

GLOBAL EXISTENCE WITH PATTERN FORMATION IN CELL AGGREGATION MODEL

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ABSTRACT. The global existence of a Keller-Segel type chemotactic aggregation model is shown together with the instability of constant steady states. Instead of the usual assumption of sensing the chemical gradient by the organisms the model assumes that the organisms change their motility depending only on the chemical density. However, the resulting model is closely related to the logarithmic model,

$$u_t = \Delta(\gamma(v)u) = \nabla \cdot \left(\gamma(v)(\nabla u - \frac{k}{v}u\nabla v) \right), \quad v_t = \varepsilon\Delta v - v + u,$$

where $\gamma(v) := c_0v^{-k}$ is the motility function. The global existence is shown for all chemosensitivity constant $k > 0$. On the other hand constant steady states are not stable only if $k > 1$. Furthermore, the threshold diffusivity $\varepsilon_1 > 0$ is found that, if $\varepsilon < \varepsilon_1$, any constant steady state is unstable and an aggregation pattern appears. Numerical simulations are given for radial cases.

1. INTRODUCTION

The purpose of this paper is to introduce a cell aggregation model that allows pattern formation without a blow up. The model consists of two equations,

$$\begin{aligned} (1) \quad & u_t = \Delta(\gamma(v)u), & \mathbf{x} \in \Omega \subset \mathbf{R}^n, t > 0, \\ (2) \quad & v_t = \varepsilon\Delta v - av + bu, & \mathbf{x} \in \Omega \subset \mathbf{R}^n, t > 0, \end{aligned}$$

where u and v are densities of cell and chemical substance, respectively. The chemical substance is produced by the cell with a rate $b > 0$, degraded with a rate $a > 0$, and diffused with a rate $\varepsilon > 0$. The domain $\Omega \subset \mathbf{R}^n$ is bounded and has a smooth boundary. The motility function $\gamma = \gamma(v)$ decreases as the density of chemical substance increases, i.e.,

$$\gamma'(v) < 0.$$

We consider the problem with a non-trivial initial value and zero-flux boundary condition,

$$(3) \quad \begin{cases} u(\mathbf{x}, 0) = f(\mathbf{x}) \geq 0, & \mathbf{x} \in \Omega, \\ v(\mathbf{x}, 0) = g(\mathbf{x}) > 0, & \mathbf{x} \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & \mathbf{x} \in \partial\Omega, \end{cases}$$

where the initial values f and g are smooth. We are going to take the same boundary and initial conditions without mentioning it anymore. For simplicity, we will consider a power law case

$$(4) \quad \gamma(v) = \frac{c_0}{v^k}, \quad c_0 > 0, \quad k > 0.$$

In this paper we show that the solution exists globally in time and the constant steady state is not stable if $\varepsilon > 0$ is small enough and $k > 1$. The difference between the model considered and classical Keller-Segel models is in the first equation. Instead of the usual chemotactic term, the motility function $\gamma = \gamma(v)$ is introduced. The basic idea of the model is that the organisms do not sense the gradient of the chemical, but simply decreases the motility if the density of the chemical substance increases. However, such a behavior produces a natural advection term, i.e., the first equation is written as

$$(1') \quad u_t = \nabla \cdot (\gamma(v)\nabla u + \gamma'(v)u\nabla v) = \nabla \cdot \left(\gamma(v)(\nabla u - \frac{k}{v}u\nabla v) \right).$$

Even though the idea behind the equation is quite different, this equation is only a small variation of typical Keller-Segel models. The original Keller-Segel model [16] can be written as

$$(5) \quad \begin{aligned} u_t &= \nabla \cdot (\mu(v)\nabla u - \chi(v)u\nabla v), \\ v_t &= \varepsilon\Delta v - a(v)v + b(v)u, \end{aligned}$$

where μ and χ satisfy

$$(6) \quad \chi(v) = (\ell - 1)\mu'(v), \quad \mu'(v) < 0.$$

In other words, the original Keller-Segel model is identical to ours if we set $\ell = 0$ and $\gamma(v) = \mu(v)$. Note that $0 < \ell < 1$ is the ratio of effective body length, i.e., the largest distance between receptors divided by the body length. Suppose that the organisms do not sense the concentration difference between two receptors or have only one receptor. Then, the ratio becomes $\ell = 0$, which is the case of this paper.

The Keller-Segel models (5) have been intensively investigated for a cell aggregation phenomenon in various forms (see [10]). Consider two relevant cases. One of most intensively studied models is the so-called minimal model when $\mu(v) = \mu_0$, $\chi(v) = \chi_0$, $a(v) = a_0$ and $b(v) = b_0$ are all positive constants, i.e.,

$$(7) \quad \begin{aligned} u_t &= \nabla \cdot (\mu_0\nabla u - \chi_0 u\nabla v), \\ v_t &= \varepsilon\Delta v - a_0v + b_0u. \end{aligned}$$

Solutions of the minimal model are global and bounded in one space dimension (see [26]). However, in multi-space dimensions, the solution blows up in a finite-time depending on the initial population size (see [9, 12–14, 19, 21, 27]). The sensitivity on the initial value is related to a following scaling property.

Definition 1.1. Let $(u, v) = (\theta_1, \theta_2)$ be a steady state solution of (5). If $(u, v) = (c\theta_1, c\theta_2)$ is also a steady state solution for any constant $c > 0$, we say that the steady states of the problem has *mass scaling invariance*.

If $\mu(v)$ and $\chi(v)$ satisfy the relation in (6), then the steady states of the original Keller-Segel equations (5) has mass scaling invariance for a wide class of diffusivity coefficients $\mu(v)$. However, by assuming constant coefficients, the connection between the diffusivity and chemosensitivity in (6) has been forgotten for the minimal model (7) and the scaling invariance property does not hold anymore. Such a simplification makes the equation look simpler. However, the loss of connection makes the analysis and behavior complicated.

Another well studied case is the so-called logarithmic model,

$$(8) \quad \begin{aligned} u_t &= \nabla \cdot (\mu_0\nabla u - \frac{\chi_0}{v}u\nabla v), \\ v_t &= \varepsilon\Delta v - a_0v + b_0u, \end{aligned}$$

which is the case that the chemosensitivity is given by $\chi(v) = \frac{\chi_0}{v}$ and all other coefficients are constant. The relation (6) is not satisfied in this case, either¹. However, the steady states of the model has the mass scaling invariance. In one space dimension solutions to the logarithmic model are bounded and exists globally in time. If the space dimension is $n > 1$, the global existence has been shown for the case with the relative chemosensitivity satisfies $\frac{\chi_0}{\mu_0} < \sqrt{\frac{2}{n}} \leq 1$. The size of initial population is not involved for this case (see [20, 30]).

The cell aggregation model in this paper, (1)-(4), can be considered as close to the logarithmic model. In this case the ratio, $\frac{\chi_0}{\mu_0}$, corresponds to the nonlinearity of the motility function γ , i.e., $k > 0$ (compare (1') and (8)), and hence we also call $k > 0$ the relative chemosensitivity. The difference is that the diffusivity in (1') is not constant and the diffusivity and the chemosensitivity satisfy the relation (6) with $\ell = 0$. From this view point we may say that the model equation of this paper is closer to the original derivation by Keller and Segel. We extend the global existence result of the logarithmic model to $k > 0$ for all dimension. We also show the instability of constant steady states when $k > 1$ and $\varepsilon > 0$ is small. Note that non-constant steady states exist when $k > 1$.

The outline of this paper is as follows. Section 2 is for the global existence of solutions. We introduce a few well-known estimates in Section 2.1 as preliminaries for later use. In Section 2.2 and 2.3, we show the global existence and boundedness of u and v . Section 3 is related to pattern formation mechanism. The structure of steady states are discussed in Section 3.1. In Section 3.2, instability conditions of a constant steady state are given. These conditions imply the possibility of convergence to a nonconstant steady state solution and hence pattern formation. Numerical simulations are given in Section 4. We close the paper in Section 5 with a few discussions and conclusions.

