Diffusive and inviscid traveling waves of the Fisher equation and nonuniqueness of wave speed

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Abstract

In this paper we present an intuitive explanation for the non-uniqueness of the traveling wave speed in the Fisher equation, showing a similar non-uniqueness property in the case of inviscid traveling waves. More precisely, we prove that traveling waves of the Fisher equation with wave speed $c > 0$ converges to the inviscid traveling wave with speed $c > 0$ as the diffusion vanishes. A complete diagram that shows the relation between the diffusive and inviscid traveling waves is given in this paper.

Keywords: Fisher equation, minimum wave speed, inviscid traveling waves

1. Introduction

It is believed that traveling wave phenomena in a reaction diffusion equation,

$$u_t = du_{xx} + \psi(u), \quad u \geq 0,\ x \in \mathbf{R},$$

are obtained by an interplay between the diffusion and the reaction. For example, there exists a unique traveling wave solution for a bistable nonlinearity case, say $\psi(u) = u(1-u)(u-a)$, $0 < a < 1$, that connects the two stable steady states, $u = 0$ and 1. However, such a traveling wave solution does not exist if $d = 0$ or $\psi = 0$. In other words, the unique traveling wave solution has been *produced* by an interplay between the two different mechanisms. However, such a belief fails when the traveling wave connects a stable steady state to an unstable one. First of all, there exist inviscid ($d = 0$) traveling waves for

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any wave speed which stand without any help of diffusion. On the other hand, diffusive (or viscous) traveling waves exist only when the wave speed is greater than or equal to a minimum speed. In other words, the diffusion does not produce traveling waves, but a gap in wave speeds. The purpose of this note is to clarify the role of each component involved in the traveling wave phenomenon that connects a stable steady state to an unstable one.

To be more specific we consider the Fisher equation case in this note, i.e., $\psi(u) = u(1-u)$, where the stable steady state is $u = 1$ and the unstable one is $u = 0$. This Fisher equation provides the phenomenon in a simplest form. If $d = 0$, there is a traveling wave for any given wave speed $c > 0$ (see Section 2). We denote it by v_c and call it an *inviscid traveling wave*. However, if $d \neq 0$, there exists a traveling wave of a speed $c > 0$ if and only if $c \geq c^* := 2\sqrt{d}$. We denote it by $u_{c,d}$ and call it a *diffusive traveling wave*. The theory of this paper is for the relation between the inviscid and diffusive traveling waves. First we show that $u_{c,d} \to v_c$ uniformly as $d \to 0$ in Theorem 1. This convergence gives an insight for the existence of a continuum of traveling wave speeds in the Fisher equation, where each fixed speed corresponds to an inviscid traveling wave as $d \to 0$. The convergence $u_{c,d} \to u_{c^*,d}$ as $c \to c*$ with a fixed d is given in Theorem 2 and the convergence of $u_{c^*(d),d}$ to a step function as $d \to 0$ is given in Theorem 3. The convergence of v_c to the same step function as $c \to 0$ directly comes from the explicit formula of v_c in (2.5). Finally, these relations of convergence among the step function, diffusive and inviscid traveling waves complete a diagram of convergence given in Figure 1, which is discussed in Section 4.

Studies of the vanishing viscosity limit are classical in hyperbolic problems of conservation laws. Such studies include the Fisher equation as a zero convection case if a monostable reaction term is added (see [9, 10, 13, 14, 18]). In particular, it was shown in [3, 4, 8] that the vanishing viscosity limit of minimum wave speeds is the minimum wave speed of the inviscid traveling wave, which is a related result to Theorem 3. The convergence relations given in this paper, Figure 1, provide a succinct insight for the full dynamics in a simplest form without convection and suggest related conjectures under the presence of nonlinear convection.

Studies of the traveling wave phenomenon of reaction diffusion equations have a long history. The Fisher equation has been introduced by Fisher [7] and by Kolmogorov, Petrovsky and Piskunov [12]. The purpose of Fisher was to perform modelling in population genetics, where the travelling wave solutions represented the spread of the advantageous gene through space. Later on, the Fisher-KPP equation was also used in ecology to model waves of an invading population (cf. Holmes et al [11]) and in wound healing, where the solutions represent healing waves of cells in the skin (cf. Sherratt and Murray [16]).

