# Bistable nonlinearity with a discontinuity and traveling waves with a free boundary

Jaywan Chung[a], Yong-Jung Kim[a,b,*]

[a] *National Institute of Mathematical Sciences, 70, Yuseong-daero 1689 beon-gil, Yuseong-gu, Daejeon 34047, Korea*
[b] *Department of Mathematical Sciences, KAIST, 291, Daehak-ro, Yuseong-gu, Daejeon 34141, Korea*

**Abstract**

We develop a traveling wave theory driven by a discontinuous bistable nonlinearity. The Allee effect is included by simply subtracting a constant from a logistic equation, which produces a discontinuity when there is no population. This discontinuity overrides the infinite propagation speed of a diffusion process and gives a finite speed. In return, we obtain a free boundary even under the presence of a linear diffusion.

*Keywords:* Traveling wave, Free boundary, Allee effect, Logistic equation, Bistable equation
*2010 MSC:* 35C07, 35R35, 35Q92

## 1. Introduction

One of the most widely accepted population models is the logistic equation:

$$u_t = r_1 u - r_2 u^2 = r_1 u(1 - u/K), \quad K = r_1/r_2, \tag{1}$$

where $u, r_1, r_2$, and $K$ are respectively the population, intrinsic growth rate, self-competition rate, and carrying capacity. The linear term gives the population growth and the quadratic one gives the negative effect of an intra-species competition (or self-competition) for limited resources. This model has a global asymptotic convergence property that, for any positive initial population $u(0) > 0$, the population approaches the carrying capacity $K$. However, the Allee effect, which is a phenomenon that a colony of biological species becomes extinct if the initial population size (or density) is less than a critical value, say $u^* > 0$, opposes such a convergence. The purpose of this paper is to develop a traveling wave theory that includes the Allee effect.

*Corresponding author
*Email addresses:* jaywan.chung@gmail.com (Jaywan Chung), yongkim@kaist.edu (Yong-Jung Kim)

One typical way to include the Allee effect is to multiply $u-u^*$ to the logistic reaction term in (1) and obtain

$$u_t = (r_1 u - r_2 u^2)(u - u^*) = -r_1 u^* u + (r_1 + r_2 u^*)u^2 - r_2 u^3,$$

where $u^*$ is the critical value inducing the Allee effect. Such a reaction term is called a bistable nonlinearity and has been studied in various contexts [1]. In this case if $u(0) < u^*$, the population converges to zero asymptotically as $t \to \infty$, but not in a finite time. Observe that the population growth in this model is of a quadratic order and the self-competition is of a cubic order. In this paper we consider a model that keeps the linear growth and the quadratic self-competition as in the logistic model (1). The key idea to obtain the Allee effect is the observation that a *an additive constant* $-r_0$ *is missing* in the logistic model and we consider

$$u_t = -r_0 + r_1 u - r_2 u^2. \tag{2}$$

Mathematically, we may understand the reaction term in (2) as a power series approximation with a constant term. Biologically the negative constant represents a population loss which is independent of the population size. Such a constant effect can be found in many places. For example, the loss of an animal herd by predators is a constant effect as long as the number of predators remains constant. The population decreases constantly if there is no reproduction since aging is the reason for the decrease. We note that the effect of enzyme in a chemical reaction process is also a constant effect, which may give further application of the reaction process (2). In fluid dynamics, gravity provides a constant effect and plays a key role in the dynamics. The study of traveling waves provides useful information in many physical and biological phenomena (see [2, 3, 4, 5]).