2. GLOBAL EXISTENCE

In this section we show the global existence of the solution of

$$(9) \quad \begin{aligned} u_t &= \Delta(\gamma(v)u), \\ v_t &= \varepsilon\Delta v - v + u, \end{aligned} \quad \mathbf{x} \in \Omega \subset \mathbf{R}^n, \quad t > 0,$$

where the degradation and production rates are set to be one, $a_0 = b_0 = 1$, for simplicity. In fact, the proof for a uniform bound of a solution does not depend on the choice of coefficients ε, a_0 and b_0 . In most of the models in the literature they are set to be one, which we follow. However, we keep $\varepsilon > 0$ since its size plays the key role related to the pattern formation in Section 3. The boundary and initial conditions are given by(3) always as mentioned above.

2.1. Preliminaries. The following four lemmas are known properties of a solution of a general system that includes our model equations (9). These properties are used in the proof for the global existence of a solution in later sections.

Lemma 2.1 (Mass conservation). *Suppose that there exists a smooth solution (u, v) for $0 < t < T_{max}$. Then,*

$$(10) \quad \|u(\cdot, t)\|_{L^1} = \|f(\mathbf{x})\|_{L^1} \quad \text{for all } t \in (0, T_{max}).$$

¹Even though (6) is not satisfied for the logarithmic model, we may say that it is almost satisfied.

This conservation of the total population is from the zero flux Neumann boundary condition and absent of a population dynamics.

Let $0 < \theta < 1$ and A be a sectional operator with $\operatorname{Re}(\sigma(A)) > 0$. Then, the fractional operator A^θ for $\theta > 0$ is defined as the inverse of

$$A^{-\theta} := \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-At} dt.$$

Let $1 < q < \infty$ be fixed. Then, the operator $-\Delta + 1$ is sectorial in $L^q(\Omega)$ and the fractional operator $(-\Delta + 1)^\theta$ is defined on a domain $D((-\Delta + 1)^\theta) \subset L^q(\Omega)$ such that

$$\|w\|_{D((-\Delta+1)^\theta)} := \|(-\Delta + 1)^\theta w\|_{L^q(\Omega)} < \infty,$$

where $0 < \theta < 1$.

Lemma 2.2 (Horstmann and Winkler [15]). *If $m \in \{0, 1\}$, $1 \leq p \leq \infty$, $1 < q < \infty$, and $m - \frac{n}{p} < 2\theta - \frac{n}{q}$, then there exists a constant $C > 0$ that satisfies*

$$(11) \quad \|w\|_{W^{m,p}(\Omega)} \leq C \|(-\Delta + 1)^\theta w\|_{L^q(\Omega)}$$

for all $w \in D((-\Delta + 1)^\theta)$. There also exist $C > 0$ and $\zeta > 0$ such that, for $q \geq p$,

$$(12) \quad \|(-\Delta + 1)^\theta e^{t(\Delta-1)} w\|_{L^q(\Omega)} \leq C t^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\zeta t} \|w\|_{L^p(\Omega)}$$

for all $w \in L^p(\Omega)$.

Next we show that the density of the signalling chemical v is bounded away from zero. This estimate gives an uniform bound of the motility function γ given by (4).

Lemma 2.3 (Hillen *et al.* [11]). *Suppose that there exists a smooth solution (u, v) for $0 < t < T_{max}$. Then, there exists a lower bound $\underline{v} > 0$ such that*

$$\inf_{\mathbf{x} \in \Omega} v(\mathbf{x}, t) \geq \underline{v} > 0 \text{ for all } t \geq 0.$$

Remark 2.4. The second equation (2) is identical to the corresponding equation of the system considered by Hillen *et al.* [11]. Even though the first equations for the population are different to each other, the proof is identical since the only thing needed from the first equation is the conservation of population.

The last lemma is from a basic estimation of the Gagliardo-Nirenberg and the Young's inequality (see [18]).

Lemma 2.5. *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|\Phi\|_{L^2(\Omega)}^2 \leq \varepsilon \|\nabla \Phi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\Phi\|_{L^1(\Omega)}^2$$

for any $\Phi \in W^{1,2}(\Omega)$.

2.2. Local existence. The main difficulties in obtaining the global existence is in the degeneracy of the parabolicity of the problem as $v \rightarrow \infty$ and unboundedness of γ as $v \rightarrow 0$. Fortunately, the boundedness of γ follows from Lemma 2.3 and we will focus on the degeneracy as $v \rightarrow \infty$. Obtaining a uniform parabolicity of the problem is equivalent to obtaining a uniform bound of the solution v . In this section we start with a modified problem.

To explicitly avoid the degeneracy of the problem, we take an approximation of the motility given by

$$\gamma_d(v) = \begin{cases} \gamma(v), & v \leq V, \\ d, & v \geq V, \end{cases}$$

where $V > 0$ is a constant such that $\gamma(V) = d$. Now we consider a modified system:

$$(13) \quad \begin{cases} u_t = \nabla \cdot (\gamma_d(v)\nabla u + \gamma'(v)u\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

where the boundary and initial conditions are not changed. Notice that only the coefficient γ in front of ∇u is replaced with γ_d and the other coefficient γ' remains as it was. The solutions $u = u^d$ and $v = v^d$ depend on the new parameter $d > 0$.

The local existence in time of a classical solution is proved by Amann's regularity theory for quasilinear parabolic problems (see [2–4]).

Theorem 2.6 (Local Existence). *Let the initial values $f \in C^0(\overline{\Omega})$ and $g \in W^{1,p}(\Omega)$ ($p > n$) be nonnegative. Then, there exists a solution of (13) in the classical sense, i.e.,*

$$u^d, v^d \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

where T_{\max} denotes the maximal existence time. This solution is unique and nonnegative. Moreover, if $T_{\max} < \infty$, then

$$(14) \quad \|u^d(\cdot, t)\|_{L^\infty(\Omega)} + \|v^d(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \quad \text{as } t \rightarrow T_{\max}.$$

Proof. To obtain the uniqueness and the existence we let $U = (u^d, v^d)$ and rewrite (13) with boundary and initial conditions as

$$(15) \quad \begin{cases} U_t = \nabla \cdot (A(U)\nabla U) + F(U), \\ \partial_\nu U = 0, \quad \text{on } \partial\Omega, \\ U(0, \cdot) = (f, g), \quad \text{in } \Omega, \end{cases}$$

where

$$A(U) = \begin{pmatrix} \gamma_d(v) & \gamma'(v)u \\ 0 & 1 \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ -v + u \end{pmatrix}.$$

The positive lower bound $\gamma_d(v)$ in lemma 2.3 implies that real parts of eigenvalues of $A(U)$ are positive and hence (15) is uniformly parabolic. Then, for this triangular diffusion matrix case, the local existence, uniqueness and blow-up criteria (14) follows from [4, Theorem 7.3] (see [6, 7, 28] for similar approaches).

Nonnegativity of the solution comes from the maximum principle for parabolic problems. The first equation of (13) is uniformly parabolic and has no reaction term. Since the initial value $f \geq 0$ is nonnegative, we have $u^d \geq 0$ for all time $t > 0$ by the maximum principle. Since $u^d \geq 0$, the second equation gives

$$v_t^d - \Delta v^d + v^d \geq 0.$$

The nonnegative initial value $g > 0$ and the maximum principle ensures that v^d is also nonnegative. \square

2.3. Boundedness. In this section we prove two theorems related to the boundedness of the solution. It is shown in Theorem 2.7 that the solution $u = u^d$ of (13) is bounded in $L^p(\Omega)$ uniformly on $d > 0$ and time t as long as the solution exists. We show that solutions $v = v^d$ is uniformly bounded, $\|v^d\|_{L^\infty(\Omega \times (0, T_{\max}))} < V$, for all $d > 0$. This implies that the solutions of the original system (9) and of the modified one (13) are identical for all small $d > 0$ such that $d < \gamma(V)$. Therefore, the global existence of (9) is obtained as soon as the uniform bound of u^d is obtained, which is done in Theorem 2.9.