2. Inviscid Traveling Waves

Consider the Fisher equation,

$$u_t = du_{xx} + u(1-u), \quad u(x,0) = u^0(x), \quad t > 0,\ x \in \mathbf{R},$$

where $u(x,t)$ is a population density and $d > 0$ is a constant diffusivity. Let $z = x - ct$ be the variable for the traveling wave solution with a speed $c > 0$. Due to the symmetric structure of the equation we may consider positive wave speed $c > 0$ and the negative one can be treated symmetrically. It is well known that, for any $c \geq c^*$, there exists a traveling wave solution of wave speed c, where the minimum wave speed is $c^* = 2\sqrt{d}$ (see [6, 15]). Let u be the travelling wave solution for a wave speed $c \geq c^*$, i.e., $u(x,t) = u(x-ct) = u(z)$. Since $u_t = -cu'$ and $\frac{\partial^2}{\partial x^2} u = u''$, the traveling wave solution satisfies

$$du'' + cu' + u(1-u) = 0, \quad z \in \mathbf{R}. \tag{2.1}$$

We are looking for a monotone traveling wave that connects the stable steady state $u = 1$ and the unstable one $u = 0$:

$$\lim_{z \to -\infty} u(z) = 1, \quad u(0) = 0.5, \quad \text{and} \quad \lim_{z \to \infty} u(z) = 0. \tag{2.2}$$

The conditions at infinity allow positive wave speeds only. Remember that the traveling wave phenomenon of the Fisher equation has translation invariance and the condition $u(0) = 0.5$ picks the symmetric one with respect to $x = 0$ and $u = 0.5$. The traveling wave solution switches from the unstable steady state $u = 0$ to the stable one $u = 1$ as the wave front passes by. There is no such a monotone traveling wave solution with a speed slower than c^*.

Remark 1. We are using the notational convention that functions are distinguished by variables and contexts such as $u(x,t)$ or $u(z)$. However, we denote the traveling waves as $u = u_{c,d}$ or $v = v_c$ to explicitly denote the dependency of parameters if needed.

Next, we recall a standard result from literature (see [1, 2, 17]).

Proposition 1. *Let $c \geq c^* := 2\sqrt{d}$. There exists a unique solution $u_{c,d} \in C^2(\mathbf{R})$ of the problem (2.1)-(2.2). It satisfies that*

1. $0 < u_{c,d} < 1$ *and* $u'_{c,d} < 0$*;*
2. *There exist positive constants, $C, p > 0$, such that*

$$\begin{aligned} 0 < 1 - u_{c,d}(z) < Ce^{-p|z|}, &\quad \textit{for} \ \ z \leq 0, \\ |u_{c,d}(z)| \leq Ce^{-p|z|}, &\quad \textit{for} \ \ z \leq 0, \\ |u'_{c,d}(z)| + |u''_{c,d}(z)| \leq Ce^{-p|z|}, &\quad \textit{for} \ \ z \in \mathbf{R}. \end{aligned}$$

Next, we introduce inviscid traveling waves. Consider the Fisher equation with $d = 0$ and an initial value v^0,

$$v_t = v(1-v), \quad v(x,0) = v^0(x), \quad t > 0, \ x \in \mathbf{R}.$$

Notice that, even if $v = v(x,t)$ is a function of two variables, this equation is simply a collection ordinary differential equations for each $x \in \mathbf{R}$, which evolve independently. It is the initial distribution $v^0(x)$ that connects them and gives the correct timing of the wave propagation. Using the traveling wave variable $z = x - ct$ we assume that $v(x,t) = v(z)$. Then, since $v_t = -cv'$, the traveling wave v satisfies

$$cv' + v(1-v) = 0, \quad z \in \mathbf{R}. \tag{2.3}$$

We are looking for a front type traveling wave that connects the stable steady state $v = 1$ and the unstable one $v = 0$, i.e.,

$$\lim_{z\to-\infty} v(z) = 1, \quad v(0) = 0.5, \quad \text{and} \quad \lim_{z\to\infty} v(z) = 0. \tag{2.4}$$

The traveling wave solution is unique only up to translation and we are choosing one by fixing $v(0) = 0.5$. Equation (2.3) is the logistic equation with a growth rate $-\frac{1}{c}$. Hence, the traveling wave solution of speed $c > 0$ is the logistic function,

$$v_c(z) = \big(1 + \exp\big(z/c\big)\big)^{-1}, \tag{2.5}$$

which satisfies the conditions in (2.4). In other words, for any speed $c > 0$, there exists a corresponding traveling wave solution driven only by the reaction term. In terms of the variables x and t, the traveling wave is written as

$$v_c(x,t) = \big(1 + \exp\big(x/c - t\big)\big)^{-1}.$$

Remark 2. Similar phenomena appear when a bistable nonlinearity, $\psi(u) = u(1-u)(u-a)$, $0 < a < 1$, is considered. In this case, inviscid traveling wave solutions that connect $u = 0$ and $u = a$, or $u = a$ and $u = 1$ can be similarly constructed. However, there is no inviscid traveling wave solution that connects two stable steady states, $u = 0$ and $u = 1$.

