A DISCRETE VELOCITY KINETIC MODEL WITH FOOD METRIC: CHEMOTAXIS TRAVELING WAVES

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Abstract. We introduce a mesoscopic scale chemotaxis model for traveling wave phenomena which is induced by food metric. The organisms of this simplified kinetic model have two discrete velocity modes, $\pm s$, and a constant tumbling rate. The main feature of the model is that the speed of organisms is constant $s > 0$ with respect to the food metric, not the Euclidean metric. The uniqueness and the existence of the traveling wave solution of the model are obtained. Unlike the classical logarithmic model case there exist traveling waves under super linear consumption rates and infinite population pulse type traveling waves are obtained. Numerical simulations are also provided.

1. Introduction

Migration behavior of biological organisms depends on food availability in many cases (see [11, 19, 29]). One way to count it quantitatively is introducing a new distance of migration related to food distribution. We define the food metric between two points in one space dimension by

$$d(a,b) = \Big| \int_a^b m(x)dx \Big|, \quad a, b \in \mathbf{R},$$

where $m(x) > 0$ is a given food distribution. If $m = 1$, it is identical to the Euclidean metric. We will consider a random walk system with a constant walk length with respect to this food metric, but not the Euclidean one. Then, the diffusion limit of a probability density distribution of the random walk system satisfies a diffusion equation,

$$u_t = \Big(D\frac{1}{m}\Big(\frac{u}{m}\Big)_x \Big)_x, \qquad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

where the subindexes denote partial derivatives and $D > 0$ is a diffusivity constant (see [10]). Clearly, it returns to the heat equation if $m = 1$.

If food is the reason for migration of a biological species, the meaningful distance that describes the migration phenomenon is not solely the Euclidean one, but is a one which is somewhere between the food and Euclidean distance. The food metric is a case when the distance is determined only by the food distribution. We consider this relatively polar case since the effect of Euclidean distance is already well studied. As an application of this diffusion operator, a chemotaxis traveling wave phenomenon has been studied in [9]. It takes Keller-Segel type equations,

$$\begin{aligned} u_t &= \Big(D\frac{1}{m}\Big(\frac{u}{m}\Big)_x \Big)_x = \Big(D\frac{1}{m^2}\Big(u_x - \frac{u}{m}m_x\Big) \Big)_x, \quad x \in \mathbf{R},\ t > 0, \\ m_t &= -\kappa(m)u, \end{aligned} \tag{1.2}$$

where u is the population density distribution and $\kappa(m)$ is a consumption rate. We assume that the consumption rate satisfies

$$\kappa(m) > 0, \quad \kappa'(m) \geq 0, \quad \text{for } m \in (0,\infty), \quad \lim_{m\to 0+} \kappa(m) = 0. \tag{1.3}$$

These hypotheses indicate natural biological relations; the organisms do not produce food ($\kappa > 0$), consume more food if there are more ($\kappa' \geq 0$), and do not consume if there is none ($\lim_{m\to 0+} \kappa(m) = 0$). The advection term in the above equation appears simply because of the food metric, not because the individual species senses the gradient of food distribution. A complete picture of traveling wave phenomena was given in [9].

The purpose of this paper is to develop a discrete velocity kinetic model for the bacterial chemotaxis phenomena based on the food metric. Let u^r and u^ℓ be fractional population densities of the species that moves to the right and to the left, respectively. Let $m(x,t) > 0$ be a temporally changing food distribution. The species migrate with a constant speed with respect to the food metric. This speed is reversely proportional to the food concentration in the Euclidean metric. Finally, we obtain a discrete velocity kinetic model:

$$
\begin{aligned}
&\partial_t u^r + \partial_x\Big(\frac{s}{m}u^r\Big) = -\lambda_0 u^r + \lambda_0 u^\ell,\\
&\partial_t u^\ell - \partial_x\Big(\frac{s}{m}u^\ell\Big) = +\lambda_0 u^r - \lambda_0 u^\ell,\\
&\partial_t m = -\kappa(m)(u^r + u^\ell),
\end{aligned}
\tag{1.4}
$$

where $\lambda_0 > 0$ is a constant turning rate (or frequency) and s is the constant speed in food metric.

