Long time asymptotics in a logistic model with a non-Fickian heterogeneous diffusion

Ohsang Kwon[1], Yong-Jung Kim[2] and Fang Li[3*]

[1]The Center for Partial Differential Equations, East China Normal University, Shanghai 200241, China; E-mails: ohsangkwon80@gmail.com

[2]Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-701, Korea; E-mail: yongkim@kaist.edu

[3]The Center for Partial Differential Equations, East China Normal University, Shanghai 200241, China; E-mail: leftfree214@gmail.com

[*]To whom correspondence should be addressed

November 25, 2011

Abstract. In this paper we study the long time asymptotics of a logistic model with a diffusion process motivated by spatial heterogeneous Brownian movement of individuals. Global stability and zero diffusion limits are obtained under such diffusion.

1. Without dynamics

1.1. General result

Consider the problem

$$\begin{cases} u_t = \Delta\left(\alpha(\frac{m(x)}{u})u\right) & \text{for } x \in \Omega,\ t > 0, \\ \nabla[\alpha(\frac{m(x)}{u})u] \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0, \end{cases} \tag{1}$$

where $\mathbf{n}$ denotes the unit outer normal vector at the boundary $\partial\Omega$ and $\alpha(s)$ represents the mobility and $m(x)$ is the density of the resources. Throughout this paper, we posit that

(A1) $\alpha(s) = d\alpha_1(s)$, where $d > 0$, $\alpha_1(s) \in C^2(0,\infty)$ is decreasing and positive. W.l.o.g., assume that $\alpha_1(1) = 1$.

(A2) $m(x) \in C^2(\bar{\Omega})$ is non-constant and $m(x) > 0$ in $\bar{\Omega}$.

Theorem 1. *Suppose that (A1) and (A2) hold. Then there exists a constant $C_0 > 0$ such that*

$$\lim_{t\to+\infty} \alpha(\frac{m(x)}{u(x,t)})u(x,t) = C_0$$

where $u(x,t)$ denotes the solution to the initial value problem (1).

Proof. W.l.o.g., assume that $d = 1$. First, it is standard to show the local existence and positivity of the solution to the problem (1).

Next, set $w = \alpha(\frac{m(x)}{u})u$. We can rewrite (1) as follows

$$\begin{cases} \left[\alpha(\frac{m(x)}{u}) - \alpha'(\frac{m(x)}{u})\frac{m(x)}{u}\right]^{-1} w_t = \Delta w & \text{for } x \in \Omega,\ t > 0, \\ \nabla w \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x) = \alpha(\frac{m(x)}{u_0})u_0 > 0. \end{cases} \tag{2}$$

Multiplying the first equation in (2) by w_t and integrating both sides over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx \\ &= -\int_\Omega \left[\alpha(\frac{m(x)}{u}) - \alpha'(\frac{m(x)}{u})\frac{m(x)}{u}\right]^{-1} w_t^2 dx \le 0, \end{aligned} \tag{3}$$

which immediately yields that

$$\int_\Omega |\nabla w(x,t)|^2 dx \le \int_\Omega |\nabla w_0(x)|^2 dx. \tag{4}$$

Moreover, by the maximum principle, it is routine to show that

$$\min_{\overline{\Omega}} w_0(x) \le w(x,t) \le \max_{\overline{\Omega}} w_0(x). \tag{5}$$

Since clearly $\lim_{s\to 0} \frac{\alpha(s)}{s} = \infty$, there exists a function $u_0(x) > 0$ such that $\alpha(\frac{m(x)}{u_0(x)})u_0(x) = \min_{\overline{\Omega}} w_0(x)$ for all $x \in \Omega$. Thus, (5) implies that $u \ge u_0$, and so there exists some constant $\ell > 0$ satisfying $\alpha(\frac{m}{u}) \ge \alpha(\frac{m}{u_0}) \ge \ell$ in Ω. Therefore, multiplying the first equation in (2) by w and integrating both sides over Ω, we have

$$\begin{aligned} \int_\Omega |\nabla w|^2 dx &= -\int_\Omega \left[\alpha(\frac{m(x)}{u}) - \alpha'(\frac{m(x)}{u})\frac{m(x)}{u}\right]^{-1} w w_t dx \\ &\le \frac{1}{\ell}\Big(\int_\Omega w^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_t^2 dx\Big)^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Thus, (5) and (6) together give that for some constant $c > 0$, independent of t,

$$c\Big(\int_\Omega |\nabla w|^2 dx\Big)^2 \le \int_\Omega w_t^2 dx.$$

Moreover, (3) implies that for sufficiently large $T > 0$ and some constant $C > 0$,

$$\int_T^\infty \int_\Omega w_t^2 dx dt \le C.$$

It follows that for some constant $C' > 0$,

$$\int_T^\infty \Big(\int_\Omega |\nabla w|^2 dx\Big)^2 dt \le C'.$$

Therefore, (3) implies that

$$\int_\Omega |\nabla w|^2 dx \to 0, \text{ as } t \to \infty. \tag{7}$$

Thus, (4) and (5) together give that

$$\|w(\cdot,t)\|_{W^{1,2}(\Omega)} \le c_1,$$

where the constant $c_1 > 0$ is independent of t. Therefore, it follows that there exist $\tilde{w} \in W^{1,2}(\Omega)$ and a sequence $t_n \to \infty$ such that

$$w(x,t_n) \rightharpoonup \tilde{w}(x) \text{ weakly in } W^{1,2}(\Omega) \tag{8}$$

as $t_n \to \infty$.

For any non-negative function $\phi \in C^1(\Omega)$ with $\nabla\phi \cdot \mathbf{n} = 0$ on $\partial\Omega$, (7) and (8) imply

$$\int_\Omega \nabla\tilde{w}\nabla\phi dx = 0.$$

Therefore $\tilde{w}$ is a *weak* solution of the problem

$$\begin{cases} \Delta w^* = 0 & \text{for } x \in \Omega, \\ \nabla w^* \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{9}$$

It is known that (9) only has constant solution, hence $\tilde{w} = C_0 > 0$.