3. Convergence Theorems

The diffusive traveling wave solution $u_{c,d}$ of the Fisher equation is defined for all $c \geq c^*$ and the inviscid one v_c is for all $c > 0$. The minimum wave speed $c^* = 2\sqrt{d}$ converges to zero as $d \to 0$. Therefore, for any given $c > 0$ we may compare v_c and $u_{c,d}$ if d is small enough. In this section we study their relations by taking the limit $d \to 0$, $c \to c^*$, or $c \to 0$. Finally, we complete the diagram of limits of traveling waves.

3.1. convergence to inviscid traveling waves

We denote by $u_{c,d}$ the unique solution of Problem (2.1)-(2.2). First, we recall that, for a positive traveling wave speed $c > 0$,

$$0 < u_{c,d}(z) < 1, \quad \text{and} \quad u'_{c,d}(z) < 0, \qquad z \in \mathbf{R}. \tag{3.1}$$

In particular,

$$\int |u'_{c,d}(z)| dz = -\int u'_{c,d}(z) dz = u_{c,d}(\infty) - u_{c,d}(-\infty) = 1$$

so that $u'_{c,d}$ is uniformly bounded in $L^1(\mathbf{R})$. Next, we multiply the ODE in (2.1) by $u'_{c,d}$ to deduce that

$$du''_{c,d}u'_{c,d} + c(u'_{c,d})^2 + u_{c,d}(1-u_{c,d})u'_{c,d} = 0.$$

Integrating it on $\mathbf{R}$ gives that

$$d\Big[\frac{(u'_{c,d})^2}{2}\Big]_{-\infty}^{+\infty} + c\int (u'_{c,d}(z))^2 dz + \Big[\frac{(u_{c,d})^2}{2} - \frac{(u_{c,d})^3}{3}\Big]_{-\infty}^{+\infty} = 0.$$

Therefore, since $\lim_{z\to\pm\infty} u'_{c,d}(z) = 0$ by Proposition 1, we have

$$\int (u'_{c,d}(z))^2 dz = 1/6c, \quad \text{or} \quad \|u'_{c,d}\|_{L^2(\mathbf{R})} = 1/\sqrt{6c}. \tag{3.2}$$

Theorem 1. *Let $c > 0$ be fixed and $u_{c,d}$ and v_c be the solutions of* (2.1)-(2.2) *and* (2.3)-(2.4)*, respectively. Then, $u_{c,d} \to v_c$ uniformly in $\mathbf{R}$ as $d \to 0$.*

Proof. To begin with, we recall the embedding $H^1(-R,R) \subset C^{1/2}([-R,R])$. Let $\alpha \in (0,1/2)$ be arbitrary. By a standard diagonal procedure, there exists a function $u_c \in C^{\alpha}_{loc}(\mathbf{R})$ and a subsequence u_{c,d_n} such that

$$u_{c,d_n} \to u_c \text{ in } C^{\alpha}_{loc}(\mathbf{R})$$

as $d_n \to 0$. We remark that $0 \le u_c \le 1$ and u_c is a decreasing function.

Next we show that u_c coincides with v_c. To do that we multiply the ODE in (2.1) by an arbitrary test function $\varphi \in C_0^\infty(\mathbf{R})$. This yields

$$d_n \int u''_{c,d_n}\varphi\, dz + c\int u'_{c,d_n}\varphi\, dz + \int u_{c,d_n}(1-u_{c,d_n})\varphi\, dz = 0,$$

that is, after integration by parts,

$$d_n \int u_{c,d_n}\varphi'' dz - c\int u_{c,d_n}\varphi' dz + \int u_{c,d_n}(1-u_{c,d_n})\varphi dz = 0. \tag{3.3}$$

Since $|u_{c,d_n}| \le 1$, we deduce that

$$d_n\Big|\int u_{c,d_n}\varphi''\Big| dz \le d_n \int |\varphi''| dz \to 0 \quad \text{as} \quad d_n \to 0.$$

