CHEMOTACTIC TRAVELING WAVES BY METRIC OF FOOD

SUN-HO CHOI* AND YONG JUNG KIM†

Abstract. The meaningful distance to biological organisms is not necessarily one measured by the Euclidean metric but possibly one by a metric that counts the amount of resources such as food. It is assumed in this paper that the distance for biological organisms is measured by the amount of food between two places. A new chemotaxis model is introduced as an application of this "metric of food". It is shown that, if the walk length of a random walk system is given by such a metric, the well-known traveling wave phenomena of the chemotaxis theory can be obtained without the typical assumption that microscopic scale bacteria may sense the macroscopic scale gradient of a chemical concentration. The uniqueness and the existence of a traveling wave solution are obtained.

Key words. chemotaxis; traveling wave; Laplace-Beltrami operator; biological diffusion

AMS subject classifications.

1. Introduction.

Since food is one of the main reasons for the migration of biological species, the distance that matters to the biological organisms is not solely the Euclidean distance, but possibly one that counts the distribution of food. For a given food concentration $m(x) > 0$ in one space dimension, the metric that measures the amount of food between two points, i.e., for $a, b \in \mathbf{R}$,

$$d(a,b) = \Big| \int_a^b m(x)dx \Big|, \qquad \text{(Metric of Food)}$$

is proposed as a candidate of such a distance and is called the *metric of food* in this paper. Introducing such a metric in biology models may give us new insights to biological phenomena and bring Riemannian geometry to mathematical biology which was the motivation of this paper.

Random migration of a biological species is often modeled by a random walk system, which converges to a diffusion equation. However, the randomness is in choosing not necessarily the position, but often the objective of the migration such as food. One way to understand such a random migration is to introduce a random walk system where the walk length is given by the metric of food, rather than by the Euclidean metric. In Section 2 a diffusion equation is derived from such a random walk system and, in one space dimension, it is given by

$$u_t = \Big(\frac{1}{m}\Big(\frac{u}{m}\Big)_x\Big)_x, \quad x \in \mathbf{R}, \quad t > 0. \tag{1.1}$$

This is the Laplace-Beltrami equation corresponding to the metric of food when the density u is taken in terms of the Euclidean metric. The purpose of this paper is to introduce this diffusion process in a chemotaxis model and show that this diffusion can explain the traveling wave phenomena without other assumptions. In reality, the correct metric could be somewhere between the metric of food and the Euclidean metric. Since the effect of the Euclidean distance is known well, we will focus on the effect of the metric of food.

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One of the fundamental assumptions in bacterial chemotaxis is that the individual species may sense the gradient of a signalling chemical concentration such as pheromone or food. It is a mysterious phenomenon that microscopic scale organisms figure out the macroscopic scale concentration distribution and find the correct direction to migrate. Instead of taking such an assumption, we simply employ the diffusion in (1.1) in a context of chemotaxis theory to explain the traveling wave phenomena of chemotaxis. The chemotaxis model proposed in this paper is[1]

$$\begin{aligned} u_t &= \Big(\frac{1}{m}\Big(\frac{u}{m}\Big)_x\Big)_x, \quad x \in \mathbf{R}, \quad t > 0, \\ m_t &= -\kappa(m)u, \end{aligned} \tag{1.2}$$

where $u \geq 0$ is the population density and $\kappa(m) \geq 0$ is the consumption rate. Even if the right side of the first equation models a pure random migration, it contains an advection phenomenon if m is not constant as in (2.3). The consumption rate satisfies three basic hypotheses for $m > 0$:

$$\kappa(m) > 0, \quad \kappa'(m) \geq 0, \quad \text{and} \quad \lim_{m\to 0} \kappa(m) = 0. \tag{1.3}$$

These hypotheses imply that species does not produce any food, but consumes more if there is more food, and does not consume at all if there is no food.

The Keller-Segel model [10] for the chemotactic traveling wave phenomenon is written as

$$\begin{cases} u_t = (\mu(m)u_x - \chi(m)u\, m_x)_x, \\ m_t = \epsilon m_{xx} - \kappa(m)u. \end{cases} \tag{1.4}$$

In the original derivation of the model, the diffusivity μ and the chemosensitivity χ are supposed to satisfy

$$\chi(m) = -(1-a)\mu'(m) \quad \text{and} \quad \mu'(m) \leq 0, \tag{1.5}$$

where $0 < a < 1$ is the ratio of effective body length, i.e., the largest distance between receptors over the body length of bacteria. However, instead of this original derivation, simplified ones are mostly used, where the diffusivity is mostly taken as a constant, $\mu = \mu_0$, and the chemosensitivity is usually taken as the Weber-Fechner law, $\chi(m) = \frac{\chi_0}{m}$. In other words, the relation (1.5) does not hold anymore. On the other hand, Eq. (1.2) can be written in the form of Keller-Segel equations with $\mu(m) = m^{-2}$ and $\chi(m) = m^{-3}$. Therefore, Eq. (1.5) is satisfied with $a = \frac{1}{2}$. Thus, the model equation in (1.2) is closer to the original Keller-Segel equation in (1.4) than other simplified ones even though the idea in its derivation is different.

The Keller-Segel model [9, 10] initiated mathematical studies on the dynamics of bacterial chemotaxis. The motivation of the model was the experimental results of Adler and his collaborators [1, 2, 3, 4]. A variety of mathematical models were introduced to explain various traveling wave phenomena [7, 11, 12, 14, 18]. Especially, many authors have studied the stability, existence, chemical diffusion limit [14], relaxation model [18], chemotactic motility on attractant concentrations [12] and general Keller-Segel's traveling wave solution [11]. In [15], authors studied for microscopic

[1]The purpose of this paper is to introduce the metric of food to biology models and to show the corresponding diffusion operator in (1.1) may give a traveling wave phenomenon.

and macroscopic chemotaxis models. We refer to [17] and [8, 13, 16] for issues on shock formation and stability, respectively.

The existence and the uniqueness of a traveling wave solution of (1.2) and the criteria to classify the types of traveling wave solutions are obtained in this paper. The traveling wave solution is unique up to translation for each given boundary conditions and wave speed. The asymptotic decay of consumption rate κ at $m = 0$ decides the types of traveling wave solutions and plays an important role on the analysis. Roughly speaking, the traveling wave solution for the food distribution m is always of the front type, i.e., $m_- \neq m_+$, and the one for population distribution u is of the front type if $\lim_{m\to 0} \kappa(m)/m^3 > 0$ or of the pulse type otherwise.

The strategy of this paper is as follows. First, we transform the PDE system (1.2) into an ODE system for a traveling wave solution (u, m) and derive necessary boundary conditions and the wave speed to have a traveling wave solution. Then, an ansatz for the population density distribution u is derived in terms of the resource distribution m, which gives a single ODE for $m(\xi)$ (See Proposition 3.4). Second, we divide the whole real line problem into forward and backward ODE problems with an initial value at $\xi = 0$, i.e., $m(0) = m_0$. Monotonicity of m and a priori convergence results of m at $\xi = \pm\infty$ are obtained, which are used to obtain the global existence. Finally, we obtain the boundary values at $\xi = -\infty$ for the wave profile (u, m). These boundary values determine whether wave profile u is of the front type or of the pulse type.

The rest of this paper is organized as follows. In Section 2, the diffusion equation (1.1) is derived using a metric induced by food. In Section 3, we develop necessary conditions for the existence of a traveling wave solution and other preliminary material. In Section 4, the existence and the uniqueness of the traveling wave solution m and u are obtained with criteria for pulse and front type wave. All types of traveling wave solutions obtained in the paper are numerically computed in Section 5. Finally, we summarize the results of this paper and further possibilities in Section 6.