The discrete velocity kinetic model (1.4) has a run phase with a speed $\frac{s}{m}$ and a tumbling phase with a tumbling rate $\lambda_0 > 0$. The advection terms in the first two equations are modeling transport phenomena, and the reaction terms on the right side are modeling transition phenomena in the moving direction, i.e., the tumbling phenomena. The third equation is the dynamics of food distribution $m > 0$, which is identical to the other case given in (1.2). In other models, the turning frequency usually depends on the chemical gradient ∇m and the chemical density m. Such a case is related to the hypothesis that the organisms sense the chemical gradient. However, in this paper, we do not take this usual hypothesis, and λ_0 is a constant. Therefore, the main finding of the paper is that the spatial heterogeneity in the velocity field can produce a traveling wave phenomenon even without the classical assumption that organisms sense the gradient.

A solution is called a traveling wave of velocity $c > 0$, if each component of the solution is a function of a moving frame variable

$$\xi = x - ct, \tag{1.5}$$

i.e.,

$$u^r(x,t) = u^r(\xi), \quad u^\ell(x,t) = u^\ell(\xi), \quad m(x,t) = m(\xi).$$

Then, the traveling wave solution satisfies ordinary differential equations,

$$
\begin{aligned}
&-c(u^r)' + \Big(\frac{s}{m}u^r\Big)' = -\lambda_0 u^r + \lambda_0 u^\ell,\\
&-c(u^\ell)' - \Big(\frac{s}{m}u^\ell\Big)' = +\lambda_0 u^r - \lambda_0 u^\ell,\\
&-cm' = -\kappa(m)(u^r + u^\ell),
\end{aligned}
\tag{1.6}
$$

where the ordinary derivatives are with respect to ξ. Denote the asymptotic limits at infinity by

$$
\begin{aligned}
&(u^r)'(\xi) \to 0, \quad (u^\ell)'(\xi) \to 0, \quad m'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty,\\
&\lim_{\xi\to\pm\infty} u^r(\xi) = u^r_\pm, \quad \lim_{\xi\to\pm\infty} u^\ell(\xi) = u^\ell_\pm, \quad \lim_{\xi\to\pm\infty} m(\xi) = m_\pm,
\end{aligned}
\tag{1.7}
$$

where the asymptotic values $u^r_\pm, u^\ell_\pm$ and $m_\pm$ are bounded and nonnegative.

Main results of this paper are the existence, uniqueness, and structure of traveling solutions which are given in the following theorem:

Theorem 1.1. *Let* $0 < c < \frac{s}{m_+}$ *and* $\kappa(m)$ *satisfy* (1.3)*. Then, the traveling wave solution of* (1.6)-(1.7) *is unique up to a translation and the followings are equivalent.*

(1) *There exists a nonnegative bounded traveling wave solution of* (1.6)-(1.7)*.*
(2) *The asymptotic limits at infinity in* (1.7) *satisfy* $0 = m_- < m_+ < \infty$, $u_+^\ell = u_+^r = 0$, *and*

$$u_-^\ell = u_-^r = \frac{c^2\lambda_0}{s^2} \lim_{m\to 0+} \frac{m^3}{\kappa(m)} < \infty. \tag{1.8}$$

The steps of the proof are as follows. The equations in (1.6) are rewritten in Section 2 using macroscopic quantities such as the total population, $u = u^\ell + u^r$, and the net flux, $J = \frac{s}{m}(u^r - u^\ell)$. Then, the system is decoupled and we obtain a single equation for m. The existence and the uniqueness of the transformed problem are shown in Section 3. Finally, Theorem 1.1 is transformed into an equivalent form in terms of u and J, Theorem 4.1, and proved in Section 4. The profiles of traveling wave solutions are provided numerically in Section 5.

