SPATIALLY NON-UNIFORM RANDOM WALK ON THE REAL LINE

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Abstract. The random walk is a mathematical counterpart of a Brownian particle system and has been used as a powerful theoretical tool to understand random dispersal phenomena. If temperature is not spatially constant, the mean free path and the traveling time for the path has variation in space. In this paper we investigate behavior of a simple random walk having spatially dependent walk length and jumping time. One of the main conclusions is that one does not sense the change of the walk length on the other side, but does sense the change of the jumping time. This also indicates that a random walk system with a non-constant time step cannot be reduced to a constant time step one even if the walk length is rescaled.

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1. Introduction

The purpose of this paper is to develop a diffusion model for a non-uniform random walk system when the walk length $\Delta x$ and the jumping time interval $\Delta t$ are spatially non-constant. Einstein’s theory of the Brownian movement [7, 8] explains the connection between the Brownian motion and the diffusion process. A random walk system is a mathematical counterpart of such phenomena, where the parameters $\Delta x$ and $\Delta t$ represent the mean free path and the mean time interval between collisions of the Brownian particles, respectively. A case with constants $\Delta x$ and $\Delta t$ or a case that these parameters are random variables with spatially constant means and variations are well understood. In this paper we will see that new features appear if $\Delta x$ and $\Delta t$ are varying in space.

Non-uniform random dispersal appears in many places and one obvious source of it is the temperature gradient. For example, the Brownian particles move more actively in a warm region and less actively in a cold region. The effect of temperature gradient such as thermal diffusion or Ludwig-Soret effect [17, 23] has been well known for a long time. There have been enormous amount of researches related to the thermal diffusion phenomenon and readers are referred to review papers on experiments and modeling [1, 6, 9, 11, 12, 20, 24]. The effect of the thermal
gradient, one of the main reasons of dispersal, has been studied for a long time [2, 18]. However, the literatures agree that there is no comprehensive or generic thermal diffusion model such as Einstein’s molecular level explanation for the homogeneous case.

Another example of non-uniform random dispersal is the migration of biological organisms. The necessity of formulating a more realistic dispersal theory beyond linear dispersal has been emphasized in mathematical ecology (see Skellam [21, 22] and Okubo & Levin [19, Chapter 5]). In many cases biological organisms change their dispersal strategy if they are starving. Also in many cases the organisms do not know where to go, but do know when they should leave. To model such a behavior, the starvation driven diffusion is developed [3, 14, 15]. Furthermore, a meaningful measure of distance for biological organisms is not simply the Euclidean distance but one which concerns food or mating. In fact Keller-Segel type traveling wave can be developed with an idea of the non-uniform random walk, but without usual assumptions on gradient [4, 25].

If the walk length and jumping time are constant, the concentration density $u$ of particles in a random walk system in one space dimension can be modeled by a linear diffusion equation

$$u_t = ku_{xx}, \quad k = \frac{|\Delta x|^2}{2\Delta t},$$

where the diffusivity constant $k$ is given by the Einstein-Smoluchowski relation. This diffusion model is valid only when the diffusivity is constant. In this paper we will derive diffusion equations that corresponds to non-uniform random walks. Heterogeneity in the walk length $\Delta x$ will be studied first and then the one in the jumping time $\Delta t$ will. These two cases turn out to be quite different. For example, one may sense the change of $\Delta t$ but not of $\Delta x$ (see Section 4.2).

In Section 2, we first consider a random walk model that the walk length is given as a spatial function $\Delta x = \frac{||\pi||}{2} b(x)$ for a constant $||\pi||$ in (4) and the jumping time interval $\Delta t = \tau$ as a constant. It is shown in Theorem 2.1 that the probability density function of the discrete random walk model converges to the solution of

$$u_t = d(b(x)b(x)u_x)_x, \quad d = \frac{||\pi||^2}{8\tau}.$$ 

Note that, even if this equation is not autonomous, we may solve it with a general initial value since, after changing the variable with

$$y(x) := \int_0^x \frac{1}{b(s)} ds,$$

the equation returns to the heat equation $u_t = du_{yy}$.

The spatial variation in the jumping time interval $\Delta t$ makes a more profound theoretical difference in comparison with the one in $\Delta x$. In particular the corresponding random walk model is not even a Markov process. In Section 3, we consider a simplest case when the jumping time consists of two values:

$$\Delta t = \begin{cases} \tau & \text{if } x < 0, \\ 2\tau & \text{if } x > 0, \end{cases}$$

where $\tau > 0$ is a constant. However, due to the discontinuity at the origin, it is challenging to develop and analyze a corresponding PDE model; it will be shown that the probability density function of the discrete random walk model converges to the solution of

$$u_t = d(b(x)b(x)\alpha(x)u_x)_x, \quad d = \frac{||\pi||^2}{8\tau},$$

where

$$\alpha(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0.5 & \text{if } x > 0. \end{cases}$$
Then, after the same variable change as above, we obtain \( u_t = d(\alpha(x)u)_{yy} \). If we set \( w = \alpha u \) and \( d = 1 \), then \( w \) satisfies

\[
w_t = \alpha w_{yy}, \quad y \in \mathbb{R}.
\]

Notice that this equation is meaningless at \( y = 0 \) due to the discontinuity of \( \alpha \). However, it is shown in Proposition 3.1 that the limit of the probability density function of the corresponding random walk system has a continuous first order derivative at the origin. Hence we are looking for a solution which is \( C^1(\mathbb{R}) \). Green’s function of this PDE with a discontinuous coefficient has been constructed in Section 3.2 explicitly. Green’s function corresponding to the diffusion equation (2) is obtained by going back to the original variable \( x \).

## 2. Random walk with a spatially non-constant walk length

In this section we develop a random walk theory with a spatial heterogeneity in the walk length \( \Delta x \). The time step \( \Delta t \) is assumed to be constant in this section. First we introduce some notations. Let spatial grid points \( x^k \) be ordered as \( x^0 = 0 \) and \( 1 \)

\[
\cdots < x^{k-1} < x^k < x^{k+1} < \cdots \quad \text{with} \quad k \in \mathbb{Z}.
\]

For a uniform random walk system, the distance between two neighboring grid points is constant. In this paper we consider a non-uniform case when \( x^{k+1} - x^k \) is not necessarily constant. We introduce a partition consisting of odd numbered grid points and its norm:

\[
\pi := \{ x^{2k+1} : k \in \mathbb{Z} \}, \quad \| \pi \| := \sup_{k \in \mathbb{Z}}(x^{2k+1} - x^{2k-1}).
\]