First we obtain a uniform L^p bound of the solution $u^d(\cdot, t)$ of the modified problem. The main idea for the estimation is to obtain an ODE inequality that provides an upper bound, where a similar approach is used in [29].

Theorem 2.7. *Let $d > 0$ be fixed and $c_0 = c_0(d) > 0$ be small enough. Let (u^d, v^d) be a solution pair of (13) in a maximal time domain $t \in (0, T_{\max})$. Then, for any $p > 1$, there exists an L^p bound $C(p; d) > 0$ of u^d , i.e.,*

$$(16) \quad \|u^d(\cdot, t)\|_{L^p(\Omega)} \leq C(p; d) \quad \text{for all } 0 < t < T_{\max} \text{ and } 1 < p < \infty.$$

Furthermore, there exists a constant $V > 0$ which is independent of $d > 0$ and satisfies

$$(17) \quad \|v^d\|_{L^\infty(\Omega \times (0, T_{\max}))} < V.$$

Proof. We will drop the parameter d of the solutions u^d and v^d in the following proof for notational simplicity, where u and v always denote solutions of the modified problem (13). Let

$$\phi(v) := \exp((1+v)^{-s}), \quad \text{and} \quad y(t) := \int_{\Omega} u^p \phi(v) d\mathbf{x},$$

where $s > 0$ is a positive constant which will be chosen later. Consider

$$\begin{aligned} \frac{1}{p} y'(t) &= \int_{\Omega} u^{p-1} \phi(v) u_t d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^p \phi'(v) v_t d\mathbf{x} \\ &= \int_{\Omega} u^{p-1} \phi(v) \nabla \cdot (\gamma_d \nabla u + \gamma'(v) u \nabla v) d\mathbf{x} \\ &\quad + \frac{1}{p} \int_{\Omega} u^p \phi'(v) (\Delta v - v + u) d\mathbf{x} \\ &= -(p-1) \int_{\Omega} u^{p-2} \phi(v) \nabla u \cdot (\gamma_d(v) \nabla u + \gamma'(v) u \nabla v) d\mathbf{x} \\ &\quad - \int_{\Omega} u^{p-1} \phi'(v) \nabla v \cdot (\gamma_d(v) \nabla u + \gamma'(v) u \nabla v) d\mathbf{x} \\ &\quad - \int_{\Omega} u^{p-1} \phi'(v) \nabla u \cdot \nabla v d\mathbf{x} - \frac{1}{p} \int_{\Omega} u^p \phi''(v) |\nabla v|^2 d\mathbf{x} \\ &\quad - \frac{1}{p} \int_{\Omega} u^p v \phi'(v) d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^{p+1} \phi'(v) d\mathbf{x}. \end{aligned}$$

Since $\phi'(v) \leq 0$ and $\gamma'(v) \leq 0$,

$$- \int_{\Omega} u^{p-1} \phi'(v) \nabla v \cdot (\gamma'(v) u \nabla v) d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^{p+1} \phi'(v) d\mathbf{x} \leq 0,$$

and the above equality becomes

$$\begin{aligned} \frac{1}{p} y'(t) &+ (p-1) \int_{\Omega} u^{p-2} \phi(v) \gamma_d(v) |\nabla u|^2 d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^p \phi''(v) |\nabla v|^2 d\mathbf{x} \\ &\leq - \int_{\Omega} u^{p-1} \phi'(v) \nabla u \cdot \nabla v d\mathbf{x} - \int_{\Omega} \gamma_d(v) u^{p-1} \phi'(v) \nabla u \cdot \nabla v d\mathbf{x} \\ &\quad - (p-1) \int_{\Omega} u^{p-1} \phi(v) \gamma'(v) \nabla u \cdot \nabla v d\mathbf{x} - \frac{1}{p} \int_{\Omega} u^p v \phi'(v) d\mathbf{x}. \end{aligned}$$

The first two terms on the right side of the inequality are estimated by the Young's inequality,

$$\begin{aligned} & - \int_{\Omega} u^{p-1} \phi'(v) \nabla u \cdot \nabla v d\mathbf{x} \\ & \leq \frac{d(p-1)}{8} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{2}{d(p-1)} \int_{\Omega} u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} & - \int_{\Omega} \gamma_d(v) u^{p-1} \phi'(v) \nabla u \cdot \nabla v d\mathbf{x} \\ & \leq \frac{d(p-1)}{8} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{2}{d(p-1)} \int_{\Omega} (\gamma_d(v))^2 u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x}. \end{aligned}$$

The next term is estimated by

$$\begin{aligned} & -(p-1) \int_{\Omega} u^{p-1} \phi(v) \gamma'(v) \nabla u \cdot \nabla v d\mathbf{x} \\ & \leq \frac{d(p-1)}{4} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{(p-1)}{d} \int_{\Omega} u^p |\gamma'(v)|^2 \phi(v) |\nabla v|^2 d\mathbf{x}. \end{aligned}$$

For $v \geq 0$,

$$-v\phi'(v) = vs(1+v)^{-s-1}\phi(v) = \frac{v}{(1+v)^{s+1}}sv\phi(v) \leq s\phi(v),$$

and the last term is bounded by

$$-\frac{1}{p} \int_{\Omega} u^p v \phi'(v) d\mathbf{x} \leq \frac{s}{p} \int_{\Omega} u^p \phi(v) d\mathbf{x} = \frac{s}{p} y(t).$$

Since $\gamma_d \geq d$, these estimates yield

$$\begin{aligned} & \frac{1}{p} y'(t) + d(p-1) \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^p \phi''(v) |\nabla v|^2 d\mathbf{x} \\ & \leq \frac{d(p-1)}{2} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{2}{d(p-1)} \int_{\Omega} (\gamma_d(v))^2 u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x} \\ & + \frac{2}{d(p-1)} \int_{\Omega} u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x} + \frac{(p-1)}{d} \int_{\Omega} u^p |\gamma'(v)|^2 \phi(v) |\nabla v|^2 d\mathbf{x} \\ & + \frac{s}{p} y(t). \end{aligned}$$

After a rearrangement, it is written as

$$\begin{aligned} & \frac{1}{p} y'(t) + \frac{d(p-1)}{2} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} + \frac{1}{p} \int_{\Omega} u^p \phi''(v) |\nabla v|^2 d\mathbf{x} \\ & \leq \frac{2}{d(p-1)} \int_{\Omega} u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x} + \frac{2}{d(p-1)} \int_{\Omega} (\gamma_d(v))^2 u^p \frac{(\phi'(v))^2}{\phi(v)} |\nabla v|^2 d\mathbf{x} \\ & + \frac{(p-1)}{d} \int_{\Omega} u^p |\gamma'(v)|^2 \phi(v) |\nabla v|^2 d\mathbf{x} + \frac{s}{p} y(t). \end{aligned}$$

Now we show that the first three terms on the right side are dominated by $\frac{1}{p} \int_{\Omega} u^p \phi''(v) |\nabla v|^2 d\mathbf{x}$.