At the end, back to (1), it is easy to see that

$$\frac{d}{dt}\int_\Omega u(x,t)dx = 0,$$

which immediately gives that $\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$. Set

$$\mathcal{F}(x,u,w) = \alpha(\frac{m(x)}{u})u - w = 0.$$

Direct computation yields that

$$\mathcal{F}_u = \alpha(\frac{m(x)}{u}) - \alpha'(\frac{m(x)}{u})\frac{m(x)}{u} > 0. \tag{10}$$

Applying the Implicit Function Theorem, we get $u = \rho(x, w)$. Hence

$$\int_\Omega \rho(x, w(x,t))dx = \int_\Omega u_0(x)dx,$$

which yields that

$$\int_\Omega \rho(x, C_0)dx = \int_\Omega u_0(x)dx.$$

Therefore, C_0 is independent of the choice of the sequence $\{t_n\}$ and the proof is complete. □

1.2. Special example

In this subsection, we want to show a very interesting example with special mobility $\alpha(s)$ in the one dimensional case. The amazing part of this example is that, based on some straightforward computations, we can provide an explicit expression of all steady state solutions of (1). More importantly, these steady state solutions not only demonstrate the rich structures of the solutions of (1), which is a sharp contract with the random diffusion case, but also reveal the interactions between the mobility, resources and the behavior of the solutions.

Now, define the mobility $\alpha(s)$ as follows

$$\alpha(s) = \begin{cases} h & \text{for } s \in [0, 1-\epsilon), \\ -\frac{h}{2\epsilon}(s-1-\epsilon) + \frac{\ell}{2\epsilon}(s-1+\epsilon) & \text{for } s \in [1-\epsilon, 1+\epsilon], \\ \ell & \text{for } s \in (1+\epsilon, \infty), \end{cases} \tag{11}$$

where $h > \ell > 0$ and $\epsilon > 0$ is very small.

Let us formulate our problem more precisely. The stationary problem for (1) in the one dimensional case is written in the following form:

$$\begin{cases} \Big(\alpha(\frac{m(x)}{u(x)})u(x)\Big)_{xx} = 0 & \text{for } x \in (0,1), \\ \Big(\alpha(\frac{m(x)}{u(x)})u(x)\Big)_x = 0 & \text{at } x = 0, 1, \end{cases} \tag{12}$$

where $\alpha(s)$ is defined in (11).

Theorem 2. *All positive solutions of (12) can be expressed explicitly. Moreover, if*

$$\frac{\max_{\bar{\Omega}} m(x)}{\min_{\bar{\Omega}} m(x)} \le \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{h}{\ell},$$

there exists a positive solution of (12) such that

$$1-\epsilon \le \frac{m(x)}{u(x)} \le 1+\epsilon$$

for all $x \in [0,1]$.

Proof. Setting $w = \alpha(\frac{m(x)}{u})u$, we have

$$\begin{cases} w_{xx} = 0 & \text{for } x \in (0,1), \\ w_x = 0 & \text{at } x = 0,1. \end{cases}$$

This immediately yields that

$$w(x) = \alpha(\frac{m(x)}{u(x)})u(x) = C,$$

where C is restricted to any positive constant since we only consider positive solutions of (12). Now we can classify all positive solutions of (12) based on the values of $C > 0$.

1. Suppose that $C \ge \frac{h}{1-\epsilon}\max_{\bar{\Omega}} m(x)$. It is routine to check that

$$u(x) \equiv \frac{C}{h}$$

   is a positive solution of (12).
2. Suppose that $C \le \frac{\ell}{1+\epsilon}\min_{\bar{\Omega}} m(x)$. Similarly, we can find a positive solution of (12),

$$u(x) \equiv \frac{C}{\ell}.$$

3. Suppose that $\frac{\ell}{1+\epsilon}\min_{\bar{\Omega}} m(x) < C < \frac{h}{1-\epsilon}\max_{\bar{\Omega}} m(x)$. Recall that $C = \alpha(\frac{m(x)}{u(x)})u(x)$, hence in this case, there exists $x \in [0,1]$ such that $1-\epsilon < \frac{m(x)}{u(x)} < 1+\epsilon$. When

$$1-\epsilon \le \frac{m(x)}{u(x)} \le 1+\epsilon,$$

   $\alpha(\frac{m(x)}{u(x)})u(x) = C$ yields that

$$-\frac{h}{2\epsilon}\left(\frac{m(x)}{u(x)} - 1 - \epsilon\right) + \frac{\ell}{2\epsilon}\left(\frac{m(x)}{u(x)} - 1 + \epsilon\right) = \frac{C}{u(x)}.$$

It then follows from direct calculations that

$$u(x) = \frac{hm(x) - \ell m(x) + 2\epsilon C}{h(1+\epsilon) - \ell(1-\epsilon)}.$$

Moreover, it is easy to see that $u(x) = C/\ell$ when $\frac{m(x)}{u(x)} > 1+\epsilon$ and $u(x) = C/h$ when $\frac{m(x)}{u(x)} < 1-\epsilon$.
Summing up, we have

$$u(x) = \begin{cases} C/\ell & \text{if } \frac{m(x)}{u(x)} > 1+\epsilon, \\ \frac{hm(x)-\ell m(x)+2\epsilon C}{h(1+\epsilon)-\ell(1-\epsilon)} & \text{if } 1-\epsilon \le \frac{m(x)}{u(x)} \le 1+\epsilon, \\ C/h & \text{if } \frac{m(x)}{u(x)} < 1-\epsilon. \end{cases} \tag{13}$$

Furthermore, notice that

$$C = \alpha(\frac{m(x)}{u(x)})u(x) = \alpha(\frac{m(x)}{u(x)})\frac{u(x)}{m(x)}m(x),$$

i.e.,

$$\frac{C}{m(x)} = \alpha(\frac{m(x)}{u(x)})\frac{u(x)}{m(x)},$$

where the right hand side is decreasing in $\frac{m(x)}{u(x)}$ according to (A1), therefore, it is clear that

$$1-\epsilon \le \frac{m(x)}{u(x)} \le 1+\epsilon$$

is equivalent to

$$\frac{C}{h}(1-\epsilon) \le m(x) \le \frac{C}{\ell}(1+\epsilon).$$

Similarly, $\frac{m(x)}{u(x)} > 1+\epsilon$ is equivalent to $m(x) > \frac{C}{\ell}(1+\epsilon)$, while $\frac{m(x)}{u(x)} < 1-\epsilon$ is equivalent to $m(x) < \frac{C}{h}(1-\epsilon)$.
Consequently, we derive the explicit solution as follows

$$u(x) = \begin{cases} C/\ell & \text{if } m(x) > \frac{C}{\ell}(1+\epsilon), \\ \frac{hm(x)-\ell m(x)+2\epsilon C}{h(1+\epsilon)-\ell(1-\epsilon)} & \text{if } \frac{C}{h}(1-\epsilon) \le m(x) \le \frac{C}{\ell}(1+\epsilon), \\ C/h & \text{if } m(x) < \frac{C}{h}(1-\epsilon). \end{cases} \tag{14}$$