We next normalize our model (2) to make comparison with other models easier. By genericity, we may assume that the right-hand side of (2) has two positive zeros $a^\pm$ such that $0 < a^- < a^+$. (If it has no positive zero, it has no biological meaning. If it has only one positive zero, it is not generic.) Then using new variables $\tilde{u} := u/a^+$ and $\tilde{t} := r_2 a^+ t$, we have $\tilde{u}_{\tilde{t}} = (\tilde{u} - \frac{a^-}{a^+})(1 - \tilde{u})$. Hence, if we denote $u^* := \frac{a^-}{a^+} < 1$, we obtain a normalized population dynamics $u_t = \psi(u)$ where

$$\psi(u) = \begin{cases} (u - u^*)(1 - u), & \text{if } u > 0, \\ 0, & \text{if } u = 0 \end{cases}$$

and $0 < u^* < 1$. The second case for $u = 0$ is added because the population should be nonnegative. Now it is clear how $\psi$ achieves the Allee effect by modifying the logistic reaction term $u(1-u)$ and it is simpler than the bistable reaction term $u(u-u^*)(1-u)$. More generally, we may consider any $\psi$ satisfying for a given $u^* \in (0,1)$,

$$\begin{cases} \psi \in C^1((0,1]), \\ \psi(0) = \psi(u^*) = \psi(1) = 0, \quad \psi'(1) < 0, \\ \psi(u) < 0, \qquad 0 < u < u^*, \\ \psi(u) > 0, \qquad u^* < u < 1, \end{cases} \tag{3}$$

and $\psi(u)$ has a discontinuity at $u = 0$, i.e.,

$$\psi(0) = 0 \quad \text{but} \quad \psi(0^+) = -u^*. \tag{4}$$

The assumption $\psi(0^+) = -u^*$ is not restrictive; if it is any negative value, we may set it by $-u^*$ using rescaling.

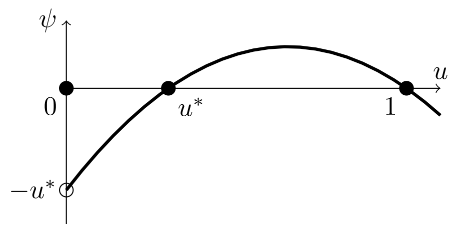



Figure 1: A sketch of the discontinuous reaction term $\psi$

In this paper we consider a traveling wave satisfying

$$\begin{cases} u_t = u_{xx} + \psi(u) & \text{for } (x,t) \in \mathbf{R} \times (0,\infty), \\ u(-\infty,t) = 1, \ u(\infty,t) = 0 & \text{for } t \geq 0, \end{cases} \tag{5}$$

where the two asymptotic limits $u = 1$ and $u = 0$ are stable steady states. We show that there exists a traveling wave solution which is unique up to spatial translation, and that the traveling wave has a free boundary. Since $\psi$ has a discontinuity at $u = 0$, the solution is understood in a weak sense. The main result of this paper is the following.

**Theorem 1** (Main Result)**.** *Suppose that $\psi$ satisfies* (3) *and* (4)*. Then, there is a unique wave speed $c \in \mathbf{R}$ and a traveling wave $u(x,t) = v(x - ct)$ satisfying* (5)*. The wave profile $v(z)$ is strictly decreasing on its support, has a free boundary at some point $z = z_0$ (i.e., $v(z_0) = 0$ and $v(z) > 0$ for all $z < z_0$), has $C^{1,\alpha}(\mathbf{R}) \cap C^{2,\alpha}(\mathbf{R} \setminus \{z_0\})$-regularity for all $0 < \alpha < 1$, and is unique up to spatial translation. The sign of the wave speed $c$ is same as the one of $\int_0^1 \psi(s)ds$.*

It is well known that a jump discontinuity at a point $u = \theta \in (0,1)$ does not change the regularity of traveling waves [6]. But in the context of subsonic detonation, when there is a jump discontinuity at $u = 1$, it was observed that a wave profile has a unity value on a half interval [7]. This observation is closely related to ours because the free boundary means a wave profile has a zero value on a half interval. Also it is worth mentioning Du and Matano [8] classified the long-time asymptotic limits for a bistable or combustion-type nonlinearities and showed that a jump discontiuity at $u = 1$ does not change the asymptotics.

