An introduction to similarity curves and self-similar solutions to convection-diffusion equations ⋆

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Abstract. In this note we introduce similarity curves which is a modified version of of the characteristics and use it to find self-similar solutions of convection-diffusion equations.

Key words. fundamental solutions; heat equation; porous medium equations; similarity solutions; Barenblatt solutions; characteristics

1. Introduction

The similarity structure of certain convection or diffusion equations are well-known. The fundamental solutions of such problems are given explicitly and called self-similar solutions. The N-waves for the Burgers equation, the Gaussian for the heat equation and the Barenblatt solution for the porous medium equation are examples. These self-similar solutions have been played key roles in the theoretical development. However, there is no systematic approach to handle these similarity structures in a single frame. In this note we introduce a method to derive similarity solution which is applicable to convection and diffusion equations.

For example, let $u(x,t)$ be the solution to a conservation law given by

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{1.1}$$

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For a fixed similarity variable ξ a curve in the xt-plane given by

$$x = x(\xi, t)$$

is called a similarity curve of the solution $u(x,t)$ if

$$\frac{d}{dt}\big(x_\xi(\xi,t)u(x(\xi,t),t)\big) = 0. \tag{1.2}$$

The similarity curve which is a resemblance of a characteristic line $x = \chi(\xi, t)$ of first order differential equations which satisfies

$$\frac{d}{dt}u(\chi(\xi,t),t) = 0.$$

However, the similarity curves are also applicable to diffusion equations.

The purpose of this paper is to investigate the similarity structure of convection and diffusion equations. Hence we are interested in the solution with a delta measure as its initial value, i.e.,

$$\rho_t + f(\rho)_x = 0, \qquad \rho(x,0) = \delta(x). \tag{1.3}$$

In particular, we consider similarity curves in a form of

$$x(\xi,t) = \xi a(t) \quad \text{for} \quad t > 0.$$

One may call this similarity curve a separable one or a uniform one in the sense that the similarity variable is expanded with a uniform rate $a(t)$. Since we are interested in the fundamental solution in this paper, we are looking for the time scale such that

$$a(0) = 0.$$

The relation (1.2) indicates that there exists a function $\varrho(\xi)$, which is called a similarity profile, that satisfies

$$\varrho(\xi) = x_\xi(\xi,t)u(x(\xi,t),t) \quad \text{and} \quad \int \varrho(\xi)d\xi = 1.$$

We also looking for a similarity profile $\varrho(\xi)$ that decays fast enough as $|\xi| \to \infty$. For example, we will construct $\varrho(\xi)$ such that

$$\varrho(\xi) = o(|\xi|) \quad \text{as} \quad |\xi| \to \infty. \tag{1.4}$$

2. Similarity curves

The fundamental solution of certain problem may have a similarity structure and is called as a similarity or a self-similar solution. In the following a modification of a characteristic curve is introduced to show the structure of a self similar solutions.

We still consider the fundamental solution $\rho(x,t)$ to the conservation law (3.1). We call this fundamental solution a *self-similar solution* if there exists a differentiable mapping $x(\xi,t)$ such that, for all $t>0$ fixed, $x(\cdot,t)$ is a one-to-one mapping from $\boldsymbol{R}$ onto $\boldsymbol{R}$,

$$x(\xi_1,t)\neq x(\xi_2,t) \quad \text{for all} \quad \xi_1\neq\xi_2 \quad \text{and} \quad t>0, \tag{2.1}$$

and

$$\frac{\partial}{\partial t}\varrho(\xi,t)=0, \quad \text{where} \quad \varrho(\xi,t)=x_\xi(\xi,t)\rho(x(\xi,t),t). \tag{2.2}$$