Let \( \beta \) be defined at even numbered grid points as

\[
\beta(x^{2k}) := x^{2k+1} - x^{2k-1},
\]

which measures the distance between odd numbered grid points. We will see in the followings that even numbered grid points and odd numbered ones are separated. The theory of this section will be developed in terms of the even numbered grid points and that is the reason for defining \( \beta \) at even numbered ones only. Similarly, we define

\[
b(x^{2k}) := \frac{x^{2k+1} - x^{2k-1}}{\| \pi \|} = \frac{\beta(x^{2k})}{\| \pi \|},
\]

which measures heterogeneity in the walk length. We assume this ratio is bounded away from zero, i.e., there exists \( c > 0 \) such that

\[
0 < c \leq b(x^{2k}) \leq 1, \quad k \in \mathbb{Z}.
\]

We will keep this lower bound \( c > 0 \) uniformly when we take the limit as \( \| \pi \| \to 0 \). We may extend these operations over the real line by defining

\[
\beta(x) := \beta(x^{2k}) \quad \text{and} \quad b(x) := b(x^{2k}) \quad \text{for} \quad x^{2k-1} \leq x < x^{2k+1}.
\]

Notice that \( \beta(x) \) is not the walk length, but the sum of two adjacent walk lengths. The reason for this is to use it in computing the probability density as in (14). If we return to a uniform random walk case with equally spaced grid points, we have \( c = 1, b(x) \equiv 1 \), and \( \Delta x = \frac{1}{2} \| \pi \| \).

### 2.1. Derivation of a PDE model

Consider a particle randomly walking along the grid points. Let \( p_0^k \) be the probability that the particle is to be placed at a grid point \( x^k \) and at initial time \( t = 0 \). Similarly, let \( p_n^k \) be such probability at time \( t_n = n \Delta t \). In a one dimensional random walk system, a particle walks to one of the two adjacent grid points with the same probability. So, for all \( n \geq 1 \) and \( k \in \mathbb{Z} \),

\[
2p_n^k = p_{n-1}^{k-1} + p_{n-1}^{k+1}.
\]

Applying this relation twice, we obtain

\[
4p_{2n}^{2k} = 2(p_{2n-1}^{2k-1} + p_{2n-1}^{2k+1}) = p_{2n-2}^{2k-2} + 2p_{2n-2}^{2k} + p_{2n-2}^{2k+2}.
\]

\(^1\)We use the sub-indices for the spatial discretization and sub-indices for the temporal one.
This relation shows that the probability at even numbered grid points and odd numbered ones evolve independently. Hence it suffices to consider the case when the particle is initially placed at one of the even numbered grid points by assuming
\begin{equation}
\sum_{k=-\infty}^{\infty} p_0^k = 1 \quad \text{and} \quad p_0^{2k+1} = 0 \quad \text{for all } k \in \mathbb{Z}.
\end{equation}

Then it holds that for all \( n \geq 1 \),
\begin{equation}
\sum_{k=-\infty}^{\infty} p_n^k = 1 \quad \text{and} \quad p_n^{2k} = p_n^{2k+1} = 0 \quad \text{for all } k \in \mathbb{Z}.
\end{equation}

Therefore, if only the even numbered time steps \( t_{2n} = 2n\Delta t \) are counted, then it is enough to consider the probability at the even numbered grid points only.

Now we derive a diffusion equation that approximates the probability distribution function. First we introduce uniform grid points \( y^k := \frac{k}{2}||\pi|| \), and let \( p(y, t) \) be a smooth function that satisfies \( p(y^k, t_n) = p_n^k \). Then, by (9),
\begin{equation}
p(y^{2k}, t_{2n}) - p(y^{2k}, t_{2n-2}) = \frac{1}{4} \left( p(y^{2k-2}, t_{2n-2}) - 2p(y^{2k}, t_{2n-2}) + p(y^{2k+2}, t_{2n-2}) \right).
\end{equation}

Divide this equation by \( 2\Delta t \) and obtain
\begin{equation}
\frac{p(y^{2k}, t_{2n}) - p(y^{2k}, t_{2n-2})}{2\Delta t} = \frac{\|\pi\|^2}{8\Delta t} \left( p(y^{2k-2}, t_{2n-2}) - 2p(y^{2k}, t_{2n-2}) + p(y^{2k+2}, t_{2n-2}) \right).
\end{equation}

The left-hand side of (11) is a first order forward approximation of \( p_t := \frac{\partial}{\partial t}p \) and the right side is a second order central approximation of \( p_{yy} := \frac{\partial^2}{\partial y^2}p \). Hence, we may write
\begin{equation}
p_t(y, t) + O(\Delta t) = d p_{yy}(y, t) + O(d\|\pi\|^2), \quad d := \frac{||\pi||^2}{8\Delta t}.
\end{equation}

To return to the original variable \( x \) we change the space variable by using \( y(x^{2k+1}) = y^{2k+1} \). Then,
\begin{equation}
\frac{dx}{dy} \approx \frac{x^{2k+1} - x^{2k-1}}{y^{2k+1} - y^{2k-1}} = \frac{\beta(x^{2k})}{||\pi||} = b(x^{2k})
\end{equation}
and the equation (12) turns into
\begin{equation}
p_t(x, t) = db(x)(b(x)p_x(x, t))_x + O(\Delta t) + O(d\|\pi\|^2).
\end{equation}

Here we abuse some notations for simplicity by writing \( p(x, t) \) instead of \( p(y(x), t) \). We will use this convention throughout the paper. Let \( u(x, t) \) be a smooth function given by
\begin{equation}
u(x, t) := \frac{p(x, t)}{\beta(x)} = \frac{p(x, t)}{||\pi||b(x)},
\end{equation}
which is called the probability density function. Then, it satisfies
\begin{equation}
u_t(x, t) = d(b(x)(b(x)u_x)_x)_x + O(\Delta t/||\pi||) + O(d||\pi||).
\end{equation}

Since \( d = \frac{||\pi||^2}{8\Delta t} \), the remainder terms can be written as
\begin{equation}u_t(x, t) = d(b(x)(b(x)u_x)_x)_x + O(||\pi||(d + 1/d)).
\end{equation}

Therefore, it is important to take
\begin{equation}
d = \frac{||\pi||^2}{8\Delta t} = O(1) \quad \text{as} \quad \Delta t \to 0,
\end{equation}
so that, for \( ||\pi|| \) small, the probability density function \( u(x, t) \) approximately satisfies a non-uniform diffusion equation,
\begin{equation}
u_t(x, t) = d(b(x)(b(x)u_x)_x)_x.
\end{equation}
Remark 2.1. The diffusivity of the diffusion equation (16) at \( x \in \mathbb{R} \) is \( k(x) := b(x)^2 d \). Since \( c \leq b(x) \leq 1 \), the diffusivity is in the range between \( c^2 d \) and \( d \). Hence we may call \( d \) the maximum diffusivity for a heterogeneous case and the diffusivity for a homogeneous case.