In particular, if

$$\frac{\max_{\bar{\Omega}} m(x)}{\min_{\bar{\Omega}} m(x)} \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{h}{\ell},$$

for all $x \in [0,1]$, then for any $C_1 > 0$ satisfying

$$\frac{\ell}{1+\epsilon}\max_{\bar{\Omega}} m(x) \leq C_1 \leq \frac{h}{1-\epsilon}\min_{\bar{\Omega}} m(x)$$

we have

$$\frac{C_1}{h}(1-\epsilon) \leq m(x) \leq \frac{C_1}{\ell}(1+\epsilon)$$

holds for all $x \in [0,1]$. Hence, due to (13) and (14), it is obvious that in this case for all $x \in [0,1]$

$$1-\epsilon \leq \frac{m(x)}{u(x)} \leq 1+\epsilon$$

and in fact, to be more specific,

$$u(x) = \frac{hm(x) - \ell m(x) + 2\epsilon C_1}{h(1+\epsilon) - \ell(1-\epsilon)} \quad \text{in } [0,1].$$

The proof is complete. □

Remark 1. In this example, the conditions ϵ is very small and

$$\frac{\max_{\bar{\Omega}} m(x)}{\min_{\bar{\Omega}} m(x)} \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{h}{\ell}$$

mean that the species is very sensitive to the sufficiency of the resources. Hence intuitively, it should result in the fact that the distribution of the species can match that of the resources very well, which corresponds to the conclusion

$$1-\epsilon \leq \frac{m(x)}{u(x)} \leq 1+\epsilon$$

for all $x \in [0,1]$, in Theorem 2.

2. With dynamics

In this section, we will incorporate population dynamics into (1). In fact, we will consider the following logistic model

$$\begin{cases} u_t = \Delta\left(\alpha(\frac{m(x)}{u})u\right) + u[m(x) - u] & \text{for } x \in \Omega,\ t > 0, \\ \nabla[\alpha(\frac{m(x)}{u})u] \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0 & \text{for } x \in \Omega, \end{cases} \tag{15}$$

where $\alpha = d\alpha_1$, and study the existence, uniqueness, global stability and asymptotic behavior of the positive stationary solutions of (15).

2.1. General result

Theorem 3. *Suppose that (A1)and (A2) hold, then (15) has at least one positive stationary solution. If in addition,*

$$\alpha_1(s) - s(s-1)\alpha_1'(s) > 0 \text{ for } 0 < s < 1, \tag{16}$$

then there exists a unique globally asymptotically stable positive stationary solution of (15).

Proof. Set $w = \alpha_1(\frac{m(x)}{u})u$. Then

$$\mathcal{F}(x,u,w) = \alpha_1(\frac{m(x)}{u})u - w = 0.$$

Direct computation yields that

$$\mathcal{F}_u = \alpha_1(\frac{m(x)}{u}) - \alpha_1'(\frac{m(x)}{u})\frac{m(x)}{u} > 0. \tag{17}$$

Applying the Implicit Function Theorem, we get $u = \rho(x,w)$. Hence (15) can be rewritten as follows

$$\begin{cases} w_t = \left[\alpha_1 - \alpha_1' \frac{m}{u}\right] [d\Delta w + \rho(m-\rho)] & \text{for } x \in \Omega,\ t > 0, \\ \nabla w \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x) & \text{for } x \in \Omega. \end{cases} \tag{18}$$

Due to the relation between (15) and (18), it suffices to show (18) has a positive steady state. To this end, we will generalize the classical upper/lower solution method.

On the one hand side, choose

$$0 < c_1 < \alpha_1(\frac{\max_{\bar\Omega} m(x)}{\min_{\bar\Omega} m(x)}) \min_{\bar\Omega} m(x)$$

and set $\underline{w}(x) \equiv c_1$. It is easy to check that

$$\underline{u}(x) = \rho(x, \underline{w}) < \min_{\bar\Omega} m(x)$$

according to the assumption imposed on the mobility $\alpha_1(s)$ and $w = \alpha_1(\frac{m(x)}{u})u$. Clearly,

$$\left[\alpha_1 - \alpha_1' \frac{m}{u}\right] [d\Delta \underline{w} + \underline{u}(m - \underline{u})] > 0$$

in Ω. Let $w_1(x,t)$ denote the solution of the initial value problem (18) with initial data $\underline{w}(x) \equiv c_1$. By (18), it is standard to check that w_{1t} satisfies that

$$(w_{1t})_t = \left[\alpha_1 - \alpha_1' \frac{m}{u_1}\right] [d\Delta w_{1t} + (m-2\rho)\rho_w w_{1t}]$$
$$+ \left[\alpha_1 - \alpha_1' \frac{m}{u_1}\right]^{-2} \alpha_1'' \frac{m^2}{u_1^3} w_{1t}^2,$$

where $u_1 = \rho(x, w_1)$, with $\nabla w_{1t} \cdot \mathbf{n} = 0$ for $x \in \partial\Omega, t > 0$. Then, the maximum principle implies that $(w_1)_t \geq 0$ for $x \in \Omega, t > 0$.

On the other hand side, choose

$$c_2 > \alpha_1 \left(\frac{\min_{\bar{\Omega}} m(x)}{\max_{\bar{\Omega}} m(x)}\right) \max_{\bar{\Omega}} m(x)$$

and set $\widehat{w}(x) \equiv c_2$. Similarly, this implies that

$$u(x) = \rho(x, \widehat{w}) > \max_{\bar{\Omega}} m(x).$$

Then let $w_2(x,t)$ denote the solution of the initial value problem (18) with initial data $\widehat{w}(x) \equiv c_2$. Similarly, it can be proved that $(w_2)_t \leq 0$ for $x \in \Omega, t > 0$.