The free boundary cannot be given by the logistic reaction term itself. Du and Lin [9] constructed a solution under a Stefan-type free boundary condition, which is a weak solution away from the free boundary. In that case the solution either spreads or vanishes depending on the size of initial support, but not initial density. Further results had been obtained using a different free boundary condition [10] or adding an advection term [11]. On the other hand, the traveling wave solution of this paper is a weak solution in the whole space and the discontinuous reaction term $\psi$ gives the Allee effect and the free boundary together. Similar kinds of free boundaries which arise from the equation itself can be observed in obstacle-type problems [12], the porous medium equation [13] and the modified KdV equation [14].

## 2. Traveling wave with a free boundary

In this section, we show that there is a unique traveling wave satisfying (5) which has a free boundary. Consider a traveling wave solution

$$u(x,t) = v(z), \quad z := x - ct,$$

where $v$ is the wave profile and $c \in \mathbf{R}$ is the wave speed. Then, from (5), the wave profile satisfies

$$v'' + cv' + \psi(v) = 0, \quad v(-\infty) = 1, \quad v(\infty) = 0. \tag{6}$$

Because any spatial translation of a wave profile $v$ is also a wave profile, we give an additional condition $v(0) = u^*$. Then we can expect the uniqueness. Since there is a discontinuity of $\psi(v)$ at $v = 0$, the solution of (6) is defined in a weak sense:

**Definition 1.** A continuous function $v \in H^1_{\text{loc}}(\mathbf{R}) \subset C^{0,1/2}_{\text{loc}}(\mathbf{R})$ and a constant $c \in \mathbf{R}$ are respectively called the traveling wave profile and wave speed of the (asymptotic) boundary problem (5) if $v(-\infty) = 1,\ v(\infty) = 0,\ v(0) = u^*$, and

$$\int -v'\phi' + cv'\phi + \psi(v)\phi \, dz = 0$$

for any compactly supported smooth function $\phi$.

The main idea to prove Theorem 1 is by approximation of the discontinuous nonlinearity $\psi$ by a sequence of continuously differentiable bistable nonlinearities $\psi_n$. Then one may find a sequence of classical wave profiles $v_n$ corresponding to $\psi_n$. With the help of an energy functional, we show that $v_n$ converges to a function $v$ which is a wave profile in Definition 1. Then the discontinuity of $\psi$ and the integrability of $\psi(v)$ confirm $v$ has a free boundary. The regularity of $v$ follows from Schauder's estimate. The uniqueness proof is similar to the classical case.

In the rest of this section, we assume that

$$\int_0^1 \psi(s)\, ds > 0,$$

which shows that the wave speed $c \in \mathbf{R}$ is positive as in the case when a traveling wave represents an expanding population. One may similarly obtain a shrinking population with a wave speed $c < 0$ if $\int_0^1 \psi(u)\, du < 0$. If $\int_0^1 \psi(u)\, du = 0$, then the previous two cases automatically imply that the wave speed is zero as in the classical case for a continuous bistable nonlinearity.

*2.1. Uniqueness and partial regularity*

In this subsection we prove the uniqueness of a wave profile satisfying Definition 1. The proof is quite similar to the one for classical traveling waves [15]. To prove the uniqueness, we will show that $0 \le v < 1$, $v \in C^{1,1/2}(\mathbf{R})$, $v \in C^{2,1/2}$ except the points where $v = 0$, and that $v$ is strictly decreasing. These properties will yield uniqueness.

Let $v \in H^1_{\text{loc}}(\mathbf{R})$ be a weak solution in Definition 1. If $v > 0$ in a compact domain $K$, then $\psi(v) \in H^1(K)$ because $\psi$ is $C^1$ there and $(\psi(v))' = \psi'(v)v' \in L^2(K)$ by the chain rule [16, Problem 5.10.17]. Also by the $L^2$-regularity theory [17, Theorem 8.13], as a weak solution of (6), $v$ is in $H^3(K) \subset C^{2,1/2}(K)$. Therefore, the weak solution is in fact $C^2$ whenever $v > 0$.