Traveling wave solutions of the diffusion model (1.2) and the kinetic model (1.4) have similar properties and shapes (compare Table 6.2, Figures 1–4, and [9, Figures 1–5]). The main difference between the two models is the existence of the uniform upper bound of the wave speed. In fact, a traveling wave solution of the diffusion model (1.2) exists for any speed $c > 0$. However, we have a traveling wave solution of the discrete kinetic model (1.4) only for $0 < c < \frac{s}{m_+}$. The numerical simulations in Figure 1–4 show that the derivative of the solution blows up as the speed c approaches to the upper bound $\frac{s}{m_+}$. This blowup phenomenon is a feature of the discrete kinetic models that have finite speed.

The bacterial chemotactic traveling wave phenomenon has been studied experimentally by Adler *et al.* [1, 2, 3, 4] and then mathematically modeled by Keller and Segel [20, 21]. The original Keller-Segel equations for the traveling wave phenomenon are written as

$$\begin{cases} u_t = (\mu(m)u_x - \chi(m)u\, m_x)_x, \\ m_t = \epsilon m_{xx} - \kappa(m)u, \end{cases}$$

where μ is called the diffusivity and χ the chemosensitivity. Here, μ and χ satisfy

$$\chi(m) = -(1-a)\mu'(m) \quad \text{and} \quad \mu'(m) \le 0, \tag{1.9}$$

where a given constant $0 < a < 1$ is the ratio of effective body length. This macroscopic model has been intensively studied in simplified forms and the relation (1.9) is mostly forgotten. (The coefficients in (1.2) satisfy this relation with $a = \frac{1}{2}$.) One may find studies on stability, existence, diffusion limit, relaxation approach, and chemotactic motility on attractant concentrations from [14, 15, 22, 26, 27, 35].

The main goal of the chemotaxis theory is to understand the advection phenomenon activated by the chemical gradient. Most theories are based on the assumption that organisms can measure the chemical gradient. There are many studies and models developed to explain how microscopic scale bacteria can measure the macroscopic scale concentration gradient (see [32, 33]). There are relatively less efforts to explain the phenomenon without such hypothesis. One possible approach to explain it without such hypothesis is to obtain a Fokker Plank type diffusion operator that may produce the advection phenomena consequentially. Cross diffusion contains such advection phenomena (see [18, 31]). The starvation driven diffusion [8] is also such a case and its consequential advection phenomenon gives a species an advantage in a competition [8, 23, 24, 25]. Such a diffusion has been applied to chemotaxis models and has produced the traveling wave and the aggregation phenomena [36, 37].

In addition, microscopic scale models are introduced and their connection to the macroscopic scale models are examined. Alt [5] proposed a biased random walk model to describe chemotactic motions. This model contains a turning dynamics toward a randomly chosen direction, and its relation to the Keller-Segel model is studied. Hillen and Othmer [13, 30] introduced kinetic equations for chemotaxis phenomena. They showed in a formal level that the diffusion limit of the kinetic equations is a Keller-Segel type equation. Chalub *et al.* [6, 7] showed the diffusion limit of a Boltzmann type kinetic equation. Hwang *et al.* [16, 17] obtained a diffusion limit for a more general case. Funaki *et al.* [12] obtained a Fokker Plank type cross-diffusion operator as a relaxation limit of a mesoscopic scale model with two sub-population groups which have different motility but do not sense the chemical gradient. Note that, even if we have a single population group, the use of food metric provides continuous variation in their migration behavior. Therefore, one may consider the model with food metric as containing continuously varying sub-population groups.