2. Random walk with metric of food. A particle jumps to one of the two adjacent *positions* of equal distance in a one dimensional random walk system. However, in modeling a random migration for biological organisms, a random walk system where a biological species jumps to one of two adjacent *preys* or *foods*, but not positions, can be meaningful. To construct such a *food based* random walk system we simply introduce a distance metric measuring the amount of food between two positions and consider a random walk system where the walk distance is given by this metric of food.

Let $m(x) > 0$ be the distribution of food or resource in one space dimension and $a, b \in \mathbf{R}$ be two distinct points. For convenience of statement we assume m is uniformly bounded and continuous. Then, the distance between the two points can be defined in a view point of resource distribution, which is

$$d(a, b) = \Big| \int_a^b m(x)dx \Big|. \tag{2.1}$$

We call this distance the *metric of food*. In this section we will introduce a diffusion equation based on this metric in two different ways.

First, we consider a random walk system in which the walk length is constant with respect to this metric of food. Let $a < b$ be adjacent jumping spots and Δy be the given constant distance in the metric of food, i.e.,

$$\int_a^b m(x)dx = \Delta y.$$

Let $\Delta x = b - a$ be the walk length in the Euclidean distance. Then, by the mean value theorem, there exists $a \le x' \le b$ that satisfies

$$\int_a^b m(x)dx = m(x')(b-a) = \Delta y, \quad \text{or} \quad \Delta x\big|_{x=x'} = \frac{\Delta y}{m(x')}.$$

In other words, the random walk system with a constant walk length in the metric of food corresponds to a random walk system with nonconstant walk length in the original Euclidean metric. The random walk model with nonconstant walk length has been considered recently (see [5, 6]), which gives

$$u_t = \Big(\frac{1}{m}\Big(\frac{u}{m}\Big)_x\Big)_x, \qquad x \in \mathbf{R},\ t > 0, \tag{2.2}$$

where a constant coefficient part has been cancelled out after normalization. Note that this equation models a purely random migration and the usual assumption of gradient sensing by individual organisms is not used. However, the heterogeneity of resource distribution naturally causes an advection phenomenon. In fact, this diffusion equation is written as

$$u_t = \Big(\frac{1}{m^2}u_x - u\frac{m_x}{m^3}\Big)_x = \Big(\frac{1}{m^2}\big(u_x - \frac{u}{m}m_x\big)\Big)_x, \tag{2.3}$$

and one may find the typical chemotaxis equation except for the nonconstant cofactor $\frac{1}{m^2}$. The idea of the derivation can be extended to a multi-dimensional space which results in

$$u_t = \nabla \cdot \Big(\frac{1}{m}\nabla\Big(\frac{u}{m}\Big)\Big), \qquad x \in \mathbf{R}^n,\ t > 0. \tag{2.4}$$

Next, we derive the same equation from the view point of Riemannian geometry. Consider a new variable $y = y(x)$ given by

$$y(x) = \int_0^x m(s)ds. \tag{2.5}$$

Then, for $a < b$,

$$y(b) - y(a) = \int_0^b m(s)ds - \int_0^a m(s)dx = \int_a^b m(s)ds = d(a,b).$$

In other words, the space $\mathbf{R}$ with the metric of food in (2.1) has been isometrically embedded to the Euclidean space by the mapping in (2.5). The volume factor of this transformation is simply the food distribution $m(x)$. Now the population density in terms of the new space of variable y is given by

$$v(y,t) = \frac{u(x,t)}{m(x)},$$

where $\frac{dy}{dx} = m(x)$. Since the corresponding diffusion in y variable is the usual random walk in the Euclidean space, the diffusion equation is simply

$$v_t = v_{yy},$$

where the usual diffusivity constant was set to be $d = 1$ after a rescaling of time. Also note that the food concentration in this rescaled space is constant since the volume factor is $m(x)$ and hence the linear diffusion obtained above is valid. If we convert the new space variable y back to the original variable x, the Laplace operator becomes

$$v_{yy} = x_y(x_y v_x)_x = \frac{1}{m}\Big(\frac{1}{m}v_x\Big)_x, \tag{2.6}$$

which is the Laplace-Beltrami operator. Therefore, $u(x,t)$ satisfies

$$\frac{1}{m}u_t = \Big(\frac{1}{m}u\Big)_t = \Big(\frac{u}{m}\Big)_{yy} = \frac{1}{m}\Big(\frac{1}{m}\Big(\frac{u}{m}\Big)_x\Big)_x,$$

which is the same random diffusion equation obtained in (2.2).

3. System of traveling waves and decoupled equation. In this section, we discuss several structural properties of the traveling wave solution of (1.2). Traveling wave solution with a given wave speed $c \in \mathbf{R}$ is written as

$$u(x,t) = u(x-ct), \quad m(x,t) = m(x-ct),$$

where we use $\xi := x - ct$ as the traveling wave variable of moving frame. We now plug it into PDE system (1.2) and obtain an ODE system for the traveling wave solution:

$$cu' = -\Big(\frac{1}{m}\Big(\frac{u}{m}\Big)'\Big)', \tag{3.1}$$

$$cm' = \kappa(m)u, \tag{3.2}$$

where the notation prime, $'$, denotes the ordinary differentiation with respect to the variable ξ. We are looking for a traveling wave solution with the following boundary conditions:

$$\begin{aligned} &u'(\xi) \to 0, \quad m'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty, \\ &u(\xi) \to u_\pm \geq 0, \quad m(\xi) \to m_\pm \geq 0 \quad \text{as} \quad \xi \to \pm\infty. \end{aligned} \tag{3.3}$$

We will first find the necessary and sufficient conditions on the boundary values that guarantee a positive traveling wave solution and then study its structure.

REMARK 3.1. *A few properties of the traveling waves can be obtained immediately. First, since $cm' = \kappa(m)u > 0$, the resource profile $m(\xi)$ is strictly increasing if and only if $c > 0$. Furthermore, if $c = 0$, there is no positive traveling wave solution. We will consider the case with $c > 0$ only, and the other case is obviously obtained due to the symmetry of the problem. Therefore, $m_+ > 0$ is a necessary condition to have a positive traveling wave solution with a positive traveling wave speed $c > 0$ since $m(\xi)$ should be an increasing function. Note that the equation* (3.2) *and the boundary conditions* (3.3) *imply $u_+ = 0$. Hence $u_+ = 0$ is also a necessary condition. We assume these necessary conditions in the rest of the paper, i.e.,*

$$u_+ = 0, \quad m_+ > 0. \tag{3.4}$$

We decouple the system (3.1)-(3.2) under the necessary boundary conditions (3.4) and find an ordinary differential equation for m. Integrating (3.1) on $[\xi,\infty)$ gives

$$\begin{aligned} c(u_+ - u(\xi)) &= -\int_\xi^\infty \Big(\frac{1}{m(\zeta)}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)'\Big)' d\zeta \\ &= -\Big[\lim_{\tau\to\infty}\frac{1}{m(\tau)}\Big(\frac{u(\tau)}{m(\tau)}\Big)' - \frac{1}{m(\xi)}\Big(\frac{u(\xi)}{m(\xi)}\Big)'\Big]. \end{aligned} \tag{3.5}$$

Since $m_+ > 0$ and we are looking for a traveling wave solution that satisfies

$$u'(\xi) \to 0, \quad m'(\xi) \to 0, \quad \text{as} \quad \xi \to \infty,$$

we have

$$\lim_{\tau\to\infty}\frac{1}{m(\tau)}\Big(\frac{u(\tau)}{m(\tau)}\Big)' = \lim_{\tau\to\infty}\frac{u'(\tau)}{m^2(\tau)} - \lim_{\tau\to\infty}\frac{u(\tau)m'(\tau)}{m^3(\tau)} = 0.$$