First we prove that $0 \le v \le 1$. Because $\psi(u)$ is not defined for $u \notin [0, 1]$, we assume that $\psi(u) < 0$ when $u > 1$ and $\psi(u) \equiv 0$ when $u < 0$. If $v > 1$ at some point, $v$ should have a local maximum due to the boundary conditions. At the local maximum, $v'' \le 0$, $v' = 0$, and $v > 1$ so that $0 \ge v'' + cv' = -\psi(v) > 0$, which is a contradiction. Hence $v \le 1$. If $v < 0$ at some point, due to the boundary conditions, $v < 0$ in an interval $(z_0, z_1)$ with $v(z_0) = v(z_1) = 0$ ($z_1$ can be infinity). But there $v'' + cv' = 0$ so the only solution satisfying the zero boundary conditions is the trivial one, which is a contraction. Hence $v \ge 0$.

Next we show that $v$ is strictly decreasing on its support. The solution of (6) corresponds to a trajectory of the phase plane,

$$\begin{cases} \frac{dv}{dz} = w, \\ \frac{dw}{dz} = -cw - \psi(v), \end{cases} \tag{7}$$

that departs from a steady state $(v, w) = (1, 0)$ and arrives at $(0, 0)$. Now we show that this trajectory does not touch $w = 0$ other than these two points. Since $v \le 1$, we have $w < 0$ near the departing point $(1, 0)$. If $v \in (u^*, 1)$, the trajectory cannot touch the $v$-axis; if then, $w = 0$ and $\frac{dw}{dz} = -\psi(v) < 0$, which is not possible. If the trajectory touches $v$-axis at $v = u^*$, we have $\frac{dv}{dz} = \frac{dw}{dz} = 0$ so the trajectory stays at $(v, w) = (u^*, 0)$ forever and does not arrive at $(0, 0)$. Finally suppose that the trajectory intersects $v$-axis at some point $v \in (0, u^*)$, where $\frac{dw}{dz} = -\psi(v) > 0$. Then $w$ becomes strictly positive at the intersection point. Therefore the trajectory goes to the positive direction in the $v$-axis so it is not possible to reach the point $(0,0)$ without touching another point in the trajectory. In summary, any trajectory that connects $(1,0)$ and $(0,0)$ gives a strictly decreasing traveling wave.

Now we are ready to prove the uniqueness. Since $v$ decreases strictly on its support, the inverse function of $v$ is well-defined for $v \in (0,1)$. Therefore, we may consider $w$ as a function of $v$, i.e., $w = w(v)$. Then the second equation in (7) is written as

$$\frac{dw}{dv} + \frac{\psi}{w} = -c.$$

Now suppose that there are two traveling waves, $v_1$ and $v_2$, with two corresponding wave speeds, $c_1$ and $c_2$. Then the difference $P := w_1 - w_2$ satisfies

$$\frac{dP}{dv} - \frac{\psi}{w_1 w_2} P = -(c_1 - c_2).$$

Using the integrating factor

$$\mu(v) := \exp\{-\int_{u^*}^{v} \frac{\psi(s)}{w_1(s) w_2(s)}\, ds\} > 0,$$

we have

$$\frac{d(\mu P)}{dv} = -(c_1 - c_2)\mu. \tag{8}$$

Note that the integrand in $\mu$ changes its sign at $v = u^*$ from minus to plus so that $\mu \le 1$, i.e., $0 < \mu(v) < 1$ for all $0 < v < 1$. Since $P(0) = P(1) = 0$, we have $(\mu P)(0) = (\mu P)(1) = 0$. Hence by Rolle's theorem, there exists $v_0 \in (0,1)$ such that $\frac{d}{dv}(\mu P)(v_0) = 0$. Therefore $c_1 = c_2$ by (8), i.e., the wave speed is unique. Furthermore, Eq. (8) with the initial condition $P(0) = 0$ gives that $\mu P = 0$ and hence $P = 0$. Hence $w_1 \equiv w_2$, which gives the uniqueness of the wave profile. □