Let

$$\begin{aligned} I_1 &:= \frac{2}{d(p-1)} \frac{(\phi'(v))^2}{\phi(v)} = \frac{2}{d(p-1)} s^2 (1+v)^{-2s-2} \phi(v), \\ I_2 &:= \frac{2}{d(p-1)} (\gamma_d(v))^2 \frac{(\phi'(v))^2}{\phi(v)} \leq \frac{2c_0^2 s^2}{d(p-1)} \underline{v}^{-2k} (1+v)^{-2s-2} \phi(v), \\ I_3 &:= \frac{(p-1)}{d} |\gamma'(v)|^2 \phi(v) = \frac{(p-1)}{d} c_0^2 k^2 v^{-2k-2} \phi(v), \\ I_4 &:= \frac{1}{p} \phi''(v) = \frac{1}{p} s(s+1) (1+v)^{-s-2} \phi(v) + \frac{1}{p} s^2 (1+v)^{-2s-2} \phi(v). \end{aligned}$$

Set the constant $s > 0$ to satisfy $s < \frac{d(p-1)}{6p}$. Then,

$$(18) \quad \frac{I_1}{\frac{1}{3}I_4} \leq \frac{\frac{2}{d(p-1)} s^2 (1+v)^{-2s-2} \phi(v)}{\frac{1}{3p} s(s+1) (1+v)^{-s-2} \phi(v)} = \frac{6ps(1+v)^{-s}}{d(p-1)(s+1)} \leq \frac{6ps}{d(p-1)} < 1.$$

We choose $s < 2k$ by taking a smaller s if needed and c_0 small enough so that

$$c_0 \leq \min \left\{ \sqrt{\frac{d(p-1)}{6ps}} \underline{v}^k, \sqrt{\frac{ds}{3p(p-1)}} \frac{1}{k} \left(\frac{\underline{v}}{1+\underline{v}} \right)^{k+1} \right\},$$

where \underline{v} is the minimum value in Lemma 2.3. Now we compute

$$(19) \quad \begin{aligned} \frac{I_2}{\frac{1}{3}I_4} &\leq \frac{\frac{2c_0^2 s^2}{d(p-1)} \underline{v}^{-2k} (1+v)^{-2s-2} \phi(v)}{\frac{1}{3p} s(s+1) (1+v)^{-s-2} \phi(v)} \\ &\leq \frac{6psc_0^2}{d(p-1)(s+1)} \underline{v}^{-2k} (1+v)^{-s} < \frac{6psc_0^2}{d(p-1)} \underline{v}^{-2k} \leq 1, \end{aligned}$$

and

$$(20) \quad \begin{aligned} \frac{I_3}{\frac{1}{3}I_4} &\leq \frac{\frac{(p-1)}{d} c_0^2 k^2 v^{-2k-2} \phi(v)}{\frac{1}{3p} s(s+1) (1+v)^{-s-2} \phi(v)} \\ &\leq \frac{3p(p-1)c_0^2 k^2 (1+v)^{s+2}}{ds(s+1) v^{2k+2}} < \frac{3p(p-1)c_0^2 k^2}{ds} \left(\frac{1+\underline{v}}{\underline{v}} \right)^{2k+2} \leq 1. \end{aligned}$$

From (18), (19) and (20), we obtain

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \phi(v) d\mathbf{x} + \frac{d(p-1)}{2} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} \leq \frac{s}{p} \int_{\Omega} u^p \phi(v) d\mathbf{x}$$

for all $t \in (0, T_{\max})$. From the Gagliardo-Nirenberg inequality, there exists $C_{GN} > 0$ such that

$$(21) \quad \|u^{\frac{p}{2}}\|_{L^2(\Omega)} \leq C_{GN} \left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\alpha} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-\alpha} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} \right),$$

where $\alpha = \frac{\frac{np}{2} - \frac{n}{2}}{\frac{np}{2} + 1 - \frac{n}{2}} \in (0, 1)$. Since $\phi(v) \leq e$, we obtain

$$(22) \quad \begin{aligned} \int_{\Omega} u^p \phi(v) d\mathbf{x} &\leq e \int_{\Omega} u^p d\mathbf{x} = e \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\ &\leq e C_{GN}^2 \left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\alpha} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-\alpha} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} \right)^2 \\ &\leq 2e C_{GN}^2 \left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\alpha} \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\alpha)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \right). \end{aligned}$$

The mass conservation property (10) gives

$$(23) \quad \|u^{\frac{p}{2}}(\cdot, t)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_{\Omega} u(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) d\mathbf{x} \quad \text{for all } t \in (0, T_{\max}).$$

Then (23) and Young's inequality yield,

$$\int_{\Omega} u^p \phi(v) d\mathbf{x} \leq C_2 \left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2\alpha} + 1 \right) \leq C_2 \|\nabla u^{\frac{p}{2}}\|_{L^2}^2 + C_3,$$

where $C_2 = 2eC_{GN}^2 \left((\int_{\Omega} f(\mathbf{x}) d\mathbf{x})^{p(1-\alpha)} + (\int_{\Omega} f(\mathbf{x}) d\mathbf{x})^p \right)$ and $C_3 = C_2(1 + \alpha(1 - \alpha))$. Then we obtain,

$$(24) \quad \begin{aligned} \frac{d(p-1)}{2} \int_{\Omega} u^{p-2} \phi(v) |\nabla u|^2 d\mathbf{x} &\geq \frac{d(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 d\mathbf{x} \\ &= \frac{2d(p-1)}{p^2} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} \\ &\geq \frac{2d(p-1)}{p^2 C_2} \left(\int_{\Omega} u^p \phi(v) d\mathbf{x} \right) - \frac{2d(p-1)C_3}{p^2 C_2}. \end{aligned}$$

Consequently, we compute

$$\frac{1}{p} y'(t) \leq - \left(\frac{2d(p-1)}{p^2 C_2} - \frac{s}{p} \right) y(t) + \frac{2d(p-1)C_3}{p^2 C_2}$$

for all $t \in (0, T_{\max})$. As long as we have $s < \frac{2d(p-1)}{p^2 C_2}$, we get

$$(25) \quad y(t) \leq e^{-Apt} y(0) + \frac{B}{A} \leq e \int_{\Omega} f^p d\mathbf{x} + \frac{B}{A},$$

where $A = \frac{2d(p-1)}{p^2 C_2} - \frac{s}{p}$ and $B = \frac{2d(p-1)C_3}{p^2 C_2}$. Therefore, y is bounded for all $t \in (0, T_{\max})$. Since $\phi(v) \geq 1$ for all $v \geq 0$, L^p -norm of u is also bounded. Therefore, there exists $C(p; d)$ such that, for all $0 < t < T_{\max}$,

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p; d),$$

which completes (16).

The above L^p estimate for u depends on d . Now we use it to obtain a uniform estimate for v (17) which is independent of d . The representation formula for v of (13) is

$$v(\cdot, t) = e^{t(\Delta-1)} g + \int_0^t e^{(t-\eta)(\Delta-1)} u(\cdot, \eta) d\eta \quad \text{for all } t \in (0, T_{\max}).$$

Let $\theta \in (\frac{1}{2}, 1)$ and choose $p > n$. Moreover, we pick $q > p$ large enough to satisfy $1 < 2\theta - \frac{n}{q}$. Then (11), (12) of Lemma 2.2 and the previous estimate for u imply

that

$$\begin{aligned}
(26) \quad \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} &\leq C \|(-\Delta + 1)^\theta v(\cdot, t)\|_{L^q(\Omega)} \\
&\leq C t^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\zeta t} \|g\|_{L^p(\Omega)} \\
&\quad + C \int_0^t (t-\eta)^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\zeta(t-\eta)} \|u(\cdot, \eta)\|_{L^p(\Omega)} d\eta \\
&\leq C t^{-\theta_0} + C \int_0^t (t-\eta)^{-\theta_0} e^{-\zeta(t-\eta)} d\eta \\
&\leq C t^{-\theta_0} + C \int_0^\infty \sigma^{-\theta_0} e^{-\zeta\sigma} d\sigma \leq C(t^{-\theta_0} + 1),
\end{aligned}$$

where $0 < t < T_{\max}$, $\theta_0 := \theta + \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$, and C is a generic constant. For a sufficiently small τ at which the L^∞ -norm of v is close to $g(\mathbf{x})$,

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C(\tau^{-\theta_0} + 1) \quad \text{for all } t \in (\tau, T_{\max}),$$

holds. We suppose that $d < 1$ and $s = s(d) = d \min\{\frac{d(p-1)}{6p}, 2k, \frac{2d(p-1)}{pC^2}\}$, i.e., s is a nondecreasing function of d . From (25),

$$\frac{B}{A} = \frac{\frac{2d(p-1)C_3}{p^2C_2}}{\frac{2d(p-1)}{p^2C_2} - \frac{s(d)}{p}}$$

is also a nondecreasing function for $d < 1$. Therefore,

$$\frac{B}{A} < \frac{\frac{2(p-1)C_3}{p^2C_2}}{\frac{2(p-1)}{p^2C_2} - \frac{s(1)}{p}}$$

which is independent of d . Therefore, the uniform bound of v is obtained indeendely of $d > 0$. \square

Remark 2.8. Note that the c_0 in Theorem 2.7 depends on d . However, the uniform estimate of v^d in (17) allows us to fix the constant c_0 independently of d as the follow. Let $d_0 = \gamma(V)$. Then, the solution of modified problem (13) and the original problem (9) are identical if $d < d_0$ since the approximation of the motility function γ_d does not play any role since the solution v is smaller than V . Therefore, our choice of c_0 is $c_0(d_0)$. However, this smallness of c_0 seems a technical issue and numerical simulations show aggregation phenomena regardless of the size of c_0 .