Now, since initially, $w_1(x,0) < w_2(x,0)$ in $\bar{\Omega}$ and for $t > 0$,

$$d\Delta w_1 + \rho(x, w_1)[m(x) - \rho(x, w_1)] \geq 0,$$
$$d\Delta w_2 + \rho(x, w_2)[m(x) - \rho(x, w_2)] \leq 0,$$

the maximum principle and Hopf boundary lemma imply that $w_1(x,t) < w_2(x,t)$ for $x \in \bar{\Omega}$ and $t > 0$.

Therefore, we have, the pointwise limits

$$\tilde{w}_1(x) = \lim_{t\to\infty} w_1(x,t),$$
$$\tilde{w}_2(x) = \lim_{t\to\infty} w_2(x,t)$$

exist and

$$w_1(x,0) \leq w_1(x,t) \leq \tilde{w}_1(x) \leq \tilde{w}_2(x) \leq w_2(x,t) \leq w_2(x,0). \tag{19}$$

Next, for any $\varphi \in C^2(\overline{\Omega})$ with $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$, multiplying the first equation in (18) with initial value $\underline{w}(x) \equiv c_1$ and integrating it over Ω, it follows that

$$\int_\Omega \frac{\rho(x, w_1(x,T)) - \rho(x, w_1(x,0))}{T}$$
$$= \frac{1}{T} \int_0^T \int_\Omega (d\Delta w_1 + \rho(x, w_1)[m(x) - \rho(x, w_1)])\, \varphi dx dt$$
$$= \int_\Omega \left\{ d\Delta\varphi \frac{1}{T} \int_0^T w_1(x,t) dt + \varphi \frac{1}{T} \int_0^T \rho(x, w_1)[m(x) - \rho(x, w_1)] dt \right\} dx.$$

Now, it is easy to see that

$$\lim_{T\to\infty}\int_\Omega \frac{\rho(x,w_1(x,T))-\rho(x,w_1(x,0))}{T}=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T w_1(x,t)dt=\tilde{w}_1(x),$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \rho(x,w_1)[m(x)-\rho(x,w_1)]dt=\rho(x,\tilde{w}_1)[m(x)-\rho(x,\tilde{w}_1)].$$

Therefore, by the Lebesgue dominated convergence theorem, we get

$$\int_\Omega d\tilde{w}_1\Delta\varphi+\rho(x,\tilde{w}_1)[m(x)-\rho(x,\tilde{w}_1)]\varphi dx=0,$$

for all $\varphi\in C^2(\overline{\Omega})$ with $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$. Similar to the arguments in [4, Page 989], we have $\tilde{w}_1(x)$ is a classical stationary solution of (18).

Similarly, it can be proved that $\tilde{w}_2(x)$ is also a classical stationary solution of (18).

Consequently, the existence of positive steady state of (15) is proved.

Suppose that $\tilde{w}_1(x)\neq\tilde{w}_2(x)$ and (16) holds. Set $\tilde{u}_1(x)=\rho(x,\tilde{w}_1)$ and $\tilde{u}_2(x)=\rho(x,\tilde{w}_2)$. Since $\tilde{w}_1(x)$ satisfies

$$\begin{cases} d\Delta\tilde{w}_1+\tilde{u}_1[m(x)-\tilde{u}_1]=0 & \text{for } x\in\Omega,\\ \nabla\tilde{w}_1\cdot\mathbf{n}=0 & \text{for } x\in\partial\Omega,\end{cases}$$

multiplying it by $\tilde{w}_2(x)$ and integrating over Ω, we get

$$\int_\Omega\left(-d\nabla\tilde{w}_1\nabla\tilde{w}_2+\tilde{w}_2(x)\tilde{u}_1[m(x)-\tilde{u}_1]\right)dx=0.$$

Similarly, using the equation satisfied by $\tilde{w}_2(x)$, we have

$$\int_\Omega\left(-d\nabla\tilde{w}_1\nabla\tilde{w}_2+\tilde{w}_1(x)\tilde{u}_2[m(x)-\tilde{u}_2]\right)dx=0.$$

Therefore, it follows that

$$\int_\Omega\alpha_1(\frac{m(x)}{\tilde{u}_2})\tilde{u}_2\tilde{u}_1[m(x)-\tilde{u}_1]dx=\int_\Omega\alpha_1(\frac{m(x)}{\tilde{u}_1})\tilde{u}_1\tilde{u}_2[m(x)-\tilde{u}_2]dx. \tag{20}$$

Obviously, (20) can be rewritten as

$$\int_\Omega\left[\frac{1-\frac{\tilde{u}_1}{m(x)}}{\alpha_1(\frac{m(x)}{\tilde{u}_1})}-\frac{1-\frac{\tilde{u}_2}{m(x)}}{\alpha_1(\frac{m(x)}{\tilde{u}_2})}\right]\alpha_1(\frac{m(x)}{\tilde{u}_1})\alpha_1(\frac{m(x)}{\tilde{u}_2})m(x)\tilde{u}_2\tilde{u}_1dx=0.$$

On the other hand side, note that $\tilde{w}_1(x) \le \tilde{w}_2(x)$ implies that $\tilde{u}_1(x) \le \tilde{u}_2(x)$ because of (17). Moreover, it is routine to check that (A1) and (16) yield that

$$\frac{d}{ds}\left(\frac{1-\frac{1}{s}}{\alpha_1(s)}\right) > 0 \text{ for all } s > 0.$$

Hence, it follows from the assumption $\tilde{w}_1(x) \ne \tilde{w}_2(x)$ that

$$\int_\Omega \alpha_1(\frac{m(x)}{\tilde{u}_2})\tilde{u}_2\tilde{u}_1[m(x)-\tilde{u}_1]dx > \int_\Omega \alpha_1(\frac{m(x)}{\tilde{u}_1})\tilde{u}_1\tilde{u}_2[m(x)-\tilde{u}_2]dx.$$

This is a contradiction to (20). Therefore, (18) has a unique positive steady state, which is globally asymptotically stable since $w_1(x,0) = c_1$ and $w_2(x,0) = c_2$ can be respectively chosen to be arbitrarily small and large.