### *2.2. Approximation with continuous bistable nonlinearities*

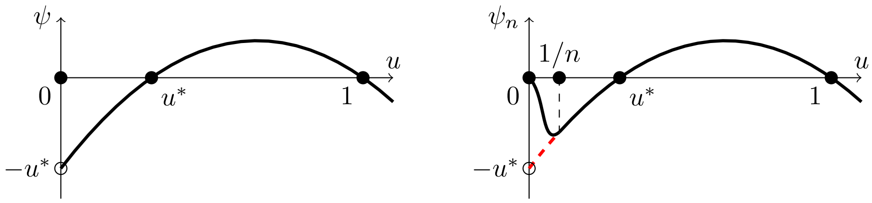
In the following subsections, we prove the existence of a wave profile. We first choose a sequence of $C^1$-functions $\psi_n$ that approximates the discontinuous one $\psi$. More precisely, we choose $\psi_n \in C^1([0,\infty))$ satisfying:

(i) $\psi_n$ is defined for all sufficiently large $n$ such that $\frac{1}{n} < u^*$,
(ii) $\psi_n(0) = 0$ and $\psi_n(u) < 0$ for all $0 < u < u^*$,
(iii) $\psi \le \psi_{n+1} \le \psi_n$ for all $u \ge 0$,
(iv) $\psi_n = \psi$ for all $u \ge \frac{1}{n}$.

These relations between $\psi$ and $\psi_n$ are illustrated in Figure 2.

For each $\psi_n$, it is well known [15] that there is a unique, strictly decreasing wave profile $v_n$ satisfying

$$v_n'' + c_n v_n' + \psi_n(v_n) = 0, \quad v_n(0) = u^*, \tag{9}$$




(a) discontinuous nonlinearity $\psi$ (b) continuous approximation $\psi_n$

Figure 2: Discontinuous $\psi$ and its approximation $\psi_n$

and boundary conditions $v_n(-\infty) = 1,\ v_n(\infty) = 0$. The wave speed $c_n > 0$ is also uniquely determined.

Then we show that the sequence $\{v_n\}$ of traveling waves converges to a unique traveling wave defined in Definition 1 as $n \to \infty$. In fact we show that there is a subsequential limit of $\{v_n\}$ and the limit is the weak solution. Therefore, the previous uniqueness result implies that the whole sequence $v_n$ actually converges. Let $\Omega$ be an open set in $\mathbf{R}$ and $K \subset \Omega$ be a compact set. Then, since $v_n$ satisfies (9), by Schauder's estimate [17, Theorem 8.32],

$$\|v_n\|_{C^{1,\beta}(K)} \lesssim_{\Omega,K} (\|v_n\|_{C^0(\Omega)} + \|\psi_n(v_n)\|_{C^0(\Omega)}) \leq 1 + \sup_{0\leq v\leq 1} |\psi(v)|,$$

where $\lesssim_{\Omega,K}$ means a positive coefficient depending only on $\Omega$ and $K$ is omitted. Hence, for any $0 < \beta \leq 1$, $\{\|v_n\|_{C^{1,\beta}(K)}\}$ is bounded. Therefore, by taking a subsequence, we can find a $v \in C^{1,\alpha}(K),\ 0 < \alpha < \beta$ such that

$$v_n \to v \quad \text{in } C^{1,\alpha}(K).$$

In fact, because $v_n$'s are bounded and decreasing, taking a further subsequence, we may show that such a limit $v$ is decreasing and can be extended to the whole domain $\mathbf{R}$.

### 2.3. Existence of a wave speed

We show that the sequence of wave speeds $\{c_n\}$ is bounded away from zero, which will imply the positivity of the wave speed $c$ for the limiting case. We will use the following lemma that estimates the gradient of $v_n$.