Therefore, $\varrho=\varrho(\xi)$ is a function of similarity variable only and is called a *similarity profile*. The variable ξ and the mapping $x(\xi,t)$ are called a *similarity variable* and a *similarity mapping*, respectively. For a fixed similarity variable ξ_0, the curve $x=x(\xi_0,t)$ is called a *similarity curve*. The relation $\frac{\partial}{\partial t}\varrho(\xi,t)=0$ is now written as

$$x_{\xi t}\rho+x_\xi x_t\rho_x+x_\xi\rho_t=0. \tag{2.3}$$

One may easily check that

$$\rho=\varrho/x_\xi,\;\; \rho_x=\varrho'/x_\xi^2-x_{\xi\xi}\varrho/x_\xi.$$

In this paper, we are mostly interested in similarity curves given in the form of

$$x(\xi,t)=\xi a(t) \quad \text{for} \quad t>0.$$

Then,

$$x_\xi=a(t),\;\; x_{\xi\xi}=0,\;\; x_{\xi t}=a'(t),\;\; x_t=\xi a'(t).$$

Substituting these to (2.3) gives

$$a'(\xi\varrho)'+a^2\rho_t=0, \qquad t>0. \tag{2.4}$$

If one can find the similarity profile $\varrho(\xi)$ and the characteristic curves $x(\xi,t)$, then the similarity solution solution $\rho(x,t)$ is simply given by the relation

$$\rho(x,t)=\varrho(\xi)/x_\xi(\xi,t), \quad x=x(\xi,t). \tag{2.5}$$

Note that after the change of variable, the total mass or the L^1-norm is preserved, i.e., for all $t>0$ fixed,

$$\int_{\xi_1}^{\xi_2}\varrho(\xi)d\xi=\int_{\xi_1}^{\xi_2}\rho(x(\xi,t),t)x_\xi(\xi,t)d\xi=\int_{x(\xi_1,t)}^{x(\xi_2,t)}\rho(x,t)dx.$$

3. conservation laws

In this section we consider the similarity structure of fundamental solutions to a conservation law. We start with a short review of characteristic lines. Let $u(x,t)$ be the solution solution to a conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{3.1}$$

The flux is assumed to be C^1 and satisfy

$$f'(0) = f(0) = 0. \tag{$H0$}$$

In fact one may obtain this hypothesis without a loss of generality after appropriate translations. A curve $x = \chi(\xi,t)$ in the xt-plane is called a *characteristic curve* if it satisfies

$$\frac{\partial}{\partial t}\chi(\xi,t) = f'(u(\chi(\xi,t),t)). \tag{3.2}$$

Then,

$$\frac{\partial}{\partial t}u(\chi(\xi,t),t) = u_t(\chi(\xi,t),t) + \frac{\partial}{\partial t}\chi(\xi,t)u_x(\chi(\xi,t),t) = 0, \tag{3.3}$$

and hence the solution $u(x,t)$ is constant on it. Since u is constant along the characteristic curve, the relation (3.2) implies that $\chi(\xi,t)$ is actually a line for any fixed ξ. The variable ξ is just a parameter to denote a specific characteristic line. However, ξ is usually considered as the spatial point at the initial time, i.e., $\xi = \chi(\xi,0)$.

Remark 1. If $\xi_1 < \xi_2$ and $f'(u_0(\xi_1)) > f'(u_0(\xi_2))$, then the two characteristic line $\chi(\xi_1,t)$ and $\chi(\xi_2,t)$ should collide. This indicates an appearance of a discontinuity of the solution, which is called a shock.

Remark 2. The characteristic line given by (3.2) and (3.3) is $x(\xi,t) = \xi + f'(u_0(\xi))t$. Then, $x_\xi(\xi,t) = 1 + f''(u_0(\xi))u_0'(\xi)t$. Hence the similarity profile ϱ given in (2.2) is

$$\varrho(\xi) = (1 + f''(u_0(\xi))u_0''(\xi)t)\rho(x(\xi,t),t).$$

However, the conserved quantity along the line is simply $\rho(x(\xi,t),t)$ and hence the characteristic line is not a self-similar one. It is also clear that

$$\int_{\xi_1}^{\xi_2} u_0(\xi)d\xi \neq \int_{x(\xi_1,t)}^{x(\xi_2,t)} u(x,t)dx.$$

The fundamental solution is an integrable solution which is a delta-sequence as $t \to 0$. Let $\rho(x,t) \geq 0$ be a positive fundamental solution to a conservation law, i.e.,

$$\rho_t + f(\rho)_x = 0, \qquad \rho(x,0) = \delta(x). \tag{3.4}$$

If the flux is convex, i.e., $f''(u) \geq 0$ for all $u \geq 0$, then one can easily check that the fundamental solution $\rho(x,t)$ is explicitly given by

$$\rho(x,t) = \begin{cases} g(x/t), & 0 < x < b(t), \\ \quad 0 \quad , & \quad \text{otherwise}, \end{cases} \tag{3.5}$$

where

$$f'(g(x)) = x \quad \text{and} \quad \int_0^{b(t)} g(x/t)dx = 1. \tag{3.6}$$

Even the fundamental solutions without the convexity assumption can be found in [4,5]. However, only a part of these fundamental solutions have similarity property. In the following we derive similarity structure of certain fundamental solutions.