2.2. Convergence to the solution of the PDE model. We have derived the diffusion equation (16) as an approximation of the difference equation (11) when the mesh grids \( x^k \)'s and the jumping time \( \Delta t \) are fixed. In the following theorem we will show that the solution of the difference equation converges to the solution of the non-uniform diffusion equation as \( \| \pi \| \) and \( \Delta t \) vanish. In this approach, the limit should be taken to the probability density function since the probability at each grid point just vanishes as the mesh becomes finer. Furthermore, one cannot take the limit as \( \| \pi \|, \Delta t \to 0 \) arbitrarily; the relation (15) should be satisfied. This kind of convergence is classical with a uniform mesh case (see Lin and Segel [16, Section 3.3]). Hence the contribution of the following theorem is its extension to a non-uniform random walk system.

In the following theorem we denote the quantities obtained from a limiting process by symbols with a bar such as \( \bar{u} \) or \( \bar{w} \). The quantities that depend on the choice of spatial and temporal mesh grids are denoted without it.

**Theorem 2.1.** Consider the random walk system \( p^k_n \) given in (3)–(10) with a relation \( \| \pi \|^2 = 8d\Delta t \) for a given constant \( d > 0 \). Let \( u^{2k}_{2n} \) be the probability density of the random particle defined by

\[
0 = \frac{p^{2k}_{2n}}{x^{2k+1} - x^{2k-1}} = \frac{p^{2k}_{2n}}{\beta(x^{2k})}
\]

and \( \bar{u} \) be the solution of

\[
\bar{u}_t = d(\bar{b}(x)\bar{u}_x)_x, \quad \bar{u}(x, 0) = \bar{f}(x).
\]

If the initial value \( \bar{f}(x) \) is bounded,

\[
\sup_k |b(x^{2k}) - \bar{b}(x^{2k})| \to 0, \quad \text{and} \quad \sup_k \left| u^{2k}_{2n} - \bar{f}(x^{2k}) \right| \to 0 \quad \text{as} \quad \Delta t, \| \pi \| \to 0,
\]

then, for any \( T > 0 \),

\[
\sup_{2n\Delta t \leq T, k \in \mathbb{Z}} \left| u^{2k}_{2n} - \bar{u}(x^{2k}, t_{2n}) \right| \to 0 \quad \text{as} \quad \Delta t, \| \pi \| \to 0.
\]

**Proof.** Introduce new grid points \( y^{2k+1} := \int_{x^{2k+1}}^{x^{2k+1}} \frac{1}{b(s)} \, ds \) and \( y^{2k} = \frac{1}{2} (y^{2k+1} + y^{2k-1}) \). Then, \( y^{2k+1} - y^{2k-1} = \int_{x^{2k-1}}^{x^{2k+1}} \left| \frac{\| \pi \|}{\beta(s)} \right| \, ds = \int_{x_{2k-1}}^{x_{2k+1}} \frac{\| \pi \|}{x^{2k+1} - x^{2k-1}} \, ds = \| \pi \| \)

and hence \( \{ y^k : k \in \mathbb{Z} \} \) is a uniform mesh grid with a constant walk length \( |\Delta y| = \frac{1}{2} \| \pi \| \). Let \( p^k_n \) be the probability for a particle to be placed at spatial grid point \( y^k \) and at time step \( t_n \). Then, we have

\[
p^{2k}_{2n} = \frac{1}{4} \left( p^{2(k-1)}_{2(n-1)} + 2p^{2k}_{2(n-1)} + p^{2(k+1)}_{2(n-1)} \right)
\]

as in (9). Since the probability at odd numbered spatial grid points at even numbered time step is zero, we consider even numbered spatial grid points at even numbered time steps only. Then, the probability density \( w^{2k}_{2n} \) for this uniform random walk is given by

\[
w^{2k}_{2n} := \frac{p^{2k}_{2n}}{y^{2k+1} - y^{2k-1}} = \frac{p^{2k}_{2n}}{\| \pi \|}.
\]

Therefore, the finite difference relation for \( w^{2k}_{2n} \) is the same as the one for \( p^{2k}_{2n} \), i.e.,

\[
w^{2k}_{2n} = \frac{1}{4} \left( w^{2(k-1)}_{2(n-1)} + 2w^{2k}_{2(n-1)} + w^{2(k+1)}_{2(n-1)} \right).
\]

Now let \( \bar{w} \) be the solution of

\[
\bar{w}_t = d\bar{w}_{yy}, \quad \bar{w}(y, 0) = \bar{b}(x)\bar{f}(x).
\]
Here the coordinate $x$ in the initial value is determined by

$$
y(x) := \int_0^x \frac{1}{b(s)} \, ds.
$$

Then, by Taylor’s theorem and the relation $2\Delta y = \|\pi\|$, we have

$$
\overline{w}_{2n}^{2k} - \overline{w}_{2(n+1)}^{2k} + O(\|\Delta t\|^2) = \frac{1}{4} \left( \overline{w}_{2(n-1)}^{2(k-1)} - 2\overline{w}_{2(n-1)}^{2k} + \overline{w}_{2(n-1)}^{2(k+1)} \right) + O(\Delta t\|\pi\|^2),
$$

where $\overline{w}_{2n}^{2k} := \overline{w}(y^{2k}, t_{2n})$. Therefore, the difference $e_{2n}^{2k} := u_{2n}^{2k} - \overline{w}_{2n}^{2k}$ satisfies

$$
|e_{2n}^{2k}| = \frac{1}{4} |e_{2(n-1)}^{2(k-1)} + 2e_{2(n-1)}^{2k} + e_{2(n-1)}^{2(k+1)}| + O(\|\Delta t\|^2) \leq \sup_k |e_{2(n-1)}^{2k}| + O(\|\Delta t\|^2).
$$

Repeatedly using this relation, we obtain

$$
|w_{2n}^{2k} - \overline{w}_{2n}^{2k}| = \sup_k |w_0^{2k} - \overline{w}_0^{2k}| + nO(\|\Delta t\|^2).
$$

Define $\bar{u}$ as the one satisfying $\bar{w}(y, t) = \bar{b}(x) \bar{u}(x, t)$, where the coordinates $x$ and $y$ are related by (20). Then,

$$
\bar{w}_t = \bar{b}(x)\bar{u}_t, \quad d\bar{w}_{yy} = d\bar{b}(x)(\bar{b}(x)(\bar{b}(x)\bar{u})_x)_x, \quad \text{and} \quad \bar{u}(x, 0) = \bar{f}(x).
$$

Therefore, this $\bar{u}$ is the solution of (17). On the other hand, the probability density $u$ for the non-uniform random walk is

$$
u_{2n}^{2k} := \frac{p_{2n}^{2k}}{x^{2k+1} - x^{2k-1}} = \frac{\|\pi\|w_{2n}^{2k}}{\beta(x^{2k})} = \frac{w_{2n}^{2k}}{b(x^{2k})}.
$$