**Lemma 1** (Gradient estimate)**.** *There are positive constants $M'$ and $M'_0$, independent of $n$, such that*

$$|v'_n| \leq M' \quad \textit{for all } n,$$

*and*

$$\liminf_{n\to\infty} |v'_n(0)| \geq M'_0 > 0.$$

*(Remember that we have set $v_n(0) = u^*$.)*

*Proof.* Consider an *energy* functional

$$E_n(z) := \frac{(v_n'(z))^2}{2} + \int_0^{v_n(z)} \psi_n(s)\, ds.$$

Then, by (9), $\frac{d}{dz}E_n(z) = v_n'(v_n'' + \psi_n(v_n)) = -c_n(v_n')^2 < 0$. Hence

$$\int_0^1 \psi_n(s)\, ds = E_n(-\infty) > E_n(z) > E_n(\infty) = 0 \quad \text{for } -\infty < z < \infty,$$

which in turn implies

$$-\int_0^{v_n} \psi_n(s)\, ds < \frac{(v_n')^2}{2} < \int_{v_n}^1 \psi_n(s)\, ds. \tag{10}$$

From the second inequality in (10),

$$\frac{(v_n')^2}{2} \le \int_{v_n}^1 |\psi_n(s)|\, ds \le \int_0^1 |\psi(s)|\, ds$$

so that $|v_n'|$ is bounded uniformly by a constant $M' := \left(2\int_0^1 |\psi(s)|\, ds\right)^{1/2}$. If $\epsilon \le v_n(z) \le u^*$, we can easily deduce from the first inequality in (10) that

$$\liminf_{n\to\infty} |v_n'(z)| \ge \left(2\int_0^{\epsilon} |\psi(s)|\, ds\right)^{1/2}.$$

In particular, if we set $v_n(0) = u^*$,

$$\liminf_{n\to\infty} |v_n'(0)| \ge \left(2\int_0^{u^*} |\psi(s)|\, ds\right)^{1/2} =: M_0',$$

which completes the proof of the lemma. □

Now we are ready to obtain the uniform estimates for the wave speeds $c_n$'s and profiles $v_n$'s.

**Proposition 1** (Uniform Estimates)**.** *There are positive constants $\underline{c}$, $\overline{c}$, and $M$ such that, for all $n$,*

$$\underline{c} \le c_n \le \overline{c}, \quad \textit{and} \quad \int |\psi_n(v_n(z))|\, dz \le M.$$

*Proof.* Multiply the equation (9) by $v_n'$ and integrate over $\mathbf{R}$ to see

$$0 = \int \frac{1}{2}[(v_n')^2]' + c_n(v_n')^2 + v_n'\psi_n(v_n)\, dz.$$

Hence we have

$$c_n \int (v_n')^2\, dz = \int_0^1 \psi_n(s)\, ds. \tag{11}$$

Since $c_n > 0$, the above asserts that $v_n' \in L^2(\mathbf{R})$ (but not uniformly in $n$). The monotonicity of $v_n$ and the gradient estimate in Lemma 1 imply

$$\int (v_n')^2\,dz \le \|v_n'\|_\infty \int |v_n'|\,dz = \|v_n'\|_\infty \int -v_n'\,dz = M'$$

so that $\|v_n'\|_{L^2}$ is uniformly bounded by $\sqrt{M'}$. The desired lower bound $\underline{c}$ is obtained by:

$$c_n = \frac{\int_0^1 \psi_n(s)\,ds}{\int (v_n')^2\,dz} \ge \frac{\int_0^1 \psi(s)\,ds}{M'} =: \underline{c} > 0.$$