Since $\rho_t = -f'(\rho)\rho_x$, substituting $\varrho = x_\xi \rho$ and $\varrho' = x_{\xi\xi}\rho + x_\xi^2 \rho_x$ into (2.3) gives

$$x_{\xi t}\varrho/x_\xi + (x_\xi x_t - x_\xi f'(\varrho/x_\xi))(\varrho'/x_\xi^2 - x_{\xi\xi}\varrho/x_\xi^3) = 0.$$

After multiplying x_ξ, we obtain

$$x_{\xi t}\varrho + (x_t - f'(\varrho/x_\xi))(\varrho' - x_{\xi\xi}\varrho/x_\xi) = 0. \tag{3.7}$$

In fact this problem is more complicate than the original one. It includes a unknown function $x(\xi,t)$ in two variables and a one variable one $\varrho(\xi)$. However, the solvability of this problem gives the information that the solution has a certain similarity structure.

Now we consider the case that the similarity mapping $x(\xi,t)$ is separable, i.e., $x(\xi,t) = \xi a(t)$. Then, $\rho_t = -f'(\rho)\rho_x = -f'(\varrho/a)\varrho'/a^2$ and hence (2.4) becomes

$$a'(t)(\xi\varrho(\xi))' - f'(\varrho(\xi)/a(t))\varrho'(\xi) = 0. \tag{3.8}$$

This problem has two unknown functions of a single variable, which is now simpler than the original one.

Suppose the exists α such that, for all a and b,

$$f'(a/b) = \frac{\alpha f'(a)}{f'(b)}. \tag{$H1$}$$

One can easily check that a power law satisfies this hypothesis. If (H1) is satisfied, then the variables in (3.8) is separated and written as

$$\frac{d}{dt}f(a(t)) = \alpha \frac{d}{d\xi}f(\varrho(\xi)) \Big/ \frac{d}{d\xi}(\xi\varrho(\xi)). \tag{3.9}$$

Therefore, the quantity in (3.9) should be a constant, say c_0. Hence, $a(t)$ should satisfy $\frac{d}{dt}f(a(t)) = c_0$ and, since $a(0) = 0$,

$$f(a(t)) = c_0 t. \tag{3.10}$$

The other ξ dependent part can be written as

$$(\alpha f(\varrho(\xi)) - c_0\xi\varrho(\xi))' = 0.$$

Since we are looking for a similarity profile which satisfies (1.4) and the flux f satisfies (H0), it gives

$$(\alpha f(\varrho(\xi)) - c_0\xi\varrho(\xi)) = 0.$$

Therefore, one can easily see that $\varrho(\xi)$ satisfies

$$\varrho(0) = 0 \tag{3.11}$$

and

$$\varrho(\xi) = 0 \quad \text{or} \quad f(\varrho(\xi))/\varrho(\xi) = \frac{c_0}{\alpha}\xi. \tag{3.12}$$

Suppose that the flux function f and $g(u) := f(u)/u$ are invertible, then

$$a(t) = f^{-1}(c_0 t), \quad \varrho(\xi) = g^{-1}(\frac{c_0}{\alpha}\xi), \text{ where } g(u) := f(u)/u. \tag{3.13}$$

Under the hypothesis (H1), one may choose c_0 arbitrary after resetting the similarity variable ξ. For example one may set $c_0 = 1$ without loss of generality and obtain

$$a(t) = f^{-1}(t), \quad \varrho(\xi) = g^{-1}(\xi/\alpha), \text{ where } g(u) := f(u)/u.$$