The proof is complete. □

Remark 2. The choices of generalized upper/solutions in the proof of Theorem 3 automatically implies that

$$\alpha_1(\frac{\max_{\bar{\Omega}} m(x)}{\min_{\bar{\Omega}} m(x)}) \min_{\bar{\Omega}} m(x) \le w \le \alpha_1(\frac{\min_{\bar{\Omega}} m(x)}{\max_{\bar{\Omega}} m(x)}) \max_{\bar{\Omega}} m(x), \quad (21)$$

where $w(x)$ denotes a positive steady state of (18).

The following result is about the uniqueness of positive stationary solution of (15) with extra conditions imposed on the resources $m(x)$.

Proposition 1. *Suppose that (A1)and (A2) hold. If m satisfies that $\nabla m \cdot \mathbf{n} \le 0$ on $\partial\Omega$ and $\Delta m + \frac{1}{2\alpha(2)}m^2 \ge 0$ in $\bar{\Omega}$, then (15) has a unique globally asymptotically stable positive steady state.*

Proof. Suppose that there exist two different positive steady states of (15), denoted by $u_1(x)$ and $u_2(x)$. Thanks to the proof of Theorem 3, we can assume that $u_1(x) \le u_2(x)$ in Ω.

Recall that $u_i(x)$, $i = 1, 2$, satisfy

$$\begin{cases} \Delta\left(\alpha(\frac{m(x)}{u_i})u_i\right) + u_i[m(x)-u_i] = 0 & \text{for } x \in \Omega, \\ \nabla[\alpha(\frac{m(x)}{u_i})u_i]\cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

It is easy to see that

$$\int_\Omega u_1[m(x)-u_1]dx = \int_\Omega u_2[m(x)-u_2]dx = 0,$$

which immediately implies that

$$\int_\Omega (u_1-u_2)[m(x)-u_1-u_2]dx = 0. \quad (22)$$

We claim that $u(x) \geq \frac{m(x)}{2}$ in Ω under the assumptions of this proposition, where $u(x)$ denotes a positive steady state of (15).

Then, this claim, $u_1(x) \leq u_2(x)$ and $u_1(x) \neq u_2(x)$ immediately imply that

$$\int_\Omega (u_1 - u_2)[m(x) - u_1 - u_2]dx > 0,$$

which contradicts (22). Hence, the uniqueness and thus global stability of the positive steady state of (15) follow.

Now, it suffices to prove the claim. For convenience, set

$$\beta(s) = \alpha(s)\frac{1}{s} \text{ and } s(x) = \frac{m(x)}{u(x)}.$$

Notice that on the boundary $\partial\Omega$,

$$0 = \nabla[\alpha(\frac{m}{u})u] \cdot \mathbf{n} = \nabla[\beta(s)m] \cdot \mathbf{n} = \left[\beta'(s)m\nabla s + \beta(s)\nabla m\right] \cdot \mathbf{n},$$

which, together with the assumption $\nabla m \cdot \mathbf{n} \leq 0$ and $\beta'(s) < 0$, yields that

$$\nabla s \cdot \mathbf{n} \leq 0 \text{ on } \partial\Omega. \tag{23}$$

Hence the problem

$$\begin{cases} \Delta\left(\alpha(\frac{m(x)}{u})u\right) + u[m(x) - u] = 0 & \text{for } x \in \Omega, \\ \nabla[\alpha(\frac{m(x)}{u})u] \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega \end{cases}$$

can be rewritten as

$$\begin{cases} \Delta\left(\beta(s)m\right) + m^2\frac{1}{s}\left[1 - \frac{1}{s}\right] = 0 & \text{for } x \in \Omega, \\ \nabla s \cdot \mathbf{n} \leq 0 & \text{for } x \in \partial\Omega. \end{cases}$$

It then follows from straightforward computations that $s(x)$ satisfies

$$\Delta s + \frac{\beta''}{\beta'}|\nabla s|^2 + \frac{2}{m}\nabla m \cdot \nabla s + \frac{\beta}{\beta' m}\left[\Delta m + \frac{m^2}{\alpha}\left(1 - \frac{1}{s}\right)\right] = 0 \tag{24}$$

in Ω.

Clearly, to prove the claim, we simply need show that

$$\max_{\bar{\Omega}} s(x) \leq 2. \tag{25}$$

Suppose that this is not true, that is, there exists $x_0 \in \bar{\Omega}$ such that

$$s(x_0) = \max_{\bar{\Omega}} s(x) > 2.$$

First, assume that $x_0 \in \Omega$. Then at x_0, it is routine to check that

$$\Delta m(x_0) + \frac{m^2(x_0)}{\alpha(s(x_0))}\left(1 - \frac{1}{s(x_0)}\right) > \Delta m(x_0) + \frac{m^2(x_0)}{2\alpha(2)} \geq 0$$

due to the assumption on $m(x)$. By (24), we have $\Delta s(x_0) > 0$, which is impossible since the maximum value of s is achieved at x_0.

Secondly, assume that $x_0 \in \partial\Omega$. Obviously, $\nabla s(x_0) \cdot \mathbf{n} \geq 0$ since

$$s(x_0) = \max_{\bar{\Omega}} s(x) > 2.$$

Thus, $\nabla s(x_0) \cdot \mathbf{n} = 0$ because of (23). Moreover, the maximum value of s is obtained at $x_0 \in \partial\Omega$ and $\nabla s(x_0) \cdot \mathbf{n} = 0$ together imply that $|\nabla s(x_0)| = 0$. Similarly, we also have

$$\Delta m(x_0) + \frac{m^2(x_0)}{\alpha(s(x_0))}\left(1 - \frac{1}{s(x_0)}\right) > 0.$$

Therefore, by (24), $\Delta s(x) > 0$ in $B_\delta(x_0) \bigcap \Omega$, where $B_\delta(x_0)$ denotes a ball centered at x_0 with radius δ and $\delta > 0$ is sufficiently small. Then $\nabla s(x_0) \cdot \mathbf{n} = 0$ contradicts to the Hopf boundary lemma.

Consequently, (25) holds and the claim is proved. □

Note that to guarantee the existence of positive stationary solutions of (15), the requirements for the mobility $\alpha(s)$ is very weak. We only require the values of $\alpha(s)$ to be positive and non-increasing. However, to have the uniqueness of positive steady states of (15), we have to impose extra technical conditions either on the mobility $m(x)$ or the resources α. We hope that these conditions can be removed.