Next, we integrate (9) from 0 to $\infty$ and obtain

$$c_n u^* + v_n'(0) = \int_0^\infty \psi_n(v_n)\,dz \le 0,$$

where the last inequality comes from the fact that $0 < v_n(z) < u^*$ for $z > 0$ and $\psi < 0$ on the range. Therefore, $c_n u^* \le |v_n'(0)| \le M'$ and hence

$$c_n \le \frac{M'}{u^*} =: \bar{c}.$$

The relation in (11) gives a lower bound for $\int (v_n')^2\,dz$, which is

$$\int (v_n')^2\,dz = \frac{1}{c_n}\int_0^1 \psi_n(s)\,ds \ge \frac{1}{\bar{c}}\int_0^1 \psi(s)\,ds.$$

Finally, we integrate (9) from $-\infty$ to 0 and obtain

$$\int_{-\infty}^0 \psi_n(v_n)\,dz = \int_{-\infty}^0 |\psi_n(v_n)|\,dz = -v_n'(0) + c_n(1-u^*).$$

Add it to the other integral from 0 to $\infty$ and obtain

$$\int_{-\infty}^\infty |\psi_n(v_n)|\,dz = -2v_n'(0) + c_n(1-2u^*).$$

This shows that integrals $\int_{-\infty}^\infty |\psi_n(v_n)|\,dz$ are uniformly bounded by a constant $M := 2M_0' + \bar{c}(1-u^*)$. □

*2.4. Structure of the limit* $v = \lim_{n\to\infty} v_n$

Finally we show that the subsequential limit $v$ is the traveling wave as asserted in Theorem 1. Then, by virtue of the uniqueness, the limit holds for the full sequence. It is left to show that (i) $v$ has a free boundary, (ii) it is a weak solution, and (iii) it is $C^{2,\alpha}$, $0 < \alpha < 1$, except the free boundary.

(i) Since the sequence of integrals $\{\int |\psi_n(v_n)|\,dz\}$ is bounded by Proposition 1, Fatou's lemma implies

$$\int |\psi(v)|\,dz \le \liminf_{n\to\infty} \int |\psi_n(v_n)|\,dz < \infty.$$

Remember that $\lim_{v\to 0+} |\psi(v)| = u^*$. Therefore, if $v(z) > 0$ for all $z > 0$, then $\int |\psi(v)|\, dz$ diverges. This implies there should be a free boundary.

(ii) For a test function $\phi$, $v_n$ satisfies $\int \big( - v_n\phi' + cv_n'\phi + \psi_n(v_n)\phi\big)\, dz = 0$. Since $v_n$ converges in $C^{1,\alpha}$ to $v$ in the support of $\phi$,

$$\int -v'\phi' + cv'\phi + \psi(v)\phi\, dz = 0.$$

The convergence of the third term follows from the dominated convergence theorem. Therefore, the limit $v$ is a traveling profile in Definition 1.

(iii) Take a point $a < \max(\mathrm{spt}\,(v))$ so that $v(a) > 0$. By taking the point sufficiently closely to the free boundary, we may obtain $v(a) \le u^*$. Then, by Lemma 1, $|v'(z)| > 0$ in a small neighborhood of $z = a$. Hence $v_n \ge v(a)/2$ in $(-\infty, a]$ for sufficiently large $n$. Let $\Omega := (-\infty, a]$. Then, for all sufficiently large $n$, Schauder's estimate [17, Theorem 6.2] gives

$$\begin{aligned}\|v_n\|_{C^{2,\beta}(\Omega)} &\lesssim_\Omega \|v_n\|_{C^0(\Omega)} + \|\psi_n(v_n)\|_{C^{0,\beta}(\Omega)}\\ &\lesssim_\Omega 1 + \|\psi_n'(v_n)v_n'\|_{C^0(\Omega)}\\ &\lesssim_\Omega 1 + \sup_{v(a)/2\le u\le 1} \psi'(u)M'.\end{aligned}$$

Therefore, by taking a subsequence, for any $0 < \alpha < \beta \le 1$, $v_n$ converge to a $C^{2,\alpha}$-function in $\Omega$, i.e., $v \in C^{2,\alpha}(\Omega)$. Because the domain $\Omega$ can be taken arbitrarily close to the support of $v$, we conclude that $v$ is $C^{2,\alpha}$ when $v > 0$.