Example 1. The power law $f(u) = u^q/q$, $q > 1$, satisfies the hypotheses with $\alpha = 1$. If one takes $c_0 = 1/q$, then $a(t) = t^{1/q}$ and the similarity mapping is given by

$$x(\xi, t) = \xi t^{1/q}.$$

The similarity profile is given by

$$\varrho(\xi) = \xi^{1/(q-1)}.$$

The only positive case that satisfies the entropy condition, $\varrho(0) = 0$, and has unit total mass is the one

$$\varrho(\xi) = \begin{cases} \xi^{1/(q-1)}, & 0 \le x \le b_0, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\int_0^{b_0} \xi^{1/(q-1)} d\xi = 1.$$

Therefore, the similarity solution $\rho(x,t)$ is given by

$$\rho(x,t) = t^{-1/q}\varrho(t^{-1/q}x).$$

On the non-zero region, the self-similar solution is given by

$$\rho(x,t) = t^{-1/q}(t^{-1/q}x)^{1/(q-1)} = (x/t)^{1/(q-1)},$$

which is the same fundamental solution given by (3.5) and (3.6). However, there are fundamental solutions that is not written in a similarity form.

4. PME type diffusion equations

In this section we consider the similarity structure of fundamental solutions to a diffusion equations:

$$\rho_t = \phi(\rho)_{xx}, \qquad \rho(x,0) = \delta(x). \tag{4.1}$$

Remind the relations

$$\varrho = x_\xi \rho, \quad \varrho' = x_{\xi\xi}\rho + x_\xi^2 \rho_x, \quad \varrho'' = x_{\xi\xi\xi}\rho + 3x_{\xi\xi}x_\xi\rho_x + x_\xi^3\rho_{xx}.$$

If one considers the separable case $x(\xi,t) = \xi a(t)$, then

$$\rho = \varrho/a(t), \quad \rho_x = \varrho'/a^2(t), \quad \rho_{xx} = \varrho''/a^3(t).$$

Since $\rho_t = \phi(\rho)_{xx} = \phi''(\rho)\rho_x^2 + \phi(\rho)\rho_{xx}$, the similarity relation (2.4) is written as

$$a^2 a'(\varrho\xi)' + \phi''(\varrho/a)(\varrho')^2 + \phi'(\varrho/a)\varrho'' a = 0. \tag{4.2}$$

Suppose that there exist α, β and γ such that, for all $a, b > 0$,

$$\phi'(a/b) = \frac{\alpha\phi'(a)}{\phi'(b)}, \quad \phi''(a/b) = \frac{\beta\phi''(a)}{\phi''(b)}, \quad \gamma\phi''(a)a = \phi'(a). \tag{H2}$$

Then, one may separate the variables in (4.2) as

$$aa'\phi'(a) = -\big[\beta\gamma\phi''(\varrho)(\varrho')^2 + \alpha\phi'(\varrho)\varrho''\big]/(\varrho\xi)'. \tag{4.3}$$

Hence $a(t)$ and $\varrho(\xi)$ are obtained by solving

$$a\phi'(a)a' = c_0, \tag{4.4}$$

$$\big[\beta\gamma\phi''(\varrho)(\varrho')^2 + \alpha\phi'(\varrho)\varrho''\big] = -c_0(\varrho\xi)'. \tag{4.5}$$

Lemma 1. *If ϕ is C^2, then*

$$\beta\gamma = \alpha.$$

Proof. □

The equation for the similarity variable ξ is

$$\alpha(\phi'(\varrho)\varrho' + \frac{c_0}{\alpha}\varrho\xi)' = 0.$$

We are considering a similarity profile that decays for $|\xi|$ large as in (1.4). Therefore, we have

$$\varrho(\xi) = 0 \quad \text{or} \quad g(\varrho(\xi)) = C - \frac{c_0}{2\alpha}\xi^2,$$

where

$$g'(u) = \phi'(u)/u.$$

Example 2 (heat equation). Consider the heat equation case $\phi(u) = u$. Then, the equation for the time scale is

$$aa' = (a^2/2)' = c_0 \quad \text{with} \quad a(0) = 0.$$

Hence, setting $c_0 = 1/2$, one obtains

$$a(t) = \sqrt{t}.$$

The equation for the similarity variable ξ is

$$\varrho'' = -\frac{1}{2}(\xi\varrho)'$$

Hence, we may set

$$\varrho' = -\frac{1}{2}\xi\varrho,$$

which gives the Gaussian, i.e., $\varrho(\xi) = Ce^{-\xi^2/4}$. Since we are looking for a solution of unit total mass, the constant C should be $C = 1/\sqrt{4\pi}$, i.e.,

$$\varrho(\xi) = \frac{1}{\sqrt{4\pi}}e^{-\xi^2/4}.$$

Using the relation in (2.5), the similarity solution to the heat equation is given by

$$\rho(x,t) = \frac{1}{\sqrt{t}}\frac{1}{\sqrt{4\pi}}e^{-(x/\sqrt{t})^2/4} = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t},$$

which is the heat kernel.