At the end of this section, we want to include a *priori* estimates for the positive steady states of (15), namely positive solution to

$$\begin{cases} d\Delta\left(\alpha_1(\frac{m(x)}{u})u\right) + u[m(x) - u] = 0 & \text{for } x \in \Omega, \\ \nabla[\alpha(\frac{m(x)}{u})u] \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{26}$$

which will be useful in proving limiting cases later.

Lemma 1. *Let $u(x)$ denote a positive solution of (26). Then*

$$\min_{\bar{\Omega}} m(x) \leq \alpha_1(\frac{m(x)}{u})u \leq \max_{\bar{\Omega}} m(x). \tag{27}$$

Proof. For convenience, set $w(x) = \alpha_1(\frac{m(x)}{u})u$. Suppose that

$$w(x_0) = \alpha_1(\frac{m(x_0)}{u(x_0)})u(x_0) = \max_{\bar{\Omega}} w(x) > \max_{\bar{\Omega}} m(x). \tag{28}$$

First, we assume that $x_0 \in \Omega$. From (26), it is easy to see that

$$u(x_0)[m(x_0) - u(x_0)] = -d\Delta w(x_0) \geq 0.$$

Thus

$$m(x_0) \geq u(x_0), \tag{29}$$

which, combined with the assumption (A1) and (28), implies that

$$u(x_0) \geq \alpha_1(\frac{m(x_0)}{u(x_0)})u(x_0) = w(x_0) > \max_{\bar{\Omega}} m(x) \geq m(x_0).$$

This obviously contradicts (29).

Now, we know $x_0 \in \partial\Omega$. If $m(x_0) \geq u(x_0)$, similarly, it follows from (A1) and (28) that $m(x_0) < u(x_0)$, which is impossible. Therefore, $m(x_0) < u(x_0)$. However, this implies that

$$\Delta w(x_0) = -u(x_0)[m(x_0) - u(x_0)] > 0.$$

Hence,

$$\Delta w > 0 \text{ in } B_\delta(x_0) \bigcap \Omega,$$

where $B_\delta(x_0)$ denotes a ball centered at x_0 with sufficiently small radius $\delta > 0$. Since according to our assumption (28) $x_0 \in \partial\Omega$ is where the maximum value of w is achieved. Thanks to the Hopf boundary lemma, a contradiction follows easily.

Consequently,

$$\alpha_1(\frac{m(x)}{u})u \leq \max_{\bar{\Omega}} m(x).$$

The other inequality in (27) can be handled in the same way. □

Obviously, Lemma 1 is an improvement of (21).

Lemma 2. *Let $u(x)$ denote a positive solution of (26). Then there exist positive constants K_1 and K_2 independent of d such that*

$$K_1 \leq \frac{m(x)}{u(x)} \leq K_2. \tag{30}$$

Proof. Define $\beta(s) = \alpha_1(s)/s$. According to the assumption (A1), it is easy to see that

$$\beta'(s) = \frac{\alpha_1'(s)}{s} - \frac{\alpha_1(s)}{s^2} < 0.$$

Hence $\beta(s)$ is invertible. Moreover, by Lemma 1, we have

$$\frac{\min_{\bar{\Omega}} m(x)}{m(x)} \leq \beta(\frac{m(x)}{u}) = \alpha_1(\frac{m(x)}{u})\frac{u(x)}{m(x)} \leq \frac{\max_{\bar{\Omega}} m(x)}{m(x)}.$$

Therefore $K_1 \leq \frac{m(x)}{u(x)} \leq K_2$, where

$$K_1 = \beta^{-1}(\frac{\max_{\bar{\Omega}} m(x)}{\min_{\bar{\Omega}} m(x)}) \text{ and } K_2 = \beta^{-1}(\frac{\min_{\bar{\Omega}} m(x)}{\max_{\bar{\Omega}} m(x)}).$$

The proof is complete. □

2.2. Limiting cases

Let $u_d(x)$ denote a positive solution of the stationary problem

$$\begin{cases} d\Delta\left(\alpha_1(\frac{m(x)}{u})u\right) + u[m(x) - u] = 0 & \text{for } x \in \Omega, \\ \nabla[\alpha_1(\frac{m(x)}{u})u] \cdot \mathbf{n} = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{31}$$

We will explore the behavior of the solution $u_d(x)$ as $d \to 0$ or $d \to \infty$.

Theorem 4. $\lim_{d\to 0} u_d = m$ *uniformly on any compact subset of* Ω.

Theorem 5. w_d *converges to a constant in* $C^{1,a}$ *as* $d \to \infty$, *where* $w_d = \alpha(\frac{m(x)}{u_d})u_d$.

The idea of the proofs of these two theorems comes from [1].
For some positive constant c, we consider the following equation

$$\begin{cases} d\Delta v + \frac{v}{\alpha_1^2(\frac{m}{u_d})}(c - v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

Then we can prove that (32) has the unique positive solution, denoted by v_d^Ω, for sufficiently small d. Because, $\widehat{v} \equiv c + 1$ is a upper solution and $\underline{v} \equiv \varphi$ is a lower solution, where $\varphi > 0$ with $\|\varphi\|_{L^\infty} = \frac{c}{2}$ is the first eigenfunction of

$$\begin{cases} \Delta\varphi + \mu\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Moreover, let v_1 and v_2 be two distinct positive solution of (32). Then we may assume that $v_1 \le v_2$ and so,

$$\begin{aligned} 0 &= \int \left\{ d\Delta v_1 + \frac{v_1}{\alpha_1^2(\frac{m}{u_d})}(c - v_1) \right\} v_2 \\ &= \int \left\{ d\Delta v_2 + \frac{v_2}{\alpha_1^2(\frac{m}{u_d})}(c - v_2) \right\} v_1. \end{aligned}$$

This implies that

$$\int \frac{v_1 v_2}{\alpha_1^2(\frac{m}{u_d})}(v_1 - v_2) = 0.$$

This is a contradiction.

Lemma 3. $\liminf_{d\to 0} v_d^\Omega \ge c$ *uniformly on any compact subset of* Ω.

Proof. We need show that for any compact $K \subset \Omega$ and every $\varepsilon > 0$, there exists $d_0 = d(K, \varepsilon) > 0$ s.t. $v_d(x) \geq c - \varepsilon$, for any $x \in K$ and $0 < d < d_0$.