**Remark 1.** The $C^2$-regularity of $v$ cannot be obtained at the free boundary $z = z_0$. Note that, from the $C^1(\mathbf{R})$-regularity of $v$ and (6),

$$v''(z_0^-) = -cv'(z_0^-) - \psi(0^+) = -\psi(0^+) < 0.$$

However, $v''(z_0^+) = 0$ since $v$ is constant for all $z > z_0$.

## References


[1] B. H. Gilding, R. Kersner, Travelling waves in nonlinear diffusion-convection reaction, Progress in Nonlinear Differential Equations and their Applications, 60, Birkhäuser Verlag, Basel, 2004. `doi:10.1007/978-3-0348-7964-4`.
URL `http://dx.doi.org/10.1007/978-3-0348-7964-4`

[2] A. I. Volpert, V. A. Volpert, V. A. Volpert, Traveling wave solutions of parabolic systems, Vol. 140 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1994, translated from the Russian manuscript by James F. Heyda.

[3] C. Yoon, Y.-J. Kim, Bacterial chemotaxis without gradient-sensing, J. Math. Biol. 70 (6) (2015) 1359–1380. `doi:10.1007/s00285-014-0790-y`.
URL `http://dx.doi.org/10.1007/s00285-014-0790-y`

[4] S.-H. Choi, Y.-J. Kim, Chemotactic Traveling Waves by Metric of Food, SIAM J. Appl. Math. 75 (5) (2015) 2268–2289. doi:10.1137/15100429X. URL http://dx.doi.org/10.1137/15100429X

[5] F. Waleffe, Homotopy of exact coherent structures in plane shear flows, Physics of Fluids (1994-present) 15 (6) (2003) 1517–1534.

[6] B. Perthame, Parabolic equations in biology. Growth, reaction, movement and diffusion, Lecture Notes on Mathematical Modelling in the Life Sciences, Springer International Publishing, 2015.

[7] H. Brezis, S. Kamin, G. Sivashinsky, Initiation of subsonic detonation, Asymptot. Anal. 24 (1) (2000) 73–90.

[8] Y. Du, H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, J. Eur. Math. Soc. (JEMS) 12 (2) (2010) 279–312. doi:10.4171/JEMS/198.
URL http://dx.doi.org/10.4171/JEMS/198

[9] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal. 42 (1) (2010) 377–405. doi:10.1137/090771089.
URL http://dx.doi.org/10.1137/090771089

[10] J. Cai, B. Lou, M. Zhou, Asymptotic behavior of solutions of a reaction diffusion equation with free boundary conditions, J. Dynam. Differential Equations 26 (4) (2014) 1007–1028. doi:10.1007/s10884-014-9404-z.
URL http://dx.doi.org/10.1007/s10884-014-9404-z

[11] H. Gu, Z. Lin, B. Lou, Long time behavior of solutions of a diffusion-advection logistic model with free boundaries, Appl. Math. Lett. 37 (2014) 49–53. doi:10.1016/j.aml.2014.05.015.
URL http://dx.doi.org/10.1016/j.aml.2014.05.015

[12] A. Petrosyan, H. Shahgholian, N. Uraltseva, Regularity of free boundaries in obstacle-type problems, Vol. 136 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012.

[13] J. L. Vázquez, The porous medium equation. Mathematical theory, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[14] P. Rosenau, What is... a compacton?, Notices Amer. Math. Soc. 52 (7) (2005) 738–739.

[15] P. C. Fife, J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal. 65 (4) (1977) 335–361.

[16] L. C. Evans, Partial differential equations, 2nd Edition, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010. doi:10.1090/gsm/019.
URL http://dx.doi.org/10.1090/gsm/019

[17] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.