Example 3 (nonlinear diffusion). Consider the nonlinear diffusion with $\phi(u) = u^m$, $1 \neq m > -1$. Then,

$$\phi'(u) = mu^{m-1}, \quad \phi''(u) = m(m-1)u^{m-2}.$$

and the constants in (H2) are

$$\alpha = m, \quad \beta = m(m-1), \quad \gamma = 1/(m-1).$$

The equation for the time scale (4.4) becomes

$$\Big(\frac{m}{m+1} a^{m+1}\Big)' = c_0 \quad \text{with} \quad a(0) = 0.$$

Hence, it is required $m \neq -1$. Note that the case $m > -1$ in the one space dimension is the mass conserving regime. Setting $c_0 = m/(m+1)$, one obtains

$$a(t) = t^{1/(m+1)}.$$

Since $g'(u) = mu^{m-2}$,

$$g(u) = \frac{m}{m-1} u^{m-1}, \quad \varrho(\xi) = \Big(C - \frac{m-1}{2m(m+1)} \xi^2\Big)_+^{1/(m-1)},$$

where the constant C should be decided by the unit mass condition,

$$\int \varrho(\xi) d\xi = 1.$$

Using the relation in (2.2), the similarity solution to the nonlinear diffusion is given by

$$\rho(x,t) = t^{\frac{-1}{m+1}} \Big(C - \frac{m-1}{2m(m+1)} x^2 t^{\frac{-2}{m+1}}\Big)_+^{1/(m-1)},$$

which is also called the Barenblatt solution.

5. P-Laplacian type diffusion equations

In this section we consider the similarity structure of fundamental solutions to p-Laplacian type diffusion equations:

$$\rho_t = \phi(\rho_x)_x, \qquad \rho(x,0) = \delta(x). \tag{5.1}$$

For the separable case $x(\xi,t) = \xi a(t)$, we have $\rho = \varrho/a(t)$, $\rho_x = \varrho'/a^2(t)$ and $\rho_{xx} = \varrho''/a^3(t)$. Hence $\rho_t = \phi'(\rho_x)\rho_{xx} = \phi(\varrho'/a^2)\varrho''/a^3$ and the similarity relation (2.4) is written as

$$aa'(\xi\varrho)' + \phi'(\varrho'/a^2)\varrho'' = 0. \tag{5.2}$$

Suppose that there exist α such that, for all $a, b \in \boldsymbol{R}$,

$$\phi'(a/b) = \frac{\alpha\phi'(a)}{\phi'(b)}. \tag{H3}$$

Then, one may separate the variables in (5.2) and obtains

$$\frac{1}{2}\big(\phi(a^2)\big)' = -\alpha(\phi(\varrho'))'/(\xi\varrho)'. \tag{5.3}$$

Hence $a(t)$ and $\varrho(\xi)$ are obtained by solving

$$\frac{1}{2}\big(\phi(a^2)\big)' = c_0, \tag{5.4}$$

$$(\phi(\varrho') + \frac{c_0}{\alpha}\varrho\xi)' = 0. \tag{5.5}$$

Therefore, the time scale $a(t)$ is given by

$$a(t) = \sqrt{\phi^{-1}(2c_0 t)}, \tag{5.6}$$

and the similarity profile $\varrho(\xi)$ should satisfy

$$\phi(\varrho') = -\frac{c_0}{\alpha}\varrho\xi.$$

Using (H3) one more time gives

$$\frac{\varrho'}{\phi^{-1}(\varrho)} = \phi^{-1}(-c_0\xi) \quad \text{or} \quad \varrho(\xi) = 0. \tag{5.7}$$