If we set

$$
u_{2n}^{2k} := \bar{u}(x^{2k}, t_{2n}) = \frac{\overline{w}_{2n}^{2k}}{b(x^{2k})},
$$

we have

$$
|u_{2n}^{2k} - \overline{u}_{2n}^{2k}| = \left| \frac{w_{2n}^{2k}}{b(x^{2k})} - \frac{\overline{w}_{2n}^{2k}}{b(x^{2k})} \right| \leq \left| \frac{w_{2n}^{2k}}{b(x^{2k})} - \frac{w_{2n}^{2k}}{b(x^{2k})} \right| + \left| \frac{\overline{w}_{2n}^{2k}}{b(x^{2k})} - \frac{\overline{w}_{2n}^{2k}}{b(x^{2k})} \right|
\leq \frac{1}{c} \left| \frac{w_{2n}^{2k}}{b(x^{2k})} \right| + \frac{1}{c^2} \left| \frac{\overline{w}_{2n}^{2k}}{b(x^{2k})} \right|.
$$

The last term vanishes uniformly in $k$ because of the assumption (18). Let $2n\Delta t < T$. Then, due to (21), the other term in the last line is bounded by

$$
\frac{1}{c} \left| \frac{w_{2n}^{2k}}{b(x^{2k})} \right| \leq \frac{1}{c} \sup_k \left| \frac{w_0^{2k}}{b(x^{2k})} \right| + nO(\|\Delta t\|^2) = \frac{1}{c} \sup_k \left| \frac{b(x^{2k})w_0^{2k}}{\bar{b}(x^{2k})\bar{u}_0(x^{2k})} \right| + O(\Delta t)
\leq \frac{1}{c} \sup_k \left( \left| b(x^{2k}) \right| \left| w_0^{2k} - \bar{u}_0(x^{2k}) \right| + \left| \bar{u}_0(x^{2k}) \right| \left| b(x^{2k}) - \bar{b}(x^{2k}) \right| \right) + O(\Delta t).
$$

This bound vanishes uniformly in $k$ as $\Delta t \to 0$ because of the assumption in (18) and the initial condition $\bar{u}(x, 0) = \bar{f}(x)$. Therefore the proof is complete.

2.3. Numerical comparison between the random walk and the PDE model. In this section we compare the discrete random walk model and the solutions of the corresponding PDE model. First we consider the models using the normalized variable $y$. The PDE for the probability density function $w$ in $y$ variable is given by

$$
w_t = dw_{yy}, \quad w(y, 0) = w_0(y), \quad -\infty < y < \infty,
$$

where the initial value is nonnegative and $d = \frac{\Delta y^2}{2\Delta t}$ for the constant walk length $\Delta y > 0$ and the jumping time $\Delta t > 0$. The solution is simply given by

$$
w(y, t) = \int \phi(y - a; t) w_0(a) \, da,
$$
where the heat kernel
\[
\phi(y;td) := \frac{1}{\sqrt{4\pi td}} e^{-y^2/4td}
\]
is the solution when the initial value is given by the Dirac delta distribution, \(p_0 = \delta\). If \(td = 1/2\), then \(\phi(\cdot;td)\) is called the standard normal distribution. Note that the formula (23) holds since the heat equation (22) is autonomous and hence Green’s function is simply \(G(y,a,t) = \phi(y-a,t)\). In Figure 1 the probability density distribution of a random walk, \(w_{2n}^k = p_{2n}^k/(2\Delta y)\), is plotted with the standard normal distribution. In this simulation we set \(\Delta y = 0.1\) and \(\Delta t = 0.1\). Then the diffusivity becomes \(d = 0.05\) and hence \(td = 0.5\) if the final time is \(t = 10\). The particle was initially placed at the origin, i.e., \(p_0^k = \delta_{k0}\), where \(\delta_{ij}\) is the Kronecker delta.

Next we consider the probability density distribution of a random walk system with a constant jumping time \(\Delta t = 0.1\) and non-uniform grid points:
\[
x^0 = 0 \quad \text{and} \quad x^k - x^{k-1} = \begin{cases} 0.1 & \text{if } k \leq 0, \\ 0.05 & \text{if } k > 0. \end{cases}
\]
The corresponding diffusion equation is
\[
u_t = d(\{b(x)(b(x)u_x)\}_x), \quad u(x,0) = u_0(x), \quad -\infty < x < \infty,
\]
where
\[
\|\pi\| = 0.2, \quad d = \frac{\|\pi\|^2}{8\Delta t} = 0.05, \quad b(x^{2k}) = \begin{cases} 1 & \text{if } k < 0, \\ 0.75 & \text{if } k = 0, \\ 0.5 & \text{if } x > 0. \end{cases}
\]
This equation is not autonomous so that there is no explicit formula such as (23). However, if we introduce a new variable,
\[
y(x) := \int_0^x \frac{1}{b(s)} ds,
\]
then \(\{y^k = y(x^k) : k \in \mathbb{Z}\}\) becomes a uniform grid and the probability density \(w(y,t)\) in the new variable satisfies (22), where the corresponding initial value is \(u_0(y(x)) = b(x)u_0(x)\). Hence
\[
u(x,t) = \frac{1}{b(x)} \frac{1}{\sqrt{4\pi td}} \int e^{-(y(x)-z)^2/4td}u_0(z) dz.
\]
The heat kernel for this non-autonomous problem is given by a rescaled Gaussian,
\[
\phi_b(x;td) := \frac{1}{b(x)} \frac{1}{\sqrt{4\pi td}} e^{-y(x)^2/4td}, \quad y(x) := \int_0^x \frac{1}{b(s)} ds.
\]
Figure 2. Random walk with a non-constant walk length. Probability density distribution of the random walk, \( u_{2n}^{2k} := p_{2n}^{2k}/(x^{2k+1} - x^{2k-1}) \), is given with the rescaled Gaussian. (a) The maximum diffusivity is \( d = 0.05 \) and the final time is given by \( t_d = 0.5 \). (b) The diffusivity is \( d = 0.0125 \) and the final time is given by \( t_d = 0.5 \).

In Figure 2 this rescaled Gaussian and the probability distribution of some discrete random walk systems are compared. These two agree well even with relatively small number of grid points. Also one may observe a discontinuity at the origin which was generated by the discontinuity in the walk length \( \Delta x \).