Letting $d_n \to 0$ in (3.3), we deduce that

$$\int \{-cu_c\varphi' + u_c(1-u_c)\varphi\} dz = 0$$

for all $\varphi \in C_0^\infty(\mathbf{R})$. This implies that u_c is a weak solution of

$$cu'_c + u_c(1-u_c) = 0. \tag{3.4}$$

We remark that the fact that $u_{c,d}(0) = 0.5$ implies that $u_c(0) = 0.5$. Therefore, $u_c \not\equiv 1$ and $u_c \not\equiv 0$. These and the uniqueness of the solution of an initial value problem for an ordinary differential equation imply that $0 < u_c < 1$, which in view of (3.4) implies that $u'_c < 0$. It follows that $u_c(-\infty) = 1$ and $u_c(+\infty) = 0$ so that u_c coincides with the unique solution v_c of Problem (2.3)-(2.4). The uniform convergence on $\mathbf{R}$ follows from Diekmann [5, Lemma 2.4, p.463]. □

3.2. convergence to the minimum speed ($c \to c^*$)

Next we prove the following result.

Theorem 2. *Let $c > c^*$ and $u_{c,d}$ be the solution of* (2.1)-(2.2). *Then,*

$$u_{c,d} \to u_{c*,d} \quad \textit{uniformly in } \mathbf{R} \quad \textit{as} \quad c \to c^*.$$

Proof. It follows from (3.1) and (3.2) that there exists a function u_{c^*} and a subsequence $u_{c_n,d}$ such that

$$u_{c_n,d} \to u_{c^*} \quad \text{uniformly in compact sets of } \mathbf{R} \quad \text{as} \quad c \to c^*.$$

Next we show that u_{c^*} coincides with the unique solution $u_{c^*,d}$ of Problem (2.1)-(2.2), which also implies that the whole sequence $u_{c,d}$ converges as $c \to c^*$. First, we multiply (2.1) by an arbitrary test function $\varphi \in C_0^\infty(\mathbf{R})$. This yields

$$d \int u''_{c_n,d} \varphi \, dz + c_n \int u'_{c_n,d} \varphi \, dz + \int u_{c_n,d}(1 - u_{c_n,d}) \varphi \, dz = 0$$

that is, after integration by parts,

$$d \int u_{c_n,d} \varphi'' dz - c_n \int u_{c_n,d} \varphi' dz + \int u_{c_n,d}(1 - u_{c_n,d}) \varphi dz = 0. \tag{3.5}$$

Letting $c_n \to c^*$ in (3.5), we deduce that

$$d \int u_{c^*} \varphi'' dz - c^* \int u_{c^*} \varphi' dz + \int u_{c^*}(1 - u_{c^*,d}) \varphi dz = 0$$

for all $\varphi \in C_0^\infty(\mathbf{R})$. This implies that u_{c^*} is a smooth function which satisfies pointwise the equation

$$du''_{c^*} + c^* u'_{c^*} + u_{c^*}(1 - u_{c^*}) = 0 \text{ on } \mathbf{R}. \tag{3.6}$$

Moreover u_{c^*} is a nonincreasing function such that

$$0 \leq u_{c^*} \leq 1, \quad u_{c^*}(0) = 0.5, \quad \text{and} \quad \|u'_{c^*}\|_{L^2(\mathbf{R})} \leq 1/\sqrt{6c^*}.$$

Next we show that $u_{c^*}(z) \to 0$ as $z \to \infty$. Integrating the equation (3.6) between 0 and R yields

$$du'_{c^*}(R) + c^* u_{c^*}(R) - du'_{c^*}(0) - \frac{c^*}{2} + \int_0^R u_{c^*}(1 - u_{c^*}) dz = 0,$$

which in turn implies that $u_{c^*}(z) \to 0$ as $z \to \infty$. Indeed, since $u'_{c^*} \in L^2(\mathbf{R})$, there exists a sequence $\{R_n\}$ such that $u'_{c^*}(R_n) \to 0$ as $R_n \to \infty$. Suppose that $u_{c^*}(R_n) \to \lambda \in (0, 1)$ as $R_n \to \infty$. Then

$$\int_0^{R_n} u_{c^*}(1 - u_{c^*}) dz \to \infty \text{ as } R_n \to \infty,$$

which contradicts the fact that

$$-du'_{c^*}(R_n) - c^* u_{c^*}(R_n) + du'_{c^*}(0) + \frac{c^*}{2}$$

tends to the bounded term $-c^*\lambda + du'_{c^*}(0) + \frac{c^*}{2}$ as $R_n \to \infty$.

The proof that $u_{c^*}(z) \to 1$ as $z \to -\infty$ is similar. Thus u_{c^*} coincides with the unique solution $u_{c*,d}$ of Problem (2.1)-(2.2) with $c = c^*$, which completes the proof of Theorem 2. □

3.3. convergence along minimum speed traveling waves

Next, instead of letting d tend zero with c fixed, we study the singular limit of the travelling wave with the minimum velocity. The following result holds.