Let $\varepsilon > 0$ be arbitrary, consider $x_0 \in K$ and fix $\rho > 0$ s.t. $B_{2\rho}(x_0) \subset \Omega$. Let $\mu_1(B_\rho(x_0))$, $\varphi_1 > 0$ with $\|\varphi_1\|_{L^\infty} = 1$ be the first eigenvalue and eigenfunction of

$$\begin{cases} \Delta\varphi + \mu\varphi = 0 & \text{in } B_\rho(x_0), \\ \varphi = 0 & \text{on } \partial B_\rho(x_0). \end{cases} \tag{34}$$

We claim that $(c - dh^2\mu_1(B_\rho(x_0)))\varphi_1(x) \leq v_d^{B_\rho(x_0)}(x) \leq v_d^\Omega(x)$, *for all* $x \in B_\rho(x_0)$ *and sufficiently small* d.

Since v_d^Ω is an upper solution of the equation satisfied by $v_d^{B_\rho(x_0)}$ in $B_\rho(x_0)$, $v_d^{B_\rho(x_0)}(x) \leq v_d^\Omega(x)$.

Assume that $dh^2\mu_1(B_\rho(x_0)) < c$ otherwise the conclusion is clear. Let $w_1 = v_d^{B_\rho(x_0)}$ and $w_2 = (c - dh^2\mu_1(B_\rho(x_0)))\varphi_1$. Then

$$\begin{cases} \Delta w_1 + \frac{w_1}{d\alpha_1^2(\frac{m}{u_d})}(c - w_1) = 0 & \text{in } B_\rho(x_0), \\ w_1 = 0 & \text{on } \partial B_\rho(x_0). \end{cases} \tag{35}$$

and

$$\begin{cases} \Delta w_2 + \mu_1(B_\rho(x_0))w_2 = 0 & \text{in } B_\rho(x_0), \\ w_2 = 0 & \text{on } \partial B_\rho(x_0). \end{cases} \tag{36}$$

Hence, it follows that

$$\begin{aligned} &\Delta w_2 + \frac{w_2}{d\alpha_1^2(\frac{m}{u_d})}(c - w_2) \\ &= -\mu_1(B_\rho(x_0))w_2 + \frac{w_2}{d\alpha_1^2(\frac{m}{u_d})}(c - w_2) \\ &= \left\{ \frac{c}{d\alpha_1^2(\frac{m}{u_d})} - \mu_1(B_\rho(x_0)) - \left(\frac{c}{d\alpha_1^2(\frac{m}{u_d})} - \frac{h^2\mu_1(B_\rho(x_0))}{\alpha_1^2(\frac{m}{u_d})} \right) \varphi_1 \right\} w_2 \\ &\geq \left(\frac{c}{d\alpha_1^2(\frac{m}{u_d})} - \frac{h^2\mu_1(B_\rho(x_0))}{\alpha_1^2(\frac{m}{u_d})} \right) (1 - \varphi_1)w_2 \geq 0. \end{aligned}$$

Since w_2 is a lower solution, $w_2 \leq w_1$. Let $d_0(x_0, \varepsilon) = \frac{\varepsilon}{2h^2\mu_1(B_\rho(x_0))}$. For every $d < d_0(x_0, \varepsilon)$, since $v_d^{B_\rho(x_0)}(x)$ is continuous, we can choose $0 < \rho(x_0, d, \varepsilon) < \rho$ s.t. $v_d^{B_\rho(x_0)}(x) \geq v_d^{B_\rho(x_0)}(x_0) - \frac{\varepsilon}{2}$, for all $x \in B_{\rho(x_0,d,\varepsilon)}(x_0)$. Then for every $x \in B_{\rho(x_0,d,\varepsilon)}(x_0)$, we have

$$\begin{aligned} v_d^\Omega(x) \geq v_d^{B_\rho(x_0)}(x) &\geq v_d^{B_\rho(x_0)}(x_0) - \frac{\varepsilon}{2} \\ &\geq [c - dh^2\mu_1(B_\rho(x_0))]\varphi_1(x_0) - \frac{\varepsilon}{2} \geq c - \varepsilon. \end{aligned}$$

Then the conclusion follows by standard compactness argument. □

We set that $w_d^{B_\rho(x_0)}$ is the solution of

$$\begin{cases} d\Delta w + \frac{w}{\alpha_1^2(\frac{m}{u_d})}\left(\alpha_1(\frac{m}{u_d})m - w\right) = 0 & \text{in } B_\rho(x_0), \\ w = 0 & \text{on } \partial B_\rho(x_0). \end{cases} \tag{37}$$

Lemma 4. *Assume $m \in C(\bar{\Omega})$. For any $x_0 \in \Omega$, there exists $\rho > 0$ and $d_0(x_0, \varepsilon) > 0$ s.t. for every $d < d_0(x_0, \varepsilon)$, $w_d(x) \geq \alpha_1(\frac{m}{u_d})m(x) - \varepsilon$ in $B_\rho(x_0)$.*

Proof. Since $m \in C(\bar{\Omega})$, there exists a constant $\rho_1 = \rho_1(x_0, \varepsilon) > 0$ s.t. $B_{\rho_1}(x_0) \subset \Omega$ and $\min_{B_{\rho_1}(x_0)} \alpha_1(\frac{m}{u_d})m \geq \alpha_1(\frac{m}{u_d})m - \frac{\varepsilon}{2}$, for all $x \in B_{\rho_1}(x_0)$. Then in $B_{\rho_1}(x_0)$, since w_d is an upper solution of (37), $w_d \geq w_d^{B_{\rho_1}(x_0)}$. Moreover, since $v_d^{B_{\rho_1}(x_0)}$ is a lower solution of (37), $w_d^{B_{\rho_1}(x_0)} \geq v_d^{B_{\rho_1}(x_0)}$, where $c = \min_{B_{\rho_1}(x_0)} \alpha_1(\frac{m}{u_d})m$. Let $\rho = \min\{\rho(x_0, d, \varepsilon), \rho_1\}$. Lemma 3 guarantees the existence of $d_0(x_0, \varepsilon)$ s.t. for every $d < d_0(x_0, \varepsilon)$ and $x \in B_\rho(x_0)$, $v_d^{B_{\rho_1}(x_0)} \geq \min_{B_{\rho_1}(x_0)} \alpha_1(\frac{m}{u_d})m - \frac{\varepsilon}{2}$.