Since $u_+ = 0$, it is derived from (3.5) that

$$-cu = \frac{1}{m}\Big(\frac{u}{m}\Big)'.$$

Substitute $u = cm'/\kappa(m)$ into the equation above which is drawn from (3.2), then multiply it by m, and one obtains

$$\Big(\frac{u}{m}\Big)' + c^2\frac{m\,m'}{\kappa(m)} = 0. \tag{3.6}$$

Integrating the equation on (ξ,∞) gives

$$\int_\xi^\infty \Big(\frac{u(\zeta)}{m(\zeta)}\Big)' d\zeta + c^2\int_\xi^\infty \frac{m(\zeta)m'(\zeta)}{\kappa(m(\zeta))}d\zeta = 0.$$

The first term becomes

$$\int_\xi^\infty \Big(\frac{u(\zeta)}{m(\zeta)}\Big)' d\zeta = \lim_{\tau\to\infty}\frac{u(\tau)}{m(\tau)} - \frac{u(\xi)}{m(\xi)} = -\frac{u(\xi)}{m(\xi)}.$$

Since $\kappa(m) > 0$ and $m(\xi)$ is monotone, we may change the variable and obtain

$$c^2\int_\xi^\infty \frac{m(\zeta)m'(\zeta)}{\kappa(m(\zeta))}d\zeta = c^2\int_{m(\xi)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta.$$

Finally, we have obtained an ansatz for u:

$$u(\xi) = c^2 m(\xi)\int_{m(\xi)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta. \tag{3.7}$$

Proposition 3.2. *Three boundary conditions,*

$$m_- = 0, \qquad m_+ > 0, \qquad u_+ = 0, \tag{3.8}$$

are necessary to have a positive traveling wave solution with a positive speed $c > 0$.

Proof. Suppose that (u,m) is a traveling wave solution of (3.1)–(3.3) with a positive speed $c>0$. It is enough to show $m_-=0$ and the other two necessary boundary conditions are already shown in Remark 3.1. Suppose that $m_->0$. Then, Eq. (3.2) gives

$$u=c\frac{m'}{\kappa(m)},$$

and hence

$$\int_{-\infty}^{\infty}u(\xi)d\xi=c\int_{-\infty}^{\infty}\frac{m'(\xi)}{\kappa(m(\xi))}d\xi=c\int_{m_-}^{m_+}\frac{1}{\kappa(\eta)}d\eta<\frac{m_+-m_-}{\kappa(m_-)}<\infty.$$

Therefore, u has a finite total population and hence $u_-=0$. To obtain a contradiction, we recall the ansatz in (3.7):

$$u(\xi)=c^2m(\xi)\int_{m(\xi)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta.$$

Taking limit as $\xi\to-\infty$ gives that

$$u_-=\lim_{\xi\to-\infty}u(\xi)=c^2\lim_{\xi\to-\infty}m(\xi)\int_{m(\xi)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta=c^2m_-\int_{m_-}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta>0,$$

which contradicts the above computation that $u_-=0$. Therefore, we conclude that $m_-=0$. □

There are four boundary values, $u_\pm$ and $m_\pm$, in the traveling wave equation. The proposition shows that, if the traveling wave speed is positive, $c>0$, then there is no freedom in choosing two boundary values, i.e., $u_+=0=m_-$. The boundary value $m_+>0$ is a free parameter, which will be used in the following decoupled equation. The last boundary value u_- will be decided by the traveling wave speed $c>0$ in Theorem 4.4. In other words, there are two parameter family of traveling wave solutions.

The ODE system (3.1)-(3.2) is decoupled by substituting (3.7) into (3.2). Then, we obtain

$$\begin{aligned}&m'(\xi)=\tfrac{1}{c}\kappa(m(\xi))h(m(\xi),m_+),\quad \xi\in\mathbf{R},\\&m(0)=m_0,\end{aligned}\tag{3.9}$$

where $0<m_0<m_+$ and

$$h(m,m_+)=c^2m\int_m^{m_+}\frac{\eta}{\kappa(\eta)}d\eta.\tag{3.10}$$

Notice that a traveling wave solution has a free parameter of translation and two of them are considered identical if one is a translation of the other. The initial condition $m(0)=m_0\in(0,m_+)$ is taken to choose one of them.

Remark 3.3. *The function h in* (3.10) *comes from the relation* (3.7)*. Since $m_-=0$ is a necessary condition, we have $m(\xi)\to0$ as $\xi\to-\infty$. Therefore,*

$$\lim_{m\to0+}c^2m\int_m^{m_+}\frac{\eta}{\kappa(\eta)}d\eta\ \Big(=\lim_{\xi\to-\infty}u(\xi)=u_-\Big)<\infty\tag{3.11}$$

is a necessary condition to have a bounded traveling wave solution. This condition restricts the range of a consumption rate κ that allows a traveling wave solution.

In the following proposition we show that the traveling wave solutions of (3.1)-(3.2) and (3.9)-(3.10) with (3.7) are identical if the consumption rate κ satisfies (3.11).

PROPOSITION 3.4. *Let $0 < m_0 < m_+$ and the consumption rate $\kappa(m)$ satisfy* (1.3) *and* (3.11). *(i) If (u, m) is a solution of* (3.1)-(3.3), *then m satisfies* (3.9)-(3.10) *after an appropriate translation if needed. (ii) If m is a solution of* (3.9)-(3.10) *and u is given by* (3.7), *then (u, m) is a solution of* (3.1)-(3.2).

Proof. (i) Let (u, m) be a solution of (3.1)-(3.3). As we have already observed, m satisfies (3.9)-(3.10) except for the initial condition $m(0) = m_0$. However, since $0 = m_- < m_0 < m_+$ and m increases continuously, there exists ξ_0 such that $m(\xi_0) = m_0$. Therefore, $m(\cdot + \xi_0)$ satisfies (3.9)-(3.10) including the initial condition.

(ii) Conversely, assume that m is a solution to (3.9)-(3.10) and u is given by (3.7). By (3.7), we have

$$-\Big(\frac{1}{m(\xi)}\Big(\frac{u(\xi)}{m(\xi)}\Big)'\Big)' = c^2\Big(\frac{1}{m(\xi)}m'(\xi)\frac{m(\xi)}{\kappa(m(\xi))}\Big)' = c^2\Big(\frac{m'(\xi)}{\kappa(m(\xi))}\Big)'. \tag{3.12}$$

Through (3.9) we obtain

$$c^2\Big(\frac{m'(\xi)}{\kappa(m(\xi))}\Big)' = cu'(\xi). \tag{3.13}$$

Thus, (3.12) and (3.13) implies that u satisfies (3.1). ☐

REMARK 3.5. *The following analysis of this paper is based on the decoupled equations* (3.7) *and* (3.9). *Another option is to use the fact that* (3.6) *is in an exact form. For example,* (3.6) *is written as*

$$\frac{u'}{m} - \frac{um'}{m^2} + c^2\frac{m\,m'}{\kappa(m)} = 0.$$

Let

$$U = \frac{1}{m}, \quad M = -\frac{u}{m^2} + \frac{c^2 m}{\kappa(m)}.$$

Then, the equation is written as

$$U du + M dm = 0,$$

where

$$\frac{\partial U}{\partial m} = \frac{\partial M}{\partial u}.$$

The traveling wave analysis in [19] is based on such an exactness property.