Theorem 2.9. *Let (u, v) be the solution of (9) with boundary and initial conditions (3). The motility function γ is given by (4) with a small $c_0 > 0$ (see Remark 2.8). Suppose that $t \in (0, T_{\max})$ is the maximal time domain for the solution. Then, there exists a constant $C > 0$ such that*

$$(27) \quad \|u\|_{L^\infty(\Omega \times (0, T_{\max}))} + \|v\|_{L^\infty(\Omega \times (0, T_{\max}))} \leq C.$$

Furthermore, the solution is global, i.e., $T_{\max} = \infty$.

Proof. The uniform estimate $\|v^d\|_{L^\infty(\Omega \times (0, T_{\max}))} \leq V$ in Theorem 2.7, which is independent of d , implies that the solution (u^d, v^d) of the modified problem (13) is identical to the original problem (9) for all $d < \gamma(V)$. Therefore, the uniform estimate of the solution v of (9) has been already obtained in Theorem 2.7. Therefore, we only need to show $\|u(\cdot, t)\|_{L^\infty} \leq C$.

We apply the iterative technique to show a uniform boundedness of u which can be found in [18]. For $p \geq 2$, we use Young's inequality and compute,

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} u^p d\mathbf{x} &= p \int_{\Omega} u^{p-1} u_t d\mathbf{x} = p \int_{\Omega} u^{p-1} \Delta(\gamma(v)u) \\
&= -p(p-1) \int_{\Omega} u^{p-2} \gamma(v) |\nabla u|^2 d\mathbf{x} - p(p-1) \int_{\Omega} u^{p-1} \gamma'(v) \nabla u \cdot \nabla v d\mathbf{x} \\
&\leq \frac{-4k_1(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + k_2 p(p-1) \int_{\Omega} u^{p-1} |\nabla u| d\mathbf{x} \\
&\leq \frac{-4k_1(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + k_2(p-1) \left(\frac{2k_1}{k_2 p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + \frac{k_2 p}{2k_1} \int_{\Omega} u^p d\mathbf{x} \right) \\
&\leq \frac{-2k_1(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + \frac{k_2^2 p(p-1)}{2k_1} \int_{\Omega} u^p d\mathbf{x},
\end{aligned}$$

where $k_1 = \gamma(\bar{v})$ and $k_2 = \max |\gamma'(v)| |\nabla v|$. The \bar{v} is a uniform upper bound of v and is obtained in theorem 2.9. Therefore,

$$\frac{d}{dt} \int_{\Omega} u^p d\mathbf{x} + p(p-1) \int_{\Omega} u^p d\mathbf{x} \leq -\frac{2k_1(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + k_3 p(p-1) \int_{\Omega} u^p d\mathbf{x},$$

where $k_3 = \frac{k_2^2}{2k_1} + 1$.

From the lemma 2.5, we use $\Phi = u^{\frac{p}{2}}$ and $\varepsilon = \frac{2}{k_3 p^2}$ to obtain

$$k_3 p(p-1) \int_{\Omega} u^p d\mathbf{x} \leq \frac{2k_1(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 d\mathbf{x} + k_4 p(p-1) \left(\int_{\Omega} u^{\frac{p}{2}} d\mathbf{x} \right)^2,$$

where $k_4 = C_\varepsilon k_3 > 0$. Therefore,

$$(28) \quad \frac{d}{dt} \int_{\Omega} u^p d\mathbf{x} + p(p-1) \int_{\Omega} u^p d\mathbf{x} \leq k_4 p(p-1) \left(\int_{\Omega} u^{\frac{p}{2}} d\mathbf{x} \right)^2.$$

Integrating (28) over $[0, t]$ for $t \in (0, T_{\max})$, we get

$$(29) \quad \int_{\Omega} u^p d\mathbf{x} \leq \int_{\Omega} f^p d\mathbf{x} + k_4 \sup_{0 \leq t \leq T_{\max}} \left(\int_{\Omega} u^{\frac{p}{2}} d\mathbf{x} \right)^2.$$

Define

$$F(p) = \max \{ \|f\|_{L^\infty(\Omega)}, \sup_{0 \leq t \leq T_{\max}} \left(\int_{\Omega} u^p d\mathbf{x} \right)^{\frac{1}{p}} \}.$$

Using (29), we obtain

$$F(p) \leq k_5^{\frac{1}{p}} F\left(\frac{p}{2}\right)$$

holds for all $p \geq 2$, where $k_5 = |\Omega| + k_4$. Taking $p = 2^i$, $i = 1, 2, \dots$, we have

$$\begin{aligned}
F(2^i) &\leq k_5^{2^{-i}} F(2^{i-1}) \\
&\leq k_5^{2^{-i} + 2^{-i+1}} F(2^{i-2}) \leq \dots \leq k_5 F(1).
\end{aligned}$$

Therefore, letting $i \rightarrow \infty$, we have

$$\|u(\cdot, t)\|_{L^\infty} \leq k_5 F(1) \leq k_5 \max \{ \|f\|_{L^\infty(\Omega)}, \|f\|_{L^1(\Omega)} \} \text{ for all } t \in (0, T_{\max}).$$

Finally, suppose that $T_\infty < 0$. Then, Theorem 2.6 implies that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \text{ as } t \rightarrow T_{\max},$$

which contradicts the previously obtained uniform boundedness. Hence, the solution is global in time. \square

3. PATTERN FORMATION

The global existence obtained in the previous section is not enough to explain the chemotactic cell aggregation phenomenon. Two more things are required. First there should exist non-constant steady states, which is well-understood and discussed in the next section. Second, the instability of a constant steady state will give an indication of cell aggregation, or of pattern formation.

3.1. Non-constant steady states. A steady state solution of (9) satisfies

$$(30) \quad \begin{cases} 0 = \Delta(\gamma(v)u), \\ 0 = \varepsilon\Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where $\gamma(v) = c_0 v^{-k}$ for $c_0 > 0$ and $k > 0$. Any constant state $(u, v) = (u_0, v_0)$ is a steady state solution if $u_0 = v_0$. The conservation of total population gives that

$$u_0 = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{x}) d\mathbf{x}.$$

In general a non-constant steady state solution can be found as follows. First, integrate the first equation of (30) with the homogenous boundary condition and obtain

$$u = \lambda v^k,$$

where λ is a positive constant. Substitute u into the second equation of (30) and obtain an elliptic equation for the steady states:

$$\varepsilon\Delta v - v + \lambda v^k = 0.$$

Then, $w(\mathbf{x}) = \lambda^{1/(k-1)} v(\mathbf{x})$ satisfies

$$(31) \quad \varepsilon\Delta w - w + w^k = 0.$$

This steady state equation has been extensively investigated (see [17, 22–25]). Solutions of (31) have *point condensation phenomena*, i.e., it tends to zero as $\varepsilon \rightarrow 0$ except at a finite number of points (see [17]). Moreover, there is a positive constant ε_0 such that, if $k > 1$ for $n = 2$ or $1 < k < \frac{n+2}{n-2}$ for $n \geq 3$, and $\varepsilon < \varepsilon_0$, then (31) has a nonconstant least energy solution which has exactly one local maximum at a point lying on the boundary (see [24]). This least energy solution is considered as the most stable one among other possible non-constant steady states.