Remark 2.2 (Invariance of the median). The probability density function of a random walk system may have different shapes depending on heterogeneity in the walk length. However, the median of the distribution cannot be changed. For example, consider a random walk system that the particle started at the origin. Then, the probability density is given by a rescaled Gaussian and satisfies

\[
\int_{0}^{\infty} u(x, t) \, dx = \frac{1}{\sqrt{4\pi t_d}} \int_{0}^{\infty} \frac{1}{b(x)} e^{-y(x)^2/4t_d} \, dx = \frac{1}{\sqrt{4\pi t_d}} \int_{0}^{\infty} e^{-s^2/4t_d} \, ds = 0.5.
\]

Hence the probability for a particle to be placed on the region where \( x < 0 \) is identical to the one where \( x > 0 \). This is the case when there is heterogeneity in \( \Delta x \) only. However, we will see in the next section that is not the case if there is heterogeneity in \( \Delta t \).

3. Random walk with a spatially non-constant travelling time

In this section we develop a random walk theory with a spatial heterogeneity in the time step \( \Delta t \). Since heterogeneity in the walk length could be handled by rescaling the space variable as shown in the previous section, we will focus on a constant walk length case:

\[ y^k = k|\Delta y|, \quad k \in \mathbb{Z}. \]

Now we will work with heterogeneity in the time step \( \Delta t \), which is the main contribution of this paper. We consider a simplest spatial heterogeneity in \( \Delta t \) given by

\[ \Delta t(y^{k+1/2}) := \begin{cases} \tau & \text{if } y^{k+1/2} < 0, \\ 2\tau & \text{if } y^{k+1/2} \geq 0, \end{cases} \]

where \( \tau > 0 \) is a constant and \( y^{k+1/2} := (y^k + y^{k+1})/2 \). We will consider \( \Delta t(y^{k+1/2}) \) as the travelling time which it takes for a particle to move from \( y^k \) to \( y^{k+1} \) or vice versa. As before, we may extend the domain of the function \( \Delta t \) over the real line by defining \( \Delta t(y) := \Delta t(y^{k+1/2}) \) for \( y^k \leq y < y^{k+1} \). Note that the value of \( \Delta t \) at a spatial grid point \( y^k \) has no meaning in our

\[ 2 \text{If we consider a non-constant } \Delta t \text{ as the waiting time for the next jump, there may exist two different particles or probabilities with different jumping moments. This makes a presentation complicate. Hence we consider it as a travelling time and the next walk starts immediately after arrival. Hence, in the region where } y > 0, \text{ one should count the probability that the particle does not arrive at a grid point yet.} \]
setting. We will focus on this two-time step random walk in this section. This simple case is actually a challenging one due to the singularity at the origin and can be used as a building block for general cases with arbitrarily heterogeneous time steps.

3.1. Derivation of a PDE model. Define

\[(25)\quad \alpha(y) := \frac{\tau}{\Delta t}, \quad \text{i.e.,} \quad \alpha(y) = \begin{cases} 1 & \text{if } y < 0, \\ 1/2 & \text{if } y \geq 0. \end{cases} \]

We consider a particle randomly walking along the grid points. Let \( p_0^k \) be the probability that the particle is placed at the grid point \( y^k \) and at the initial time \( t = 0 \). Similarly, let \( p_n^k \) be the probability at time \( t_n = n\tau, n > 0 \). Then, the probability \( p_n^k \) for \( n \geq 2 \) is computed by

\[(26)\quad 2p_n^k = \begin{cases} p_{n-1}^k + p_{n-1}^{k+1} & \text{if } k < 0, \\ p_{n-1}^k + p_{n-2}^k & \text{if } k = 0, \\ p_{n-2}^k + p_{n-2}^{k+1} & \text{if } k > 0. \end{cases} \]

Applying this relation twice obtain

\[(27)\quad 2w_n^k = \begin{cases} w_{n-1}^k + w_{n-1}^{k+1} & \text{if } k < 0, \\ w_{n-1}^k + w_{n-2}^k & \text{if } k = 0, \\ w_{n-2}^k + w_{n-2}^{k+1} & \text{if } k > 0. \end{cases} \]

Notice that even numbered grid points and odd numbered ones are not separated in this case because of the boxed terms in the second and the fourth cases. Now we repeat the derivation process as in the previous section. Let

\[ w_n^k = \frac{p_n^k}{\Delta y} \]

Then, the relation in (26) gives

\[(28)\quad 2w_n^k = \begin{cases} w_{n-1}^k + w_{n-1}^{k+1} & \text{if } k < 0, \\ w_{n-1}^k + w_{n-2}^k & \text{if } k = 0, \\ w_{n-2}^k + w_{n-2}^{k+1} & \text{if } k > 0. \end{cases} \]

Fix

\[ d = \frac{|\Delta y|^2}{2\tau} \]

as a constant \( d > 0 \) and consider the limit as \( \tau \to 0 \). The argument on convergence in the previous section shows that \( w_n^k \) converges to a limit \( w(x, t) \) as \( \tau \to 0 \) and satisfies

\[ w_t = dw_{yy} \quad \text{for } y < 0, \quad w_t = \frac{1}{2} dw_{yy} \quad \text{for } y > 0. \]

Using the definition of \( \alpha(y) \) in (25) we may write it as

\[ w_t = \alpha(y)dw_{yy} \quad \text{for } y \neq 0. \]

This equation has meaning only if \( y \neq 0 \) due to the discontinuity of \( \alpha \) at the origin \( y = 0 \). Notice that the three cases with \( k = -1, 0, 1 \) are forgotten since \( \Delta y \) is of microscopic scale, which is also the reason for the disconnected domain of the equation. In the following proposition we show that the limit \( w \) is continuously differentiable even at the origin, i.e.,

\[ \lim_{y \to 0^-} w(y, t_0) = \lim_{y \to 0^+} w(y, t_0), \quad \lim_{y \to 0^-} w_y(y, t_0) = \lim_{y \to 0^+} w_y(y, t_0). \]
Proposition 3.1. Let $w_{\pm}$ be the limit of $w_{n}^{\pm}$ as $\tau \to 0$ and $n \to \infty$, where the limit is taken under fixed $d = \frac{\Delta y^2}{2\tau}$ and fixed $t_0 = n\tau$. Then, $w_- = w_+$. Furthermore, if

$$
\lim_{\tau \to 0} \frac{w_n^1 - w_n^0}{\Delta y} = w'_-, \quad \lim_{\tau \to 0} \frac{w_n^0 - w_n^{-1}}{\Delta y} = w'_-,
$$

then $w'_- = w'_+$. Therefore, the limit function $w$ is in $C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$.