Therefore, $w_d \geq v_d^{B_{\rho_1}(x_0)} \geq \min_{B_{\rho_1}(x_0)} \alpha_1(\frac{m}{u_d})m - \frac{\varepsilon}{2} \geq \alpha_1(\frac{m}{u_d})m(x) - \varepsilon$ for all $x \in B_\rho(x_0)$. □

Lemma 5. *Assume $m \in C(\bar{\Omega})$. Then, for any small $\varepsilon > 0$, there exists $d_0 > 0$ s.t. $0 \leq w_d(x) \leq \alpha_1(\frac{m}{u_d})m + \varepsilon$ in Ω for every $0 < d < d_0$.*

Proof. Let $w \in C^2$ s.t. $\alpha_1(\frac{m}{u_d})m + \frac{\varepsilon}{2} \leq w \leq \alpha_1(\frac{m}{u_d})m + \varepsilon$ for all $x \in \Omega$ and $\frac{\partial w}{\partial \nu}|_{\partial\Omega} \geq 0$. Then, there exists $d_0(\varepsilon) > 0$ s.t. for any $d < d_0(\varepsilon)$

$$\begin{aligned} & d\Delta w + \frac{w}{\alpha_1^2(\frac{m}{u_d})}\left(\alpha_1(\frac{m}{u_d})m - w\right) \\ \leq & d\Delta w + \frac{w}{\alpha_1^2(\frac{m}{u_d})}\left(\alpha_1(\frac{m}{u_d})m - \alpha_1(\frac{m}{u_d})m - \frac{\varepsilon}{2}\right) \\ \leq & d\Delta w - \frac{\varepsilon w}{2\alpha_1^2(\frac{m}{u_d})} \leq 0. \end{aligned}$$

So w is an upper solution. Therefore $0 \leq w_d(x) \leq w(x) \leq \alpha_1(\frac{m}{u_d})m + \varepsilon$ in Ω for every $0 < d < d_0$. □

Theorem 4 follows immediately from Lemmas 4 and 5. Next, we consider the limiting solution when $d \to \infty$.

Proof of Theorem 5. We denote by $w_d = \bar{w} + \psi$, where $\bar{w} = \frac{1}{|\Omega|}\int_\Omega w_d dx$ and $\int_\Omega \psi dx = 0$.

Set
$$C = 2\max_{\bar{\Omega}} m(x).$$
Thanks to Lemma 1, we have $\|w_d\|_{L^\infty} \le C/2$, $\|\bar{w}\|_{L^\infty} \le \|w_d\|_{L^\infty} \le C/2$ and $\|\psi\|_{L^\infty} = \|w_d - \bar{w}\|_{L^\infty} \le C$.

Now consider the equation satisfied by ψ,
$$\begin{cases} d\Delta\psi + \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\left(m - \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\right) = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$
Hence,
$$\begin{aligned} & d\int_\Omega |\nabla\psi|^2 \\ = & \int_\Omega \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\left(m - \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\right)\psi \\ = & \bar{w}\int_\Omega \frac{m\psi}{\alpha_1(\frac{m}{u_d})} - \bar{w}^2\int_\Omega \frac{\psi}{\alpha_1^2(\frac{m}{u_d})} + \int_\Omega \left(m\alpha_1(\frac{m}{u_d}) - 2\bar{w} - \psi\right)\frac{\psi^2}{\alpha_1^2(\frac{m}{u_d})}. \end{aligned}$$
This, combined with Lemma 2, implies that
$$\begin{aligned} d\int_\Omega |\nabla\psi|^2 \le & \frac{\bar{w}}{l}\|m\|_{L^2}\|\psi\|_{L^2} + \bar{w}^2\|\alpha_1^{-2}(\frac{m}{u_d})\|_{L^2}\|\psi\|_{L^2} \\ & + \frac{\|m\alpha_1(\frac{m}{u_d}) - 2\bar{w} - \psi\|_{L^\infty}}{l^2}\int_\Omega \psi^2, \end{aligned}$$
where $\ell = \alpha_1(K_2)$.

Since $\mu_1 := \inf\{\frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} : \int_\Omega v dx = 0, v \neq 0, \frac{\partial v}{\partial\nu}|_{\partial\Omega} = 0\} > 0$,
$$\mu_1 d\int_\Omega \psi^2 \le d\int_\Omega |\nabla\psi|^2 \le c\|\psi\|_{L^2} + c\int_\Omega \psi^2.$$
When $d > 0$ is large such that $c < \frac{1}{2}d\mu_1$, we get $\|\psi\|_{L^2} \le \frac{c}{d}$. And this implies that $\|\nabla\psi\|_{L^2} \le \frac{c}{d}$. Therefore, we have $\|\psi\|_{H^1} \le \frac{c}{d}$ and thus $\|\psi\|_{L^{q_1}} \le \frac{c}{d}$, where $q_1 = \frac{2n}{n-2}$. Since ψ satisfies that
$$\begin{cases} \Delta\psi - \psi = \frac{1}{d}\frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\left(m - \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\right) - \psi & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{39}$$
and
$$\|\frac{1}{d}\frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\left(m - \frac{\bar{w}+\psi}{\alpha_1(\frac{m}{u_d})}\right) - \psi\|_{L^{q_1}} \le \frac{c}{d},$$

we obtain that $\|\psi\|_{W^{2,q_1}} \le \frac{c}{d}$. Moreover, from the embedding theorem, it follows that $\|\psi\|_{L^{q_2}} \le \frac{c}{d}$, where $q_2 = \frac{nq_1}{n-2q_1}$. By this iteration, we get $\|\psi\|_{W^{2,q}} \le \frac{c}{d}$, for any $1 < q < \infty$. This implies that $\|\psi\|_{C^{1,a}} \le \frac{c}{d}$ and so, $\psi \to 0$ in $C^{1,a}$ as $d \to \infty$. Therefore, w_d converges to a constant in $C^{1,a}$ as $d \to \infty$. $\square$

Remark 3. Since $d\Delta w_d + \rho(x, w_d)[m(x) - \rho(x, w_d)] = 0$, it follow that $\int_\Omega \rho(x, w_d)[m(x) - \rho(x, w_d)]dx = 0$. Therefore, as $d \to \infty$, w_d converges to a constant C_1, where $\int_\Omega \rho(x, C_1)[m(x) - \rho(x, C_1)]dx = 0$.

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