Proposition 3.1 (Existence of non-constant steady states). *Let $k > 1$ if $n = 1, 2$, or $1 < k < \frac{n+2}{n-2}$ if $n \geq 3$. There exists $\varepsilon_0 > 0$ depending on the domain Ω such that there exists a non-constant steady state of (30) whenever $\varepsilon < \varepsilon_0$.*

Remark 3.2. It is assumed $1 < k < \frac{n+2}{n-2}$ for $n \geq 3$ since it is the case in the literature. However, we may still find convergence to non-constant steady states numerically for $k > \frac{n+2}{n-2}$.

If the chemosensitivity $k < 1$, there is no non-constant steady state solution for any $\varepsilon > 0$.

Theorem 3.3 (Nonexistence of non-constant steady states). *Let $0 < k \leq 1$. Then, the only nonnegative solution of (31) is constant.*

Proof. Let w be a nonnegative solution of (31) and let $w = 1 + \xi$. Then, ξ satisfies

$$(32) \quad \varepsilon \Delta \xi - (1 + \xi) + (1 + \xi)^k = 0,$$

Multiply (32) by ξ , integrate it over Ω , and obtain

$$\varepsilon \int_{\Omega} |\nabla \xi|^2 d\mathbf{x} + \int_{\Omega} (1 + \xi)\xi - (1 + \xi)^k \xi d\mathbf{x} = 0.$$

Since $(1 + \xi)\xi - (1 + \xi)^k \xi \geq 0$ for $\xi > -1$, the equality holds for only $\xi = 0$. Therefore, $\xi \equiv 0$, i.e., w is a constant for any $\varepsilon > 0$. \square

3.2. Instability of a constant steady state. In this section, we investigate the instability conditions for a constant steady state solution.

Theorem 3.4 (Instability of a constant steady state). *Let μ_1 be the principal eigenvalue of $-\Delta$ in Ω , $k > 1$, and $\varepsilon_1 = \frac{k-1}{\mu_1}$. If $\varepsilon < \varepsilon_1$, any nontrivial constant steady state solution $(u, v) = (u_0, v_0)$ of (30) is unstable.*

Proof. Let a nontrivial constant state (u_0, v_0) be a steady state solution. Then,

$$v_0 = u_0.$$

We denote $u = u_0 + u_1$, $m = v_0 + v_1$ and linearize (30) by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_t = A(u_0, v_0) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix},$$

where

$$A(u_0, v_0) = \begin{pmatrix} \gamma(v_0)\Delta & \gamma'(v_0)u_0\Delta \\ 1 & \varepsilon\Delta - 1 \end{pmatrix}.$$

Therefore, the local stability of the constant steady state is determined by the eigenvalues of $A(u_0, v_0)$. Let $\{\mu_i\}_{i=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be eigenvalues of $-\Delta$ and $A(u_0, v_0)$, respectively, under the domain Ω . Then λ_j satisfies

$$\lambda_j^2 + (\mu_i(\gamma(v_0) + \varepsilon) + 1)\lambda_j + \mu_i\gamma(v_0)(\varepsilon\mu_i + 1) + \gamma'(v_0)u_0\mu_i = 0.$$

Let

$$\begin{aligned} P_i &:= \mu_i(\gamma(v_0) + \varepsilon) + 1, \\ Q_i &:= \mu_i\gamma(v_0)(\varepsilon\mu_i + 1) + \gamma'(v_0)u_0\mu_i. \end{aligned}$$

From relations $u_0 = v_0$ and $\gamma'(v_0)v_0 = -k\gamma(v_0)$, we rewrite Q_i as

$$Q_i = \varepsilon\gamma(v_0)\mu_i^2 + (\gamma(v_0) + \gamma'(v_0)v_0)\mu_i = \gamma(v_0)\mu_i(\varepsilon\mu_i + (1 - k)).$$

Now, λ_j is written as

$$\lambda_j = -\frac{-P_i \pm \sqrt{P_i^2 - 4Q_i}}{2}.$$

Since P_i is positive, the steady state solution is unstable if $Q_i < 0$ for some μ_i . Since $\mu_i \geq \mu_1 > 0$ for all $i > 1$ it is enough to check the sign of Q_1 . Since $\mu_i > 0$, we have

$$Q_1 < 0 \Leftrightarrow \varepsilon\mu_1 + (1 - k) < 0 \Leftrightarrow \varepsilon < \frac{k-1}{\mu_1} \text{ and } k > 1.$$

Therefore, if $k > 1$ and $\varepsilon < \varepsilon_1 := \frac{k-1}{\mu_1}$, the nontrivial constant steady state $(u, v) = (u_0, v_0)$ is unstable. \square

4. NUMERICAL SIMULATIONS

In this section we study the aggregation behavior of the suggested model through numerical simulations. These simulation results agree with theoretical conclusions of this paper. Furthermore, they provide other interesting features of the model and several conjectures. In the numerical simulations we will consider radial solutions with two different initial values. Consider

$$(33) \quad \begin{cases} u_t = \Delta(\gamma(v)u), \\ v_t = \varepsilon\Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0, \quad \text{on } \partial\Omega, \\ u(\mathbf{x}, 0) = 1, \\ v(\mathbf{x}, 0) = v_\pm(\mathbf{x}) := 1 \pm 0.01 \times \cos(|\mathbf{x}|), \end{cases}$$

where the motility function γ is given with $k = 2$ and $c_0 = 1$, i.e.,

$$\gamma(v) = v^{-2},$$

and the domain is the unit ball,

$$\Omega = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1\}.$$

Note that the aggregation pattern of steady states is independent of the initial total population since steady states of the above problem have mass scaling invariance.

In this example the initial cell population is distributed uniformly, $u_0 = 1$. Two initial values for the signalling chemical density $v(\mathbf{x}, 0)$ is considered, $v_\pm(\mathbf{x})$. The case with $v_+(\mathbf{x})$ is when the initial signal of aggregation started in the middle of the domain. This is a usual case of aggregation that the signal starts from one point and the population in a neighboring region responds. The case with $v_-(\mathbf{x})$ is when the signal starts from the circular boundary of the domain at the same time. Under the radial symmetry assumption we will observe an aggregation (or an exodus) toward the circular boundary.

4.1. Aggregation profile toward the origin. In this section we consider the initial value,

$$u(\mathbf{x}, 0) = 1, \quad v(\mathbf{x}, 0) = v_+(\mathbf{x}) := 1 + 0.01 \times \cos(|\mathbf{x}|),$$

which represents the case when the chemical signal started in the middle of the domain. It has been proved in Theorem 3.4 that there exists $\varepsilon_1 > 0$ such that any constant steady state is unstable if $\varepsilon < \varepsilon_1$. In fact, we will observe numerically that there exists a threshold diffusivity $\varepsilon_1 > 0$ such the stability of a constant steady state is divided precisely by this diffusivity.

In Figure 1(a) aggregation profiles of cell density are given when the solution has reached a steady state. Numerical computation for (33) is done using a matlab pde solver. Four different diffusivity constants,

$$\varepsilon = 0.128, 0.064, 0.016, \text{ and } 0.004,$$

are tested. We observed aggregation phenomena for the first three cases. The solution converges to a steady state that has a peak at the origin, which is the least energy solution of the elliptic problem. Furthermore, as $\varepsilon \rightarrow 0$ the steady state solution converges to a delta distribution. However, if the diffusivity is large, say $\varepsilon = 0.128$, the steady states becomes constant as in the figure. We have observed that the stability of a constant steady state is divided by $\varepsilon_1 \cong 0.1023$ under the

above simulations. Note that the first eigenvalue of the Laplace operator in the domain is π^2 and hence the threshold diffusivity becomes $\varepsilon_1 = \frac{k-1}{\mu_1} \cong 0.1023$, which agrees with the numerical experiment.