4. Existence and uniqueness in traveling wave solutions.

4.1. Global existence and uniqueness. In this section, we study the uniqueness and the existence of a traveling wave solution (u, m) to (3.1)-(3.3) where the boundary conditions are given at terminal points $\xi = \pm\infty$. However, the decoupled system (3.9)-(3.10) has an interior value condition which naturally has the uniqueness property. Remember that the boundary conditions in (3.8) are necessary ones for the existence of a traveling wave solution and hence we assume them in this section. Therefore, the only undecided boundary value is $u_- \geq 0$.

LEMMA 4.1 (Uniqueness and local existence). *Let $0 < m_0 < m_+$ and κ satisfy (1.3) and (3.11). There exists $\epsilon_0 > 0$ such that (3.9)-(3.10) has a solution for $|\xi| < \epsilon_0$ and the solution is unique.*

Proof. We consider a domain $0 \leq m \leq m_+$. Let

$$f_1(m) = \kappa(m)m\int_m^{m_+}\frac{\eta}{\kappa(\eta)}d\eta, \quad f_2(m) = m\int_m^{m_+}\frac{\eta}{\kappa(\eta)}d\eta, \quad \text{and} \quad f_3(m) = \frac{m^2}{\kappa(m)}.$$

Clearly, f_i's are positive on $(0, m_+)$ and $f_i(m_+) = 0$ for $i = 1, 2, 3$. Since $m\int_m^{m_+}\frac{\eta}{\kappa(\eta)}d\eta$ is bounded, (3.11), we have $f_1(0) = 0$. Therefore, f_1 has a positive maximum, $M_1 > 0$, on $(0, m_+)$. Similarly, since $\lim_{\xi\to 0} f_2(\xi)$ is bounded, f_2 has a positive maximum, $M_2 > 0$, on $[0, m_+)$. f_3 is not necessarily bounded. However, for any $\epsilon > 0$, it is bounded on (ϵ, m_+) and let $M_{3,\epsilon}$ be the maximum on $[\epsilon, m_+)$. We choose $\epsilon = \frac{m_0}{2}$ in the estimates below.

Consider an open interval $I := \big(-\frac{m_0}{2cM_1}, \frac{m_+ - m_0}{cM_1}\big)$ and a subspace of real valued smooth functions defined on it,

$$\Omega := \Big\{m \in C^1(I) : m(0) = 0,\ m'(\xi) \geq 0,\ \frac{m_0}{2} < m < m_+\Big\}.$$

For $m \in \Omega$, we define an operator $L(m)(\xi)$ as

$$L(m)(\xi) := m_0 + \frac{1}{c}\int_0^\xi \kappa(m(\zeta))h(m(\zeta), m_+)d\zeta.$$

We will use the Picard iteration method which is a standard way to obtain local existence and uniqueness of fixed points for such integral operators.

◦ Invariance property ($L : \Omega \mapsto \Omega$): First, we show $L(\Omega) \subset \Omega$. It is clear that

$$\frac{d}{d\xi}L(m)(\xi) \geq 0 \quad \text{and} \quad L(m)(0) = m_0.$$

We have

$$\begin{aligned} L(m)(\xi) &= m_0 + \frac{1}{c}\int_0^\xi \kappa(m(\zeta))h(m(\zeta), m_+)d\zeta \\ &= m_0 + c\int_0^\xi \Big(\kappa(m(\zeta))m(\zeta)\int_{m(\zeta)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta\Big)d\zeta \ \leq\ m_0 + \xi c M_1. \end{aligned}$$

Therefore, for $0 < \xi < \frac{m_+ - m_0}{cM_1}$, we have $L(m)(\xi) < m_+$. Let $-\frac{m_0}{2cM_1} < \xi < 0$. Then,

$$L(m)(\xi) \ = \ m_0 - c\int_\xi^0 \Big(\kappa(m(\zeta))m(\zeta)\int_{m(\zeta)}^{m_+}\frac{\eta}{\kappa(\eta)}d\eta\Big)d\zeta \ \geq\ m_0 + \xi c M_1 > \frac{m_0}{2}.$$

Therefore, $L(m) \in \Omega$ and hence $L(\Omega) \subset \Omega$.

∘ Contraction property: Let $m, \widetilde{m} \in \Omega$. By a direct calculation,

$$
\begin{aligned}
|L(m) - L(\widetilde{m})| &\le \frac{1}{c}\Big| \int_0^{\xi} \kappa(m(\zeta))h(m(\zeta), m_+)d\zeta - \int_0^{\xi} \kappa(\widetilde{m}(\zeta))h(\widetilde{m}(\zeta), m_+)d\zeta \Big| \\
&\le \frac{1}{c}\int_0^{|\xi|} \big|\kappa(m(\zeta))h(m(\zeta), m_+) - \kappa(\widetilde{m}(\zeta))h(\widetilde{m}(\zeta), m_+)\big| d\zeta \\
&\le \frac{1}{c}\int_0^{|\xi|} \big|\kappa(m(\zeta)) - \kappa(\widetilde{m}(\zeta))\big| \, \big|h(m(\zeta), m_+)\big| d\zeta \\
&\quad + \frac{1}{c}\int_0^{|\xi|} \big|\kappa(\widetilde{m}(\zeta))\big| \, \big|h(m(\zeta), m_+) - h(\widetilde{m}(\zeta), m_+)\big| d\zeta \\
&=: A_1 + A_2.
\end{aligned}
$$

Since the consumption rate κ is C^1, it is Lipschitz continuous and hence

$$
\|\kappa\|_{Lip} := \sup_{0 \le x, y \le m_+} \frac{|\kappa(x) - \kappa(y)|}{|x - y|} < \infty.
$$

The upper bound of f_2 gives

$$
|h(m(\zeta), m_+)| = c^2 m(\zeta) \int_{m(\zeta)}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta \le c^2 M_2 < \infty.
$$

We obtain the following estimate for A_1 by combining these two:

$$
\begin{aligned}
A_1 := \frac{1}{c}\int_0^{|\xi|} \big|\kappa(m(\zeta)) - \kappa(\widetilde{m}(\zeta))\big| \, \big|h(m(\zeta), m_+)\big| d\zeta \\
\le c\|\kappa\|_{Lip} \, M_2 \int_0^{|\xi|} \big|m(\zeta) - \widetilde{m}(\zeta)\big| d\zeta.
\end{aligned} \tag{4.1}
$$

The monotonicity of κ and the upper bound of m and $\widetilde{m}$ allow

$$
\begin{aligned}
\kappa(\widetilde{m}(\zeta)) &\le \kappa(m_+), \\
\|h(\cdot, m_+)\|_{Lip} &\le \sup_{\frac{m_0}{2} \le m \le m_+} \left(\int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + \frac{m^2}{\kappa(m)} \right) \\
&\le \int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3, m_0/2} < \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_2 &\le \frac{1}{c}\int_0^{|\xi|} |\kappa(\widetilde{m}(\zeta))| \, \|h(\cdot, m_+)\|_{Lip} |m(\zeta) - \widetilde{m}(\zeta)| d\zeta \\
&\le \frac{\kappa(m_+)}{c}\Big(\int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3, m_0/2} \Big) \int_0^{|\xi|} |m(\zeta) - \widetilde{m}(\zeta)| d\zeta.
\end{aligned} \tag{4.2}
$$

It follows from (4.1) and (4.2) that

$$
\begin{aligned}
&|L(m)(\xi) - L(\widetilde{m})(\xi)| \\
&\le \left(c\|\kappa\|_{Lip} \, M_2 + \frac{\kappa(m_+)}{c}\Big(\int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3, m_0/2} \Big) \right) \int_0^{|\xi|} |m(\zeta) - \widetilde{m}(\zeta)| d\zeta \\
&\le \left(c\|\kappa\|_{Lip} \, M_2 + \frac{\kappa(m_+)}{c}\Big(\int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3, m_0/2} \Big) \right) |\xi| \sup_{-\xi < \zeta < \xi} |m(\zeta) - \widetilde{m}(\zeta)|.
\end{aligned}
$$

If we take the domain I so small that, for any $\xi \in I$,

$$|\xi| < \frac{1}{c\|\kappa\|_{Lip}\ M_2 + \frac{\kappa(m_+)}{c}\left(\int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3,m_0/2}\right)},$$

then L becomes a contraction mapping on Ω.