Proof. From the relation for $k = -1$ in (27) we obtain

$$
4w_n^{-1} = 2w_n^{-2} + 2w_n^0 = 2w_n^{-2} + w_n^{-1} + w_n^1.
$$

Taking the limit as $\tau \to 0$ gives

$$
4w_- = 2w_- + w_- + w_+.
$$

Therefore, $w_- = w_+$. Next subtract the relation for $k = 0$ in (27) from the one for $k = -1$ to obtain

$$
2(w_n^1 - w_n^0) = 2w_n^{-2} - w_n^{-1} + w_n^0 - w_n^{-1} + w_n^1 - w_n^{-1},
$$

The last term vanishes since

$$
\frac{w_n^1 - w_n^{-1}}{\Delta y} = \frac{w_n^1 - w_n^{-1}}{2d\tau} \frac{|\Delta y|^2}{\Delta y} \to 0
$$

as $\tau \to 0$. Hence, taking the limit as $\tau \to 0$ in (29) gives

$$
2w'_- = w'_+ + w'_-,
$$

or $w'_- = w'_+$. \qed

Remark 3.1. Notice that the vanishing limit in (30) is invalid for second order derivative since $\frac{\Delta y}{|\Delta y|}$ does not vanish. Therefore continuity in the proposition is valid only up to the first order derivative and $w_{yy}$ has a discontinuity at $y = 0$.

Next we consider the probability density function that includes the probability for a particle to be placed between grid points, but not at a grid point. Notice that $w_n^k$ is the probability density that a particle arrives at $y^k$ at time $t_n = n\tau$. However, the travelling time in the region where $y > 0$ is $2\tau$ and hence there is a chance for a particle to be in the middle of its way. Including such a chance, the probability density function is given by

$$
v(y, t) = \begin{cases} 
    w(y, t) & \text{if } y < 0, \\
    2w(y, t) & \text{if } y > 0,
\end{cases}
\text{ or } v(y, t) = \frac{1}{\alpha(y)} w(y, t).
$$

This relation can be justified only when the probability at grid points and the probability in the middle of the way are identical in the region where $y > 0$. They are almost identical if $\Delta t$ and $\Delta x$ are small enough. Substitute $w = \alpha(y)v$ into (28) and obtain

$$
v_t = d(\alpha v)_{yy}, \quad y \in \mathbb{R}, \quad d := \frac{|\Delta y|^2}{2\tau}.
$$

Now we change the variable using (20) as in the previous section. Then the density becomes $u(x, t) = v(y(x), t)/b(x)$ and satisfies

$$
u_t = d(b(\alpha u)_{x})x, \quad x \in \mathbb{R}, \quad d := \frac{|\Delta x|^2}{2\tau}.
$$

Remember that $w(y, t) = b(x(y))\alpha(y)u(x(y), t)$ is the relation between $w$ and $u$ and it is $w$ that has $C^1(\mathbb{R})$ regularity.
3.2. Green’s function for the two-time step PDE model. The purpose of this section is to find Green’s function for the diffusion model (31) in an explicit form. For simplicity we consider the case when there is no heterogeneity in $\Delta x$, i.e., $b(x) \equiv 1$, and the diffusivity is $d = 1$. Going back to (28), we will actually find $G = G(y,a,t)$ that solves

$$
\begin{cases}
G_t = \alpha(y) G_{yy}, \\
G(y,0) = \alpha(y) \delta(y-a).
\end{cases}
$$

In the followings we will look for Green’s function such that $G(\cdot, a, t) \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ for each $a \in \mathbb{R}$ and $t > 0$ since a meaningful solution satisfies this regularity as shown in Proposition 3.1.

3.2.1. When $a = 0$. In this section, we will find $w(y,t) := G(y,0,t)$ explicitly that solves

$$
\begin{cases}
w_t = \alpha(y) w_{yy}, \\
w(y,0) = \alpha(y) \delta(y).
\end{cases}
$$

It is tricky to find a solution due to the discontinuity of the coefficient $\alpha$ at the origin. We employ a similar technique used in [5]. First, we split the real line $\mathbb{R}$ into two regions, $\{y > 0\}$ and $\{y < 0\}$. In the region $\{y > 0\}$, we solve an initial-boundary value problem:

$$
\begin{cases}
w_t = \frac{1}{2} w_{yy} & \text{if } y > 0, \ t > 0, \\
w(y,0) = \frac{1}{2} f(y) & \text{if } y > 0, \\
w(0,t) = g(t) & \text{if } t > 0,
\end{cases}
$$

where the value $g(t)$ at the boundary $y = 0$ is unknown and will be determined later. Notice that we have replaced the initial data $\delta$ by an arbitrary function $f(y)$ for now. This will be replaced by a Dirac delta sequence and the unknown boundary value $g$ will be decided from a limiting process. One may find the solution to the initial-boundary value problem in [13, p18 and p22], which is

$$
w(y,t) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty \frac{1}{2} f(\xi) (e^{-(\xi-y)^2/2t} - e^{-(\xi+y)^2/2t}) \, d\xi
+ \int_0^t g'(\eta) \text{erfc}\left(\frac{y}{\sqrt{2(t-\eta)}}\right) \, d\eta + g(0) \text{erfc}\left(\frac{y}{\sqrt{2t}}\right) \text{ if } y > 0.
$$

For $y < 0$, $w$ satisfies

$$
\begin{cases}
w_t = w_{yy} & \text{if } y < 0, \ t > 0, \\
w(y,0) = f(y) & \text{if } y < 0, \\
w(0,t) = g(t) & \text{if } t > 0.
\end{cases}
$$

Notice that, by choosing the same boundary value $g(t)$, the solution $w$ is automatically continuous. One can similarly find the solution

$$
w(y,t) = -\frac{1}{2\sqrt{\pi t}} \int_0^\infty f(-\xi) (e^{-(\xi-y)^2/4t} - e^{-(\xi+y)^2/4t}) \, d\xi
+ \int_0^t g'(\tau) \text{erfc}\left(\frac{-y}{2\sqrt{\tau-t}}\right) + g(0) \text{erfc}\left(\frac{-y}{2\sqrt{t}}\right) \text{ if } y < 0.
$$

To determine $g(t)$, we use the assumption that the solution $w$ is $C^1(\mathbb{R})$ in the $y$ variable, i.e.,

$$
w_y(0+, t) = w_y(0-, t) \quad \text{for all } t > 0.
$$
Therefore the solution

\[ w(y, t) = \begin{cases} \\
\frac{1}{2\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \left( (\xi - y)e^{-(\xi-y)^2/2t} + (\xi + y)e^{-(\xi+y)^2/2t} \right) d\xi \\
- \frac{1}{4\sqrt{\pi t}} \int_{0}^{t} g'(\tau) e^{-y^2/2(\tau-t)} d\tau \\
- \frac{1}{4\sqrt{\pi t}} g(0) e^{-y^2/2t} \\
+ \frac{1}{\sqrt{\pi}} f(0) e^{-y^2/4t} d\tau + \frac{1}{\sqrt{\pi}} g(0) e^{-y^2/4t} \\
\end{cases} \] 

if \( y > 0 \),

\[ \frac{1}{\sqrt{\pi t}} \int_{0}^{t} g'(\tau) e^{-y^2/2(\tau-t)} d\tau + \frac{1}{\sqrt{\pi}} g(0) e^{-y^2/4t} \] 

if \( y < 0 \),

the continuous differentiability of \( w \) implies that

\[ \frac{1}{t^{3/2}} \int_{0}^{\infty} f(\xi) \xi e^{-\xi^2/2t} d\xi + \frac{1}{2t} \int_{0}^{\infty} f(-\xi) \xi e^{-\xi^2/2t} d\xi \]

\[ = (\sqrt{2} + 1) \int_{0}^{t} g'(\tau) \frac{1}{\sqrt{t-\tau}} d\tau + (\sqrt{2} + 1) \frac{g(0)}{\sqrt{t}} \]

for all \( t > 0 \).