In Figure 1(b) the maximum population at each time is displayed on a time interval $0 < t < 100$. The global existence and the uniform boundedness proved in this paper are numerically observed from this picture. We can observe that the maxima of steady states are almost doubled when the diffusivity decreases 1/4 times smaller.

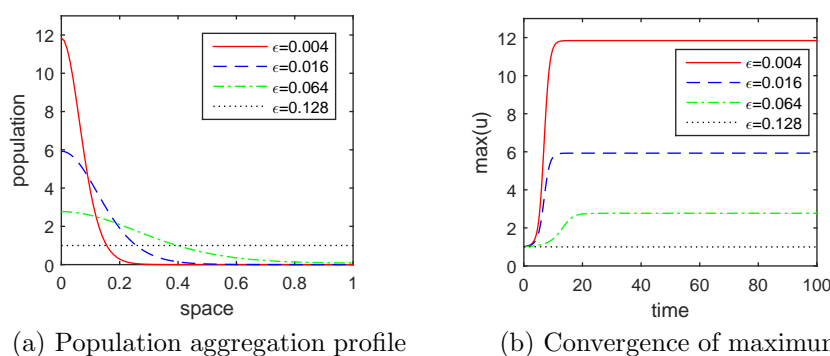


FIGURE 1. Numerical simulation of (33) with v_+ in one space dimension.

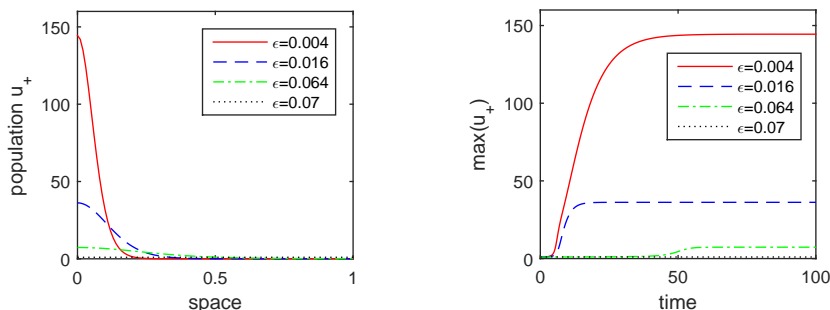


FIGURE 2. Numerical simulation of (33) with v_+ in two space dimensions.

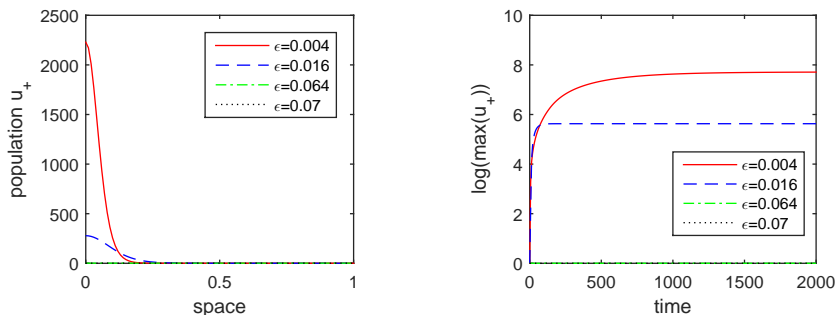


FIGURE 3. Numerical simulation of (33) with v_+ in three space dimensions.

In Figure 2 similar computations are given for radial symmetric solutions in two space dimensions. Four different diffusivity constants,

$$\varepsilon = 0.07, 0.064, 0.016, \text{ and } 0.004,$$

are tested. We observe aggregation phenomena for the first three cases. Therefore, we may expect from the simulation that the threshold diffusivity that separates the stability of a constant steady state is between $\varepsilon = 0.064$ and $\varepsilon = 0.07$. It has been observed from a detailed numerical simulations that the stability of a constant steady state is divided by $\varepsilon \cong 0.068$. This is almost the same value of ε_1 in the theorem (see Remark 4.1). We can also observe that the numerical solution is bounded and the maximum is about reversely proportional to the diffusivity. Similar numerical simulations are given in Figure 3 for radially symmetric steady states in three space dimensions with $\Omega = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| < 1\}$. The corresponding threshold diffusivity becomes smaller which is about $\varepsilon_1 \cong 0.05$.

Remark 4.1 (Finding the threshold diffusivity for radial domain). Let us consider the radial problem for $n \geq 2$. For a radial case, the Laplace operator is written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r},$$

where $r = |\mathbf{x}|$. Thus, the eigenvalue problem

$$-\Delta v = \mu v$$

can be reformulated as

$$\frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r} \frac{\partial v}{\partial r} + \mu v = 0,$$

where $v(|\mathbf{x}|) = v(r)$. Take a variable $\rho = \sqrt{\mu}r$ and obtain

$$v_{\rho\rho} + \frac{n-1}{\rho} v_{\rho} + v = 0.$$

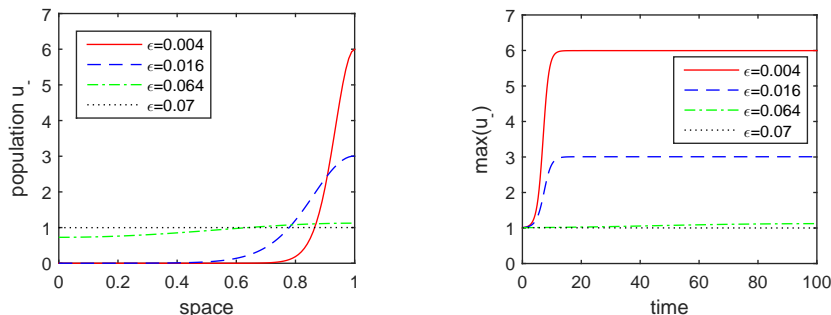
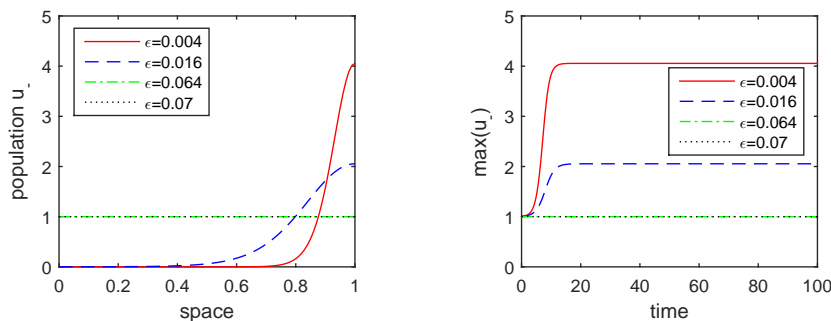
Now consider $n = 2$. Then, the above is Bessel's differential equation of order 0 when the Neumann boundary condition, $\frac{\partial v}{\partial r}|_{r=1} = 0$, is given. Therefore, $J_0'(\sqrt{\mu}) = 0$, where $J_0(\cdot)$ is the Bessel function of the first kind of order 0. Moreover, $\{\sqrt{\mu_j}\}$ is a set of zeros of $J_0'(\cdot)$ and $\sqrt{\mu_1}$ is the first zero which is approximately 3.8317. Hence, $\mu_1 \cong 14.6819$ and the threshold diffusivity is $\varepsilon_1 = \frac{k-1}{\mu_1} \cong 0.068$. This threshold diffusivity can be observed from numerical simulations in Figure 2. One may observe aggregation with $\varepsilon = 0.064$, but not with $\varepsilon = 0.07$.

4.2. Exodus profile toward perimeter. In this section we consider the initial value

$$u(\mathbf{x}, 0) = 1, \quad v(\mathbf{x}, 0) = v_-(\mathbf{x}) = 1 - 0.01 \times \cos(|\mathbf{x}|),$$

which represents the case when the chemical signal starts from the perimeter of the domain simultaneously. This assumption and the resulting phenomena are possible because of the radial symmetry assumption and could not be found in reality. However, this case gives us another interesting aspect of the model equations.