∘ (Local existence and uniqueness): Let

$$\epsilon_0 = \min\left\{\frac{m_0}{2cM_1}, \frac{m_+ - m_0}{cM_1}, \frac{1}{c\|\kappa\|_{Lip}\ M_2 + \frac{\kappa(m_+)}{c}\left(\int_{\frac{m_0}{2}}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta + M_{3,m_0/2}\right)}\right\},$$

$$I_0 = (-\epsilon_0, \epsilon_0),$$

$$\Omega_0 := \left\{m \in C^1(I_0) : m(0) = 0,\ m'(\xi) \geq 0,\ \frac{m_0}{2} < m < m_+\right\}.$$

Then, by the previous invariance and contraction properties of the operator L, there exists a fixed point of L in Ω_0 uniquely. Let $L(m) = m \in \Omega_0$. Then, one can easily see that $m'(\xi) = \frac{1}{c}\kappa(m(\xi))h(m(\xi), m_+)$ for $\xi(-\epsilon_0, \epsilon_0)$. □

Lemma 4.2. *If the consumption rate κ satisfies (1.3), then*

$$\lim_{m\to 0+} \frac{1}{c}\kappa(m)h(m, m_+) = 0. \tag{4.3}$$

Proof. If the consumption rate κ satisfies

$$\lim_{m\to 0+} \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta < \infty,$$

then the lemma is clear from (1.3). Let us assume

$$\lim_{m\to 0+} \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = \infty. \tag{4.4}$$

By L'Hospital's theorem and (4.4), we have that

$$\begin{aligned}
0 &\leq \lim_{m\to 0+} \frac{1}{c}\kappa(m)h(m, m_+) \\
&= \lim_{m\to 0+} c\kappa(m)m \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = \lim_{m\to 0+} \frac{c\int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta}{\frac{1}{\kappa(m)m}} \\
&= \lim_{m\to 0+} \frac{-c\frac{m}{\kappa(m)}}{-\frac{\kappa'(m)m+\kappa(m)}{(\kappa(m)m)^2}} = \lim_{m\to 0+} \frac{cm^3\kappa(m)}{\kappa'(m)m + \kappa(m)}.
\end{aligned}$$

Since $\kappa'(m)m \geq 0$, we obtain

$$\lim_{m\to 0+} \frac{cm^3\kappa(m)}{\kappa'(m)m + \kappa(m)} \leq \lim_{m\to 0+} \frac{cm^3\kappa(m)}{\kappa(m)} = 0.$$

Therefore, the limit in (4.3) holds. □

Theorem 4.3 (Global Existence). *Let m_+ and c be positive constants and κ satisfy* (1.3) *and* (3.11). *Then, for any $m_0 \in (0, m_+)$, there exists $m = m(\xi)$ that* *satisfies* (3.9)-(3.10). *Furthermore, the solution is unique,* $0 < m(\xi) < m_+$ *for all* $\xi \in \mathbf{R}$,

$$m(\xi) \to 0 \quad \textit{as} \quad \xi \to -\infty, \quad \textit{and} \quad m(\xi) \to m_+ \quad \textit{as} \quad \xi \to \infty.$$

Proof. The uniqueness and the local existence has been obtained in Lemma 4.1. Let (T_-, T_+) be the maximal domain of the solution. The continuity of κ and the mean value theorem provide

$$\lim_{m\to m_+} \frac{\kappa(m)h(m,m_+)}{m_+ - m} = \lim_{m\to m_+} \frac{\kappa(m)}{m_+ - m} c^2 m \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = c^2 m_+^2.$$

This indicates that $\kappa(m)h(m,m_+) \cong c^2 m_+^2 (m - m_+)$ when m is close to m_+. Suppose that there exists $\xi_0 > 0$ such that $m(\xi_0) = m_+$. Then, after separating the variables in the decoupled equation (3.9), we obtain

$$\frac{\xi_0}{c} = \int_0^{\xi_0} \frac{1}{c} d\xi = \int_{m_0}^{m_+} \frac{1}{\kappa(m)h(m,m_+)} dm = \infty, \tag{4.5}$$

where the integral has a singularity at $m = m_+$. This is a contradiction and we may conclude $m(\xi) < m_+$ for all $\xi > 0$.

Next we will show that $T_+ = \infty$ and $m(\xi) \to m_+$ as $\xi \to \infty$. If not, there are three possibilities:

$$\begin{aligned}
&A. \qquad T_+ < \infty,\ m(T_+) < m_+,\\
&B. \qquad T_+ < \infty,\ m(T_+) = m_+,\\
&C. \qquad T_+ = \infty,\ m(\xi) \to m_\infty < m_+ \text{ as } \xi \to \infty.
\end{aligned}$$

Case A is not possible since we may apply the local existence argument with $m_0 = m(T_+) < m_+$ and the initial time T_+. Therefore, the domain (T_-, T_+) is not the maximal one anymore. Since $m(\xi) < m_+$ for all $\xi \in \mathbf{R}$, Case B is not possible, neither. Rewrite (4.5) as

$$\int_0^{\xi} \frac{1}{c} d\xi = \int_{m_0}^{m(\xi)} \frac{1}{\kappa(m)h(m,m_+)} dm.$$

If $m(\xi) \to m_\infty < m_+$ as $\xi \to \infty$, then the left side diverges and the right side is bounded as $\xi \to \infty$. Hence Case C is not possible.

Now we consider the domain $\xi < 0$. Note that

$$\lim_{m\to 0} \frac{\kappa(m)h(m,m_+)}{m} = \lim_{m\to 0} \frac{\kappa(m)}{m} c^2 m \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta \le \lim_{m\to 0} c^2 \int_m^{m_+} \eta \, d\eta = \frac{c^2 m_+^2}{2}.$$

This indicates that $\kappa(m)h(m,m_+) \lesssim \frac{1}{2} c^2 m_+^2 m$ when m is close to 0. Suppose that there exists $\xi_0 < 0$ such that $m(\xi_0) = 0$. Then, similarly,

$$\frac{\xi_0}{c} = \int_0^{\xi_0} \frac{1}{c} d\xi = \int_{m_0}^{0} \frac{1}{\kappa(m)h(m,m_+)} dm = -\infty, \tag{4.6}$$

which is a contradiction. Therefore, $m(\xi) > 0$ for all $\xi \in \mathbf{R}$. The last part is to show that $T_- = -\infty$ and $m(\xi) \to 0$ as $\xi \to -\infty$. If not, there are three possibilities:

$$\begin{aligned}
&D. \qquad T_- > -\infty,\ m(T_-) > 0,\\
&E. \qquad T_- > -\infty,\ m(T_-) = 0,\\
&F. \qquad T_- = -\infty,\ m(\xi) \to m_- > 0 \text{ as } \xi \to -\infty.
\end{aligned}$$

Similarly, Case D is not possible due to the local existence property. Case E is not possible since $m(\xi) > 0$ for all $\xi \in \mathbf{R}$. Rewrite (4.6) as

$$\int_{\xi}^{0} \frac{1}{c}\, d\xi = \int_{m(\xi)}^{m_0} \frac{1}{\kappa(m)h(m, m_+)} dm.$$

If $m(\xi) \to m_- > 0$ as $\xi \to -\infty$, then the left side diverges and the right side is bounded as $\xi \to \infty$. Hence Case F is not possible, neither. □

4.2. Necessary and sufficient conditions for traveling waves. We find necessary and sufficient conditions on the boundary value for the existence of traveling wave solutions in this section. We have shown in Theorem 4.3 that there always exists a unique traveling wave solution m for each $m(0) = m_0$ if the consumption rate satisfies (1.3) and (3.11). Hence, our concern is the existence of the other component u. The monotonicity of m implies that m is always of the front type. However, the types of population distribution u is decided by the consumption rate κ and is either of the front or the pulse type.