Example 4 (P-Laplacian). Consider the p-Laplacian equation with $\phi(u_x) = |u_x|^{p-2}u_x$, $p > 1$. Then, ϕ is increasing and invertible. The constant α in (H3) is $\alpha = p-1$. Set $c_0 = 1/2$, then $a(t)$ in (5.6) becomes

$$a(t) = t^{\frac{1}{2(p-1)}}.$$

For the positive similarity profile $\varrho \geq 0$, Eq. (5.7) becomes

$$\frac{\varrho'}{\phi^{-1}(\varrho)} = \Big(\frac{p-1}{p-2}\varrho^{\frac{p-2}{p-1}}\Big)' = -\Big|\frac{1}{2}\xi\Big|^{\frac{1}{p-1}}\operatorname{sign}(\xi).$$

Integrating the equation gives

$$\frac{p-1}{p-2}\varrho^{\frac{p-2}{p-1}} = C - \frac{p-1}{p}2^{\frac{-1}{p-1}}|\xi|^{\frac{p}{p-1}}.$$

Therefore, the similarity profile is

$$\varrho(\xi) = \Big(C - 2^{\frac{-1}{p-1}}\frac{p-2}{p}|\xi|^{\frac{p}{p-1}}\Big)_+^{\frac{p-1}{p-2}},$$

where the constant C is decided by the unit mass relation

$$\int \varrho(\xi)d\xi = 1.$$

Using the relation in (2.5), the similarity solution to the p-Laplacian nonlinear diffusion is given by

$$\rho(x,t) = t^{\frac{-1}{2(p-1)}} \left(C - 2^{\frac{-1}{p-1}} \frac{p-2}{p} |x|^{\frac{p}{p-1}} t^{\frac{-p}{2(p-1)^2}} \right)_+^{\frac{p-1}{p-2}},$$

which is sometimes called a Barenblatt-type solution. Note that this similarity solution is valid for $2 \neq p > 1$. If $p = 2$, then the equation is the heat equation and the Gaussian is the right one.

6. A non-power law case

The examples we have considered so far are all power laws. It is clear that a power law satisfies Hypotheses (H0–H3) and separates the variables from the equations (3.8), (4.2) and (5.2). The similarity solutions obtained in the technique we developed in this paper are identical to the well-known ones such as the Gaussian and Barenblatt-type solutions.

Hence it is natural to ask if a non-power law case may have a similarity structure. In fact all the known cases of similarity of L^1 solution are based on the power law. It will be great if the method developed in this paper helps to find out such similarity structure. It will be also good if the method may give even a negative answer. In this section we consider an example without the power law. Consider the nonlinear diffusion equation

$$u_t = \phi(u)_{xx}\big(:= (u+u^2)_{xx}\big), \quad u(x,0) = \delta(x). \tag{6.1}$$

In this example the linear diffusion of the heat equation and the nonlinear diffusion of the porous medium equation with $m = 2$ coexist. Then,

$$\phi'(u) = 2u + 1, \qquad \phi''(u) = 2,$$

and the equation (2.4) is written as

$$a^2 a'(\xi\varrho)' + 2(\varrho\varrho')' + a\varrho'' = 0. \tag{6.2}$$

We are looking for a solution such that $a(0) = 0$, $\varrho(\xi) \to 0$ fast enough as $|\xi| \to \infty$ and $\varrho(-\xi) = -\varrho(\xi)$.

The variables in equation (6.2) are not separable. However, there is a better way. Remember that, in the choice of the time scale $a = a(t; c_0)$, there is a freedom to choose a constant c_0. We may also choose a time $t > 0$. Taking these two choices we may assume that there exists c_0 and $t_0 > 0$ such that

$$a(t_0; c_0) = 1, \qquad a'(t_0, c_0) = 1. \tag{6.3}$$

Substituting these to the equation (6.2) give a equation for the similarity profile $\varrho(\xi)$:

$$(\xi\varrho)' + 2(\varrho\varrho')' + \varrho'' = 0.$$

We are looking for a profile that decays for $|\xi|$ large and hence we obtain

$$\xi\varrho + 2\varrho\varrho' + \varrho' = 0. \tag{6.4}$$

All we have to do is to solve this ODE and then substitute its solution to (6.2) to obtain $a(t)$.

Can we find one? Explicitly? Is there a theory for existence? If we can not find it explicitly, can we graph it numerically?

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