The numerical simulation with above initial values is identical to the previous case if the space dimension is $n = 1$. Hence, we start with numerical simulations in two space dimensions, which is given in Figure 4. One may observe that the maximum of the solution appears along the perimeter of the domain, $r = 1$. The maximum value is smaller than in the other case since the value is distributed

FIGURE 4. Numerical simulation of (33) with v_- in two space dimensions.FIGURE 5. Numerical simulation of (33) with v_- in three space dimensions.

along the perimeter. The same threshold diffusivity $\varepsilon_1 \cong 0.068$ divides the regimes of convergence to a constant state and to an aggregation state.

Simulations in three space dimensions are given in Figure 5. The observed phenomenon is similar to the previous case. The same threshold diffusivity $\varepsilon_1 \cong 0.05$ divides the stability and instability regimes of constant steady states. It is also observed that the maximum is doubled if the diffusivity ε becomes 1/4 times smaller in both two and three space dimensions.

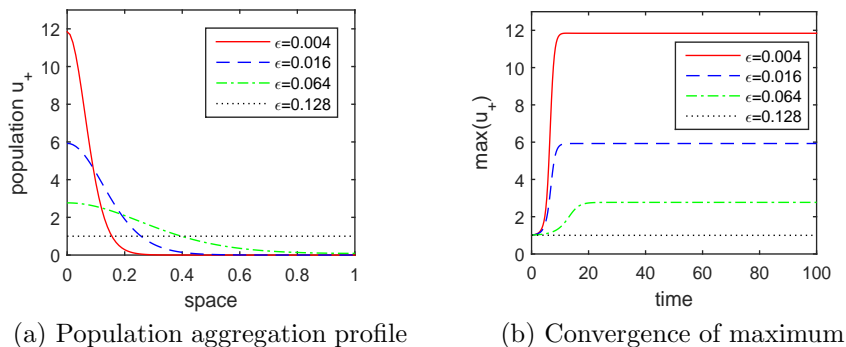
4.3. Logarithmic model. We may rewrite the first equation of (33) with $k = 2$ as

$$u_t = \nabla \cdot \left(v^{-2} \left(\nabla u - \frac{2}{v} u \nabla v \right) \right).$$

After deleting the common factor, v^{-2} , we obtain the usual logarithmic model. In this section we compare the numerical simulation results in previous sections to the numerical solutions of

$$(34) \quad \begin{cases} u_t = \nabla \cdot \left(\nabla u - \frac{2}{v} u \nabla v \right), \\ v_t = \varepsilon \Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0, \quad \text{on } \partial\Omega. \end{cases}$$

This logarithmic model is one of most studied Keller-Segel chemotaxis models. The cancellation of such a common factor v^{-1} does not make difference in the shape of the steady states since Neumann boundary conditions are given. The cancellation may speed up or slow down the convergence to the steady state depending on its size.



(a) Population aggregation profile (b) Convergence of maximum

FIGURE 6. Numerical simulation of (34) with initial values (35) in one space dimension. (Figures will be replaced.)

In Figure 6(a) aggregation profiles of cell densities are given when the same initial values used for Figure 1 are taken, i.e.,

$$(35) \quad u(\mathbf{x}, 0) = 1, \quad v(\mathbf{x}, 0) = v_+(\mathbf{x}) := 1 + 0.01 \times \cos(|\mathbf{x}|).$$

The same four different diffusivity constants,

$$\varepsilon = 0.128, 0.064, 0.016, \text{ and } 0.004,$$

are tested. We may find that the steady states are identical to the one in Figure 1(a) and that the threshold diffusivity is also $\varepsilon_1 = 0.1023$. In fact, the instability analysis for the logarithmic model gives the same criterion for the threshold diffusivity. In Figure 6(b) the maximum population is similarly displayed on a time interval $0 < t < 100$. We may observe even the speed of evolution is almost identical. The simulations for the other cases with dimensions $n = 2, 3$ and the initial values v_{\pm} show the same phenomena and the figures are omitted.

5. DISCUSSION

Since K.B. Raper discovered in 1935 that *Dictyostelium discoideum*, a social amoeba, develops the cellular slime mold, the phenomena took attention of many theoretical biologists. This model system provides in a simplest way how a single cell organism makes complex multicellular patterns. When food dwindles, one of the cells starts to secrete cyclic Adenosine Monophosphate (cAMP) and attracts the other cells. Then, nearby amoebae begin to aggregate and eventually form a fruiting body. The whole process consists of several stages and we are interested in the first one that cells start to aggregate.

5.1. Relation between diffusivity and chemosensitivity. In this paper we have investigated the cell aggregation phenomenon using a chemotaxis model (1)-(4), which can be considered as a special case of the original Keller-Segel model (5)-(6). In most chemotaxis models, the relation (6) between the diffusivity and chemosensitivity is forgotten and simplified versions are used. However, the disconnection between the two coefficients seems to make the analysis more complicated but not simpler. We have proved the global existence of the system (1)-(4) for all $k > 0$, where the system satisfies the relation with $\ell = 0$. We also showed that constant steady states are unstable if $k > 1$ and the diffusivity ε is less than a threshold

value $\varepsilon_1 = \frac{k-1}{\mu_1}$, where μ_1 is the principal eigenvalue of the Laplace operator on the domain.

The model equation (1) (or equivalently (1')) is a special case of (5) satisfying (6) with $\ell = 0$. In other words, the ratio of effective body length is zero and hence the gradient of the chemical substance is not measured. However, the model is very closely related to the logarithmic model (8) (see a discussion in Section 4.3). The global existence and uniform boundedness of the logarithmic model are proved when the relative chemosensitivity satisfies $\frac{\chi_0}{\mu_0} < \sqrt{\frac{2}{n}}$, which corresponds to the case $k < \sqrt{\frac{2}{n}}$ (see [1, 8, 30]). However, this is a regime without any non-constant steady state for space dimensions $n \geq 2$ and a non-constant pattern may appear for $k > 1$ (see Theorem 3.3).

The minimal model (7) has quite different properties. There is a threshold value for the total mass in space dimensions $n \geq 2$, where the solution blows up in finite time if the initial mass is above the threshold. This indicates that the minimal model does not possess the mass scaling invariance in Definition 1.1. Notice that the Keller-Segel equation (5)-(6) satisfies the mass scaling invariance for most cases (for example, when μ is a combination of power laws). The logarithmic model (8) also satisfies the mass scaling invariance even though it does not satisfy (6). The model of this paper, (1)-(4), also possesses the property.

5.2. Aggregation versus wave propagation. Wave propagation toward food is another chemotactic phenomenon and a similar idea of this paper has been used to explain a traveling wave phenomenon by the authors [31]. In the aggregation model the motility of organisms is suppressed when the chemical substance is produced by the organisms. In the wave propagation model, the motility is enhanced to find food if food is not enough. Starvation driven diffusion is introduced to model such a motility increase when there is not enough food relative to the population size (see [5]). Consider a ratio

$$\alpha = \frac{v}{u}.$$

If v represents the amount of food and u the population, then α is the average amount of food for each individual. Then, the motility of the organisms is a decreasing function of α . Now consider a ratio-dependent type model as the following:

$$(36) \quad \begin{cases} u_t = \Delta(\gamma(\alpha)u), \\ v_t = \varepsilon\Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0, \quad \text{on } \partial\Omega, \\ u(\mathbf{x}, 0) = f(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \Omega, \\ v(\mathbf{x}, 0) = g(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega, \end{cases}$$

where the motility function γ is the function of the ratio $\alpha = \frac{v}{u}$ and is given by

$$(37) \quad \gamma(\alpha) = c_0\alpha^{-k},$$

where $c_0 > 0$ and $k > 0$. Using similar derivation in the previous section, we obtain the following elliptic equation for a steady states of (36).

$$(38) \quad \varepsilon\Delta w - w + w^k = 0,$$

where $\hat{k} = \frac{k}{k+1} < 1$. This is the case of Theorem 3.3 that there is no non-constant solution. Therefore, there is no aggregation phenomenon if a ratio dependent motility function is taken.

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