THEOREM 4.4. *Suppose that the consumption rate κ satisfies* (1.3) *and* (3.11). *There exists a traveling wave solution (u, m) of* (3.1)-(3.3) *of wave speed $c > 0$ if and only if the boundary conditions satisfy $0 = m_- < m_+ < \infty$, $u_+ = 0$ and*

$$u_- = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)} < \infty. \tag{4.7}$$

This traveling wave solution is unique up to a translation. Furthermore, if

$$\int_0^{m_+} \frac{1}{\kappa(\eta)} d\eta < \infty, \tag{4.8}$$

it has a finite total population

$$\int_{\mathbf{R}} u(\xi) d\xi = c \int_0^{m_+} \frac{1}{\kappa(\eta)} d\eta. \tag{4.9}$$

Proof. ($\Rightarrow$) Let (u, m) be a traveling wave solution of (3.1)-(3.3). We have shown in Proposition 3.2 that $0 = m_- < m_+ < \infty$ and $u_+ = 0$ are necessary conditions. Hence, it is left to show the other boundary condition. First consider a case that $\int_0^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = \infty$. Then, by the relation (3.7) and L'Hospital's theorem, we obtain

$$\begin{aligned}\lim_{\xi\to-\infty} u(\xi) &= \lim_{m\to 0+} c^2 m \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta \\ &= \lim_{m\to 0+} c^2 \frac{\int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta}{\frac{1}{m}} = \lim_{m\to 0+} c^2 \frac{\frac{m}{\kappa(m)}}{\frac{1}{m^2}} = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)}.\end{aligned}$$

Therefore, (4.7) is a necessary condition for the case. Next consider the other case that $\int_0^{m_+} \frac{\eta}{\kappa(\eta)} d\eta < \infty$. Then,

$$\lim_{\xi\to-\infty} u(\xi) = \lim_{m\to 0+} c^2 m \int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = 0.$$

Hence it is left to check $\lim_{m\to 0+} \frac{m^3}{\kappa(m)} = 0$ to show (4.7) is a necessary condition. Suppose that there exists $\epsilon > 0$ such that $\lim_{m\to 0+} \frac{m^3}{\kappa(m)} \geq \epsilon$. Then, $\int_m^{m_+} \frac{\eta}{\kappa(\eta)} d\eta \geq$

$\int_0^{m_+} \epsilon\eta^{-2}d\eta = \infty$. Therefore, there is no such constant $\epsilon > 0$ and hence $\lim_{m\to 0+} \dfrac{m^3}{\kappa(m)} = 0$.

($\Leftarrow$) To show $0 = u_+ = m_- < m_+ < \infty$ and (4.7) are sufficient conditions, we find a traveling wave solution with these boundary conditions. Let m be the unique solution of (3.9) given by Theorem 4.3 and define

$$u(\xi) = c^2 m(\xi) \int_{m(\xi)}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta.$$

Now we check the boundary conditions. First, we have, from Theorem 4.3,

$$\lim_{\xi\to\infty} m(\xi) = m_+, \quad \lim_{\xi\to -\infty} m(\xi) = 0.$$

Therefore, we also have

$$\lim_{\xi\to\infty} u(\xi) = \lim_{\xi\to\infty} c^2 m(\xi) \int_{m(\xi)}^{m_+} \frac{\eta}{\kappa(\eta)} d\eta = 0.$$

The computations in the previous part of this proof give

$$\lim_{\xi\to -\infty} u(\xi) = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)} = u_-.$$

Therefore, by Proposition 3.4 and these boundary values, the pair (u, m) satisfy (3.1)-(3.3).

(Uniqueness) The uniqueness of solution m that satisfies (3.9) has been obtained in Lemma 4.1 and the relation (3.7) gives u in a unique way. By Proposition 3.4, this pair (u, m) is the unique traveling wave solution of (3.1)-(3.3).

(Finite total population) Eq. (3.2) gives

$$u = c\frac{m'}{\kappa(m)},$$

and hence

$$\int_{\mathbf{R}} u(\xi)d\xi = c\int_{\mathbf{R}} \frac{m'(\xi)}{\kappa(m(\xi))} d\xi = c\int_0^{m_+} \frac{1}{\kappa(\eta)} d\eta.$$

Therefore, the population of the traveling wave solution is finite if $\int_0^{m_+} \frac{1}{\kappa(\eta)} d\eta < \infty$ and the total population is given by (4.9). ☐

Remark 4.5. *Consider a consumption rate given by a power law $\kappa(m) = m^p$ for $p > 0$. If $p < 3$, the left boundary value u_- is $u_- = \lim_{m\to 0+} c^2 \frac{m^3}{\kappa(m)} = 0$ for any wave speed $c > 0$. Hence, the traveling wave for the population u is of pulse type. If $p = 3$, then $u_- = c^2$, which gives a font type traveling wave. If $p > 3$, there is no bounded traveling wave solution. If $p < 1$, then (4.8) is satisfied and hence the traveling wave solution has a finite population for any wave speed $c > 0$. Usually, the consumption is not doubled even if the amount of food is doubled. Hence, the regime $p < 1$ is more realistic in the context of chemotaxis.*

Remark 4.6. *Similar traveling wave phenomena when the dispersal was modeled by a starvation driven diffusion, $u_t = (\gamma(\frac{m}{u})u)_{xx}$, have been recently studied (see [19]). The motility γ is a departing probability depending on $s = \frac{m}{u}$ and the type of a* *traveling wave is decided by γ and κ together. For example, if $\gamma(s) \to \infty$ as $s \to 0$ and $\int_0^{m_+} \frac{1}{k(\eta)} d\eta < \infty$, the population of the traveling wave has a finite value, $c \int_0^{m_+} \frac{1}{k(\eta)} d\eta$, where the boundary conditions should be $0 = u_- = u_+ = m_- < m_+ < \infty$ for all $c > 0$. In other words, the two theories give a similar conclusion. For the infinite total population case, there are some similarities and differences between the two theories. The equation (4.7) gives a relation among the boundary value u_-, wave speed $c > 0$ and the consumption rate $\kappa(m)$. The corresponding relation for the starvation driven diffusion case is $\lim_{m\to 0} \gamma'\left(\frac{m}{u_-}\right) k(m) = -c^2$. Here, we have many choices for γ. In particular, if we choose $\gamma(s) = \frac{1}{2} s^{-2}$, we obtain a relation*

$$u_-^3 = c^2 \lim_{m\to 0} \frac{m^3}{\kappa(m)}, \tag{4.10}$$

which gives the same criterion for the types of traveling waves as (4.7). However, the corresponding boundary value u_- is the cubed root of the other.

5. Numerical simulations. In this section, we compare the shapes of traveling wave solutions numerically when the consumption rate is given by a power law

$$\kappa(m) = m^p, \quad p > 0.$$

Four cases are tested with $p = 0.5, 2, 3$ and 4. We simply follow the steps in previous sections numerically. We first construct m by numerically solving (3.9) with

$$m_0 = 0.1, \quad m_+ = 1, \quad c = \sqrt{0.5} \quad \text{or} \quad \sqrt{2}.$$

The population distribution u is obtained from m by the relation in (3.7). Notice that the boundary condition m_+ is used in computation. However, one may observe that the other boundary conditions $u_+ = 0, m_- = 0$ and $u_- = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)}$ are satisfied by the numerical solutions.

