf{R})}, \max_{0\le t} |u(0,t)|\Big\} < \infty.$$

$\square$

Appendix: Heat propagation from a point source



(a) $t = 0.1$ (b) $t = 1$ (c) $t = 3$

(d) $t = 5$ (e) $t = 10$ (f) $t = 15$

(g) $t = 20$ (h) $t = 50$ (i) $t = 300$

Fig. 1. Solid lines are the spatial profile of the solution at a given time t and dashed lines are the steady state w in (1.4). These figures are from the explicit formula (3.1), but not from a numerical simulation. The solution approaches to the steady state from below.

The solution u of the Burgers equation with a point source (1.3), which is given in Theorem 3.2, is explicit except the boundary condition $g(t)$. For the case with zero initial value, $u_0 \equiv 0$, the boundary condition $g(t)$ was explicitly given by

$$g(t) = e^{-t/4}\, I_0(t/4), \tag{2.10}$$

where I_0 is the modified Bessel function of the first kind. In this section we will numerically display how the solution evolves to the steady state w in (1.4). The solution is explicitly given by

$$u(x,t) = \begin{cases} -2\frac{R_x(x,t)}{R(x,t)}, & 0 \leq x, \\ -2\frac{L_x(x,t)}{L(x,t)}, & x < 0, \end{cases}$$

where R, L, R_x and L_x are given in (2.2), (2.3), (2.4) and (2.5), respectively, and

$$\Theta_0 \equiv 1.$$

Since both the numerator and the denominator vanish as $t \to \infty$, it is inconvenient to numerically compute the solution directly from the formula for t large. Instead, a modified formula obtained by multiplying $\sqrt{t}$ to both the numerator and the denominator is used:

$$u(x,t) = \begin{cases} -2\frac{\sqrt{t}\,R_x(x,t)}{\sqrt{t}\,R(x,t)}, & 0 \leq x, \\ -2\frac{\sqrt{t}\,L_x(x,t)}{\sqrt{t}\,L(x,t)}, & x < 0. \end{cases}$$

Then by Lemmas 3.3 and 3.4, both the numerator and the denominator converge to nonzero numbers so that numerical computation becomes robust.

In Figure 1, the solution of (1.3) with $u_0 \equiv 0$ is drawn at each given time along with the steady state w. One may observe that the point source generates heat constantly near the origin and the nonlinear convection transports the heat to the right, preventing the solution from blowing up.

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