Theorem 3. *Let $u_{c,d}$ be the solution of (2.1)-(2.2) and $c^*(d) = 2\sqrt{d}$. Then, $u_{c^*(d),d} \to w$ pointwise in $\mathbf{R}$ as $d \to 0$, where w is a step function given by*

$$w(0) = 0.5 \quad \text{and} \quad w = \chi_{(-\infty,0)} \quad \text{if} \quad z \neq 0. \tag{3.7}$$

Proof. The function $u_{c^*(d),d}$ satisfies the equation,

$$du'' + 2\sqrt{d}u' + u(1-u) = 0, \quad z \in \mathbf{R},$$

together with the boundary conditions in (2.2). Set $z = \sqrt{d}\,y$ and

$$u_{c^*(d),d}(z) = \phi(y) = \phi(\frac{z}{\sqrt{d}}). \tag{3.8}$$

Then ϕ satisfies $\phi'' + 2\phi' + \phi(1-\phi) = 0$ on $\mathbf{R}$ and the same conditions in (2.2). It follows from (3.8) that $u_{c^*(d),d}(z) \to w(z)$ for all $z \in \mathbf{R}$ as $d \to 0$. □

3.4. limit along inviscid traveling waves

The convergence of the inviscid traveling waves v_c to the step function w in (3.7) as $c \to 0$ is easily obtained from the explicit formula (2.5), i.e., $v_c \to w$ pointwise in $\mathbf{R}$ as $c \to 0$, where w is

4. Result and Discussion

Combining the previous convergence results we obtain a complete diagram of convergence given in Figure 1. First, we find a parabola, $c = 2\sqrt{d}$, in (c,d)-plane which shows the minimum traveling wave speed for a given diffusivity $d > 0$. The area on its right side is the regime with diffusive traveling waves $u_{c,d}$. However, there is no traveling wave corresponding to the parameters in the other regime.

The authors remind of the existence of inviscid traveling waves for any wave speed $c > 0$ and demonstrated that the traveling wave speeds beyond the minimum speed is actually generated by reaction, but not diffusion. Theorem 1 is the one that shows uniform convergence of $u_{c,d}$ to v_c as $d \to \infty$ with c fixed. The uniform convergence of $u_{c,d}$ to the minimum speed traveling wave $u_{c^*(d),d}$ as $c \to c^*(d)$ with d fixed is shown in Theorem 3. Both of $u_{c^*(d),d}$ and v_c converges pointwise to the step function w in (3.7) as $d \to 0$ or $c \to 0$, which is shown in Theorem 2 and by the explicit formula (2.5).

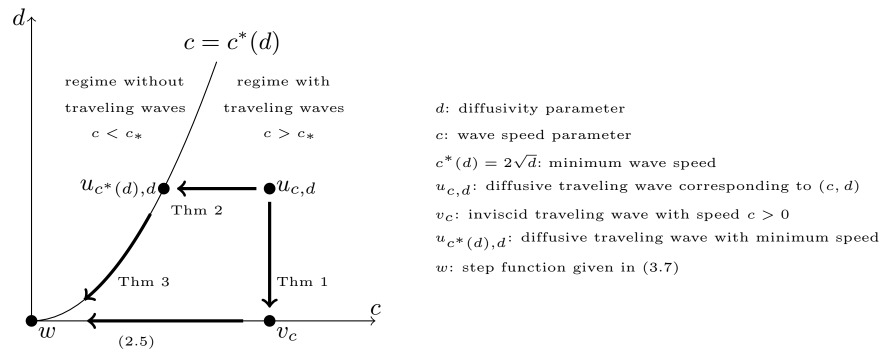

Figure 1: This figure shows the convergence relations among inviscid and diffusive traveling waves of the Fisher equation. The convergences to w are pointwise and to others are uniform.

5. Conclusion

In this paper we find the reason why there are many traveling wave speeds of a reaction diffusion equation when the traveling wave connects a stable steady state to an unstable one. We took the Fisher equation as a model case and shown that the existence of inviscid (or zero diffusivity) traveling wave for any wave speed $c > 0$, denoted by v_c, is the reason for the non-uniqueness. In fact, we have shown that a diffusive traveling wave of speed $c > 0$, denoted by $u_{c,d}$, converges to v_c uniformly as $d \to 0$, which confirms the conclusion. In Figure 1 a complete diagram of their relation is given. The relation between the diffusive and inviscid traveling waves provides a new view for the traveling wave phenomenon with a monostable linearity.

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