∘ (Case #1. $\kappa(m) = m^p$, $0 < p < 1$: pulse type with finite total population): Consider the case $\kappa(m) = m^{1/2}$. Then,

$$h(m, m_+) = c^2 m \int_m^1 \eta d\eta = \frac{2}{3} c^2 m (1 - m^{3/2}),$$

and, therefore, m satisfies

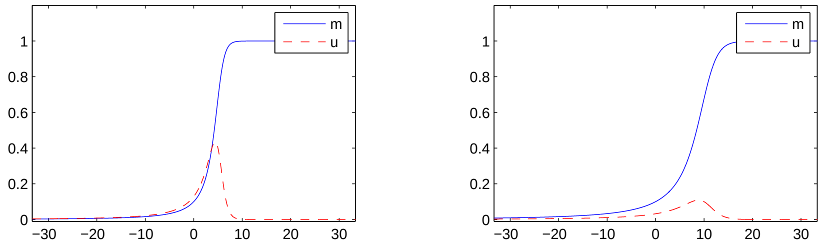
$$m' = cm^2(1 - m^{3/2}), \quad m(0) = 0.1, \quad \xi \in \mathbf{R}.$$

The solution of this ODE and the population u given by (3.7) are computed numerically and given in Figure 5.1. One may observe that the traveling wave solution satisfies the necessary boundary conditions, i.e.,

$$\lim_{\xi\to\pm\infty} u(\xi) = 0, \quad \lim_{\xi\to-\infty} m(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to\infty} m(\xi) = 1. \tag{5.1}$$

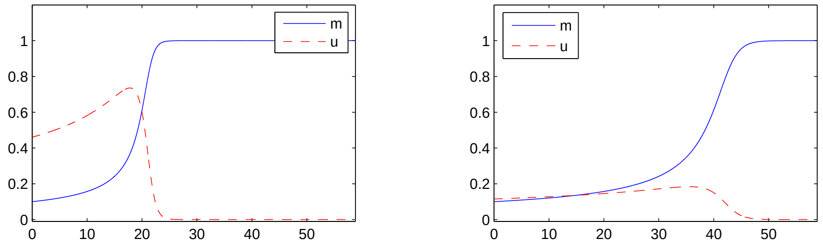
The total population given by the relation (4.9) becomes

$$\int_{\mathbf{R}} u(\xi) d\xi = c \int_0^1 \eta^{-1/2} d\eta = 2c. \tag{5.2}$$

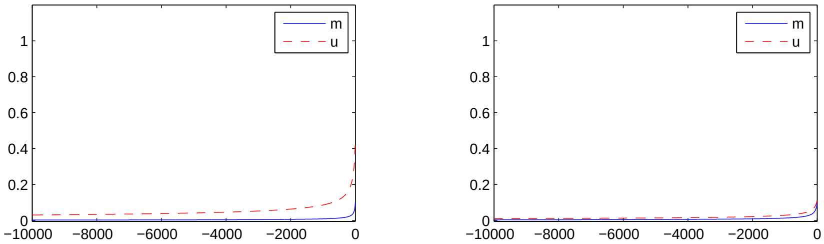


(a) $c = \sqrt{2}$ (b) $c = \sqrt{0.5}$

Fig. 5.1: Pulse type traveling wave with $\kappa(m) = m^{1/2}$

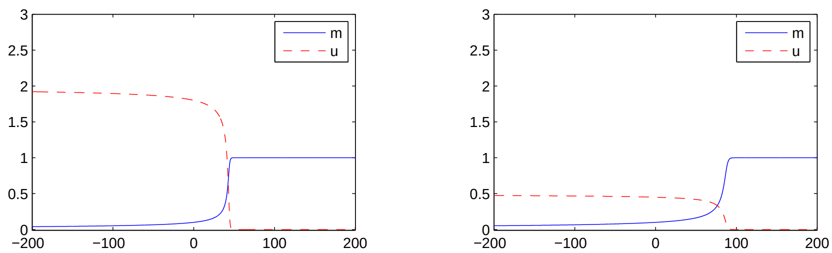


(a) $c = \sqrt{2},\ \xi > 0$ (b) $c = \sqrt{0.5},\ \xi > 0$



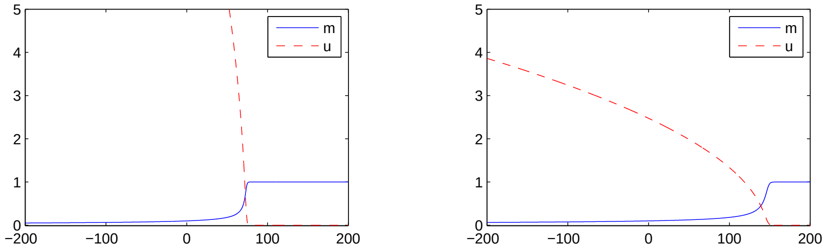
(c) $c = \sqrt{2},\ \xi < 0$ (d) $c = \sqrt{0.5},\ \xi < 0$

Fig. 5.2: Pulse type traveling wave with $\kappa(m) = m^2$



(a) $c = \sqrt{2}$ (b) $c = \sqrt{0.5}$

Fig. 5.3: Front type traveling wave with $\kappa(m) = m^3$



(a) $c = \sqrt{2}$ (b) $c = \sqrt{0.5}$

Fig. 5.4: Nonexistence of bounded traveling waves with $\kappa(m) = m^4$

The total population of numerical solutions in Figures 5.1(a) and 5.1(b) are approximately 2.8275 and 1.4137, respectively, which is a good approximation of the relation (5.2).

∘ (Case #2. $\kappa(m) = m^p$, $1 \le p < 3$: pulse type with infinite total population): Consider the case $\kappa(m) = m^2$. Then,

$$h(m, m_+) = c^2 m \int_m^1 \frac{1}{\eta} d\eta = -c^2 m \ln(m).$$

Therefore, m satisfies

$$m' = -cm^2 \ln(m), \quad m(0) = 0.1, \quad \xi \in \mathbf{R}.$$

The solution of this ODE and the population u are computed numerically and are given in Figure 5.2. This is a case that

$$u_- = c^2 \lim_{m \to 0+} \frac{m^3}{\kappa(m)} = c^2 \lim_{m \to 0+} m = 0, \qquad \int_0^1 \frac{1}{k(\eta)} d\eta = \int_0^1 \eta^{-2} d\eta = \infty.$$

Hence, the traveling wave is a pulse type, but the total population is infinite. One may observe that the traveling wave solution satisfies the necessary boundary conditions (5.1). However, the convergence rate is very slow as $\xi \to -\infty$. In Figures 5.2(c) and 5.2(d) the tails of these traveling waves are given. Notice the scale of x-axis in comparison with Figures 5.2(a) and 5.2(b). Observe that the convergence speed of the solution when $\xi \to -\infty$ is extremely slower than the one when $\xi \to \infty$.