This relation gives the boundary value \( g(t) \) implicitly for any given initial value \( f(y) \). For Green’s function case, the corresponding initial value is the Dirac delta distribution and we may find \( g \) explicitly. First, take a Dirac delta sequence in the place of the initial value: for example, \( f(y) = \frac{n}{\sqrt{2}} \chi_{[-1/n, 1/n]}(y) \). Then

\[\frac{n}{\sqrt{2\sqrt{t}}} (1 - e^{-\frac{1}{2n^2}t}) + \frac{n}{2\sqrt{t}} (1 - e^{-\frac{1}{4n^2}t}) = (\sqrt{2} + 1) \int_{0}^{t} g'(\tau) \frac{1}{\sqrt{t-\tau}} d\tau + (\sqrt{2} + 1) \frac{g(0)}{\sqrt{t}}.\]

Taking the Laplace transformation yields

\[ \frac{n}{\sqrt{2}} \sqrt{\frac{s}{2\pi}} (1 - e^{-\sqrt{2s}/n}) + \frac{n}{2\sqrt{s}} (1 - e^{-\sqrt{s}/n}) \]

\[ = (\sqrt{2} + 1)(sG(s) - g(0)) \sqrt{\frac{n}{2\pi}} + (\sqrt{2} + 1) g(0) \sqrt{\frac{n}{2\pi}}, \]

or

\[ G(s) = \frac{n}{2(\sqrt{2} + 1)s} \left( \frac{1}{\sqrt{2}} (1 - e^{-\sqrt{2s}/n}) + (1 - e^{-\sqrt{s}/n}) \right), \]

and the inverse Laplace transform gives

\[ g(t) = \frac{n}{2} (\sqrt{2} - 1) \left( \frac{1}{\sqrt{2}} \text{erf} \left( \frac{\sqrt{2}}{2n\sqrt{t}} \right) + \text{erf} \left( \frac{1}{2n\sqrt{t}} \right) \right). \]

Therefore, the boundary condition \( g(t) \) for Green’s function is obtained by taking the limit as \( n \to \infty \):

\[ g(t) = \frac{\sqrt{2} - 1}{\sqrt{\pi t}}. \]

Therefore the solution \( w \) of (32) is given by

\[ w(y, t) = \begin{cases} \\
\frac{1}{\sqrt{2n\pi}} \int_{y/\sqrt{2n\pi}}^{y} \frac{\mu y^{-y^2/2(\tau-t)}}{\sqrt{\pi/(\tau-t)}} d\eta = \frac{2(\sqrt{2} - 1)}{\pi} \int_{y/\sqrt{2}}^{\infty} \frac{e^{-y^2/2t}}{\sqrt{t-y^2/2t^2}} d\eta \\
\frac{1}{\sqrt{2n\pi}} \int_{-y/\sqrt{2n\pi}}^{-y} \frac{(\tau-t)e^{-y^2/4(\tau-t)^2}}{\sqrt{\pi/(\tau-t)^2}} d\eta = \frac{2(\sqrt{2} - 1)}{\pi} \int_{-y/2\sqrt{t}}^{\infty} \frac{e^{-y^2/4t}}{\sqrt{t-y^2/4t^2}} d\eta \\
\end{cases} \]

if \( y > 0 \),

\[ \frac{1}{\sqrt{2n\pi}} \int_{-y/\sqrt{2n\pi}}^{y} \frac{\mu y^{-y^2/2(\tau-t)}}{\sqrt{\pi/(\tau-t)}} d\eta = \frac{2(\sqrt{2} - 1)}{\pi} \int_{y/\sqrt{2}}^{\infty} \frac{e^{-y^2/2t}}{\sqrt{t-y^2/2t^2}} d\eta \]

if \( y < 0 \).

Using the definition of \( \alpha(y) \) in (25), Green’s function \( G(y, a = 0, t) = w(y, t) \) can be written in a combined form

\[ G(y, a = 0, t) = \frac{2(\sqrt{2} - 1)}{\pi} \int_{|y|/\sqrt{4\alpha(y)t}}^{\infty} \frac{e^{-y^2}}{\sqrt{t-y^2/4\alpha(y)t^2}} d\eta, \quad y \in \mathbb{R}. \]
3.2.2. When \( a \neq 0 \). Now we move the initial Dirac delta distribution from the origin to some other point. Then the corresponding equation becomes

\[
\begin{align*}
    w_t &= \alpha(y) w_{yy}, \\
    w(y,0) &= \alpha(y) \delta(y-a),
\end{align*}
\]

where \( a \neq 0 \). One may solve this problem in the same manner by dividing the spatial domain into two parts, \( \{y > 0\} \) and \( \{y < 0\} \). For the case when \( a > 0 \), the corresponding boundary value is

\[
g(t) = \frac{\sqrt{2} - 1}{\sqrt{\pi t}} e^{-a^2/2t}
\]

and Green’s function is

\[
G(y, a > 0, t) := \begin{cases} 
    \frac{2(\sqrt{2} - 1)}{\pi} \int_{|y|/\sqrt{2t}}^{\infty} \exp\left( -\frac{a^2/2}{t-y^2/2\eta^2} \right) \frac{e^{-\eta^2}}{\sqrt{1-y^2/4\eta^2}} d\eta & \text{if } y > 0, \\
    + \frac{1}{2\sqrt{2t}} \left( e^{-(y-a)^2/2t} - e^{-(y+a)^2/2t} \right) & \text{if } y < 0.
\end{cases}
\]

For the case when \( a < 0 \), we have

\[
g(t) = \frac{\sqrt{2} - 1}{\sqrt{\pi t}} e^{-a^2/4t}
\]

and

\[
G(y, a < 0, t) := \begin{cases} 
    \frac{2(\sqrt{2} - 1)}{\pi} \int_{|y|/\sqrt{2t}}^{\infty} \exp\left( -\frac{a^2/4}{t-y^2/4\eta^2} \right) \frac{e^{-\eta^2}}{\sqrt{1-y^2/4\eta^2}} d\eta & \text{if } y > 0, \\
    + \frac{1}{2\sqrt{2t}} \left( e^{-(y-a)^2/4t} - e^{-(y+a)^2/4t} \right) & \text{if } y < 0.
\end{cases}
\]

Note that the formulas for \( a > 0 \) and \( a < 0 \) are almost identical; especially, the integrands for the case when \( a < 0 \) is achieved just by simply replacing ‘\( a^2/2 \)’ in the exponent for the case when \( a > 0 \) by ‘\( a^2/4 \)’.