∘ (Case #3. $\kappa(m) = m^p$, $p = 3$: front type with infinite total population): Consider a case with $\kappa(m) = m^3$. Then,

$$h(m, m_+) = c^2 m \int_m^1 \frac{1}{\eta^2} d\eta = c^2(1 - m).$$

Therefore, m satisfies

$$m' = cm(1 - m), \quad m(0) = 0.1, \quad \xi \in \mathbf{R}.$$

The graphs of u and m are given in Figure 5.3 and one may observe that the traveling wave solution satisfies the necessary boundary conditions (5.1) except the one for $\lim_{\xi \to -\infty} u(\xi)$. The limit u_- is given by the relation (4.7),

$$u_- = \lim_{\xi \to -\infty} u(\xi) = c^2 \lim_{m \to 0+} \frac{m^3}{\kappa(m)} = c^2.$$

Hence, the population distribution u is a decreasing front type wave. One may observe the limit $u_- = c^2$ from Figures 5.3(a) and 5.3(b). Remember that the front type traveling wave is obtained only from the cubic nonlinearity such as $\kappa(m) = m^3$.

◦ (Case #4. $\kappa(m) = m^p$, $p > 3$: nonexistence of bounded traveling waves): Consider a case with $\kappa(m) = m^4$. Then,

$$h(m, m_+) = c^2 m \int_m^1 \frac{1}{\eta^3} d\eta = \frac{1}{2m} c^2 (1 - m^2).$$

Therefore, m satisfies

$$m' = \frac{c}{2}(1 - m^2), \quad m(0) = 0.1, \quad \xi \in \mathbf{R}.$$

The solution of this ODE and the population u given by (3.7) are computed numerically and given in Figure 5.4 and one may observe that the traveling wave solution satisfies the necessary boundary conditions (5.1) except the one for $\lim_{\xi\to-\infty} u(\xi)$. The limit u_- is given by the relation (4.7),

$$u_- = \lim_{\xi\to-\infty} u(\xi) = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)} = c^2 \lim_{m\to 0+} \frac{1}{m} = \infty.$$

One may observe from Figures 5.4(a) and 5.4(b) that the solution u diverges as $\xi \to -\infty$.

6. Conclusions. In this paper, we have examined an idea that migration distance that matters to biological organisms is not necessarily the Euclidean distance, but possibly one that also counts the amount of food. As an extreme example, the metric of food is introduced which counts the amount of food only and is defined by (2.1). This metric is used to derive a diffusion equation (1.1) that models a nontraditional random walk system where the walk length is given by the metric of food, but not by the metric of space. As an application example of such a diffusion process, a new chemotaxis model (1.2) has been studied in this paper. Most of chemotaxis models are based on the mysterious assumption[2] that microscopic scale organisms may sense the macroscopic scale concentration gradient. In our model considered, we took only the random diffusion operator without the usual assumption. However, if the food distribution is not constant, the new diffusion operator produces advection phenomena.

The main objective of this paper is to investigate the traveling wave solution of the new chemotaxis model that satisfies (3.1)-(3.3). It is shown in Theorem 4.4 that the necessary and sufficient conditions to have a bounded traveling wave solution with a positive traveling wave speed $c > 0$ are

$$0 = u_+ = m_- < m_+ < \infty \quad \text{and} \quad u_- = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)} < \infty, \tag{6.1}$$

where κ is the consumption rate that satisfies (1.3). The uniqueness of the traveling wave solution is also proved. The food distribution m is always of the front type. On the other hand the type of population distribution u depends on the consumption rate κ. For example, if $\kappa(m) = m^p$, the population distribution u is of the pulse type

[2]Keller and Segel took this assumption and called it mysterious in their seminal papers.

with $0 < p < 3$ and is of the front type with $p = 3$. However, if $p > 3$, there is no bounded traveling solution. Furthermore, if

$$\int_0^{m_+} \frac{1}{\kappa(m)} dm < \infty \quad \big(\text{or } 0 < p < 1\big), \tag{6.2}$$

then the traveling wave has a finite total population. The relation among the total population, the wave speed, and the boundary value $m_+ > 0$ is given by (4.9). In conclusion, we may say that the traveling wave phenomenon can be obtained only from metric of food[3].

Introducing a new distance metric to biology models connects the Riemannian geometry and mathematical biology as we have observed in this paper for one space dimension. Applying this idea to a multi-dimensional space may be quite challenging and the theory of Riemannian geometry may provide us useful tools (see Appendix A). We took a diffusion operator in a simplest form derived from food distribution independent of time. The resulting traveling wave equation is in an exact form which simplifies the analysis (see Remark 3.5). One may develop a similar diffusion theory using a different distance metric and time dependent food distribution, which can be more appropriate to chemotaxis models.

Appendix A. Metric of food in $\mathbf{R}^n$ and Laplace-Beltrami operator.

In this section we briefly discuss about the possibility and limitation of applying the idea of Riemannian geometry to multi-dimensions. The distance between two given points $a, b \in \mathbf{R}^n$ is defined as

$$d(a,b) = \inf_{\gamma} \int_0^1 m(\gamma(t))|\gamma'(t)|dt, \tag{A.1}$$

where γ is a smooth curve such that $\gamma(0) = a$ and $\gamma(1) = b$. The curve that gives the infimum is called the geodesic connecting the two points. The metric tensor g_{ij} and the volume factor corresponding to the distance function above are

$$g_{ij} = m^2 \delta_{ij}, \quad \sqrt{\det(g_{ij})} = m^n,$$

where δ_{ij} is the Kronecker delta. Notice that the volume factor is not $m(x)$, but $m^n(x)$. This implies that the concentration of food with respect to the metric of food is m^{n-1} which is constant only when $n = 1$. Having constant concentration with respect to the new metric is a requirement to take the Laplace-Beltrami operator as a random diffusion operator with respect to the given metric. Otherwise, the Laplace equation on the manifold is not the corresponding diffusion.

Now we consider a metric that gives a constant concentration of food:

$$d(a,b) = \inf_{\gamma} \int_0^1 \sqrt[n]{m(\gamma(t))}\ |\gamma'(t)|dt. \tag{A.2}$$

Then, the metric tensor g_{ij}, the volume factor, and the inverse metric tensor g^{ij} are

$$g_{ij} = m^{2/n} \delta_{ij}, \quad \sqrt{\det(g_{ij})} = m, \quad g^{ij} = m^{-2/n} \delta_{ij}.$$

[3]We could not obtain the chemotactic aggregation phenomenon using the approach of this paper. It seems that aggregation requires more than a random diffusion.

In this case the volume factor is m and hence the concentration distribution with respect to the new metric is constant. Let $v(x,t)$ be the population density, where the density is in terms of the new metric. Then, v satisfies the linear diffusion equation and the corresponding Laplace-Beltrami operator is

$$\Delta_g v = \frac{1}{\sqrt{\det(g_{ij})}}\partial_i\Big(\sqrt{\det(g_{ij})}\, g^{ij}\partial_j v\Big) = \frac{1}{m}\nabla\cdot\Big(m^{1-\frac{2}{n}}\nabla v\Big).$$

This is the relation that corresponds to (2.6) for one space dimension. The density in the Euclidean distance is simply obtained by multiplying the volume factor,

$$u(x,t) = m(x)\, v(x,t).$$

Then, the random diffusion equation becomes

$$u_t = \nabla\cdot\Big(m^{1-\frac{2}{n}}\nabla\Big(\frac{u}{m}\Big)\Big) \qquad \text{in } \mathbf{R}^n. \tag{A.3}$$

Remember that the diffusion equation (A.3) is the one when the metric is given by (A.2). If the metric is given by (A.1), the Laplace-Beltrami operator does not correspond to the random diffusion. However, the diffusion equation (2.4) is the one derived from the random walk model when the walk distance is given by the metric in (A.1).

Acknowledgements: The authors would like to thank Prof. Young-Heon Kim of the University of British Columbia, Canada for the valuable discussion on the Laplace-Beltrami operator related to this work during his visit to KAIST Mathematics Research Station (KMRS) of KAIST in Daejeon, Korea.

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