3.3. **Numerical comparison between the random walk and the PDE model.** We compare the probability density function of the two-time step discrete random walk system and the explicit Green’s function in (33). In Figure 3 Green’s function and the probability distribution of the discrete random walk system are compared. One may observe a discontinuity at the origin which comes from the discontinuity in the travelling time \( \Delta t \). These two agree well even with a small number of grid points. The both cases with \( \Delta x = 0.1 \) or \( \Delta x = 0.05 \) are given in the figures.

Notice difference between Figure 2 and Figure 3. The probability density distribution in Figure 2 are basically obtained by gluing two normal distributions of different variations. Hence the probability of one side, \( P\{x < 0\} \), is the same as the other side \( P\{x > 0\} \). However, the profiles in Figure 3 do not follow the Gaussian distribution and even the probability for \( x < 0 \) is different from the one of the other side, i.e., \( P\{x < 0\} \neq P\{x > 0\} \), which will be shown in the following section.

4. **Conclusion and discussion**

Under presence of spatial heterogeneity in the walk length \( \Delta x \) and in the jumping time \( \Delta t \), random walk systems and the corresponding PDE models have been studied. Thermal diffusion of Brownian particles or starvation driven dispersal of biological organisms are possible applications. In this paper we have considered essential differences among uniform and non-uniform random walk systems. We also observed that the spatial heterogeneity in the jumping time \( \Delta t \) has a profound difference from the one in \( \Delta x \) and they are unexchangeable.
Figure 3. Random walk with a non-constant time step. Probability distribution of the random walk is given with the explicit solution $u = w/\alpha$, where $w$ is given by (23). The time step is $\Delta t = 0.1$ for $x < 0$ and $\Delta t = 0.2$ for $x > 0$. The diffusivity is $d = 0.05$ on $x < 0$ and the final time is given by a relation $td = 0.5$.

4.1. Numerical comparison with Fick’s law. The diffusivity is given by the relation $k = |\Delta x|^2/2|\Delta t|$ in one space dimension. For the case with a constant jumping time interval $\Delta t$, the diffusivity can be written as $k = db(x)^2$ using the notation in the diffusion model (16). The original Fick’s law [10] is for the constant diffusivity case. For a non-constant diffusivity $k = k(x)$, Fick’s diffusion law usually refers to

\begin{equation}
  u_t = (k(x)u_x)_x
\end{equation}

which is different from the diffusion model (16) we have derived in this paper. We will numerically compare the solutions of these diffusion equations to the probability density distribution of the discrete random walk system. For numerical simulations we take an initial value

\begin{equation}
  u_0(x) = \begin{cases} 
  1 & \text{if } -2 < x < 2, \\
  0 & \text{otherwise}.
\end{cases}
\end{equation}

The time step is fixed with $\Delta t = 0.1$. We consider two sets of random walk grids:

\begin{equation}
  x^k = \begin{cases} 
  0.1k & \text{if } k \leq 0, \\
  0.05k & \text{if } k > 0,
\end{cases}
\end{equation}

and

\begin{equation}
  x^0 = 0, \quad x^k = \begin{cases} 
  x^{k-1} + 0.08 \times (1.5 + \sin(x^{k-1})) & \text{if } k > 0, \\
  x^{k+1} - 0.08 \times (1.5 + \sin(x^{k+1})) & \text{if } k < 0.
\end{cases}
\end{equation}

There is a discontinuity in the variation of the walk length in the case of (37). The other case of (38) has gradually changing walking length.

For the above random walk systems one may easily compute the diffusivity $k(x)$ and the coefficient $b(x)$ in (13). In Figure 4 solutions of these two diffusion models are given with the probability density distribution of the discrete random walk system. From these examples one may find that the solutions of the new diffusion law (16) gives the probability density distribution of the discrete random walk system correctly. However, Fick’s law gives something else. In conclusion the new diffusion law (16) is the one that explains the non-uniform random walk system.
4.2. Sensibility of $\Delta t$ versus insensibility of $\Delta x$. Let $w(y, t) = G(y, 0, t)$ be Green’s function with a Dirac delta distribution placed at $a = 0$, which is given by the formula (33),

$$w(y, t) = \frac{2(\sqrt{2} - 1)}{\pi} \int_{|y|/\sqrt{4\alpha(y)t}}^{\infty} \frac{e^{-\eta^2}}{\sqrt{t - y^2/4\alpha(y)\eta^2}} d\eta, \quad y \in \mathbb{R}.$$  

By Fubini’s theorem, we may observe that

$$\int_{0}^{\infty} w(y, t) \, dy = \frac{\sqrt{2} - 1}{\sqrt{2}\pi} \int_{0}^{t} \frac{1}{\sqrt{t - \tau}} \tau^{3/2} \int_{0}^{\infty} ye^{-y^2/2\tau} \, dy \, d\tau$$

$$= \frac{\sqrt{2} - 1}{\sqrt{2}\pi} \int_{0}^{t} \frac{1}{\sqrt{t - \tau}} \sqrt{\tau} \, d\tau = \frac{\sqrt{2} - 1}{\sqrt{2}}.$$  

We similarly compute

$$\int_{-\infty}^{0} w(y, t) \, dx = \sqrt{2} - 1.$$  

The probability density function in $y$ variable is given by $v(y, t) = w(y, t)/\alpha(y)$ and satisfies

$$\sqrt{2} \int_{-\infty}^{0} v(y, t) \, dy = \int_{0}^{\infty} v(y, t) \, dy. \quad (39)$$

The relation (39) implies that the probability for a particle to be placed on $x > 0$ is $\sqrt{2}$ times greater than the one on $x < 0$. Remember that, for the case with spatial heterogeneity in walk length only, the two probabilities are identical as shown in (24). One may clearly observe this phenomenon by comparing Figure 2 and Figure 3.

In conclusion we may say that people in the region where $x < 0$ may sense a change in jumping time $\Delta t$ of the other side where $x > 0$. However, they cannot sense a change in walk length $\Delta x$. This observation implies that heterogeneity in time step cannot be reduced to one in walk length through any kinds of rescaling.

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