2. Total mass and net flux

The fractional densities, u^ℓ and u^r, are microscopic scale level quantities that cannot be observed in a macroscopic scale level. We will first transform the system (1.6) in terms of macroscopic quantities of total mass and net flux, which are defined by

$$\begin{aligned} u(x,t) &= u^r(x,t) + u^\ell(x,t), \\ J(x,t) &= \frac{s}{m(x,t)}u^r(x,t) - \frac{s}{m(x,t)}u^\ell(x,t). \end{aligned} \tag{2.1}$$

We consider traveling wave solutions that has $m(x,t) > 0$. Thus, these linear equations are invertible and hence solving (u, J, m) and (u^r, u^ℓ, m) are equivalent as long as the corresponding asymptotic limits are equivalent.

Next we derive equations for the new variables. Adding up the first two equations in (1.4) gives the continuity equation,

$$\partial_t u + \partial_x J = 0.$$

The difference between the two equations gives

$$\partial_t(u^r - u^\ell) + \partial_x\Big(\frac{s}{m}u\Big) = -2\lambda_0(u^r - u^\ell).$$

The time derivative of J becomes

$$\begin{aligned} \partial_t J &= \frac{s}{m}\partial_t(u^r - u^\ell) - \frac{s}{m^2}(u^r - u^\ell)\partial_t m \\ &= -\frac{s}{m}\partial_x\Big(\frac{s}{m}u\Big) - \frac{s}{m}2\lambda_0(u^r - u^\ell) - \frac{s}{m^2}(u^r - u^\ell)\partial_t m \\ &= -\frac{s}{m}\partial_x\Big(\frac{s}{m}u\Big) - 2\lambda_0 J - \frac{J}{m}\partial_t m. \end{aligned}$$

It follows from the third equation in (1.4) that

$$\partial_t J = -\frac{s}{m}\partial_x\Big(\frac{s}{m}u\Big) - 2\lambda_0 J + \frac{J}{m}\kappa(m)u.$$

In summary, the three equations for u^r, u^ℓ and m are written as a system for u, J and m: for $x \in \mathbf{R}$ and $t > 0$,

$$\begin{aligned}
&\partial_t u + \partial_x J = 0,\\
&\partial_t J + \frac{s}{m}\partial_x\Big(\frac{s}{m}u\Big) = -2\lambda_0 J + \frac{\kappa(m)}{m}uJ, \qquad (2.2)\\
&\partial_t m = -\kappa(m)u.
\end{aligned}$$

Let $c \geq 0$ be the traveling wave speed and $\xi = x - ct$ be the variable of the moving frame. Traveling wave solutions with the wave speed c are written as

$$u(x,t) = u(\xi), \quad J(x,t) = J(\xi), \quad m(x,t) = m(\xi),$$

and satisfy a system of ordinary differential equations,

$$-cu' + J' = 0, \qquad (2.3)$$

$$-cJ' + \frac{s}{m}\Big(\frac{s}{m}u\Big)' = -2\lambda_0 J + \frac{\kappa(m)}{m}uJ, \qquad (2.4)$$

$$-cm' = -\kappa(m)u. \qquad (2.5)$$

We are looking for a traveling wave solution with the following conditions at infinity:

$$u'(\xi) \to 0, \quad J'(\xi) \to 0, \quad m'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty, \qquad (2.6)$$

$$u(\xi) \to u_\pm, \quad J(\xi) \to J_\pm, \quad m(\xi) \to m_\pm \quad \text{as} \quad \xi \to \pm\infty. \qquad (2.7)$$

Notice that we are considering a traveling wave solution without oscillation for a large $|\xi|$ by assuming (2.6). The goal of the analysis of this paper is to find sufficient and necessary conditions of the six asymptotic limits and wave speed,

$$u_\pm \geq 0, \quad m_\pm \geq 0, \quad J_\pm \geq 0, \quad c > 0,$$

for the existence of a traveling wave solution.

The integration of the continuity equation (2.3) on (ξ, ∞) gives

$$J(\xi) - cu(\xi) = C_0, \qquad (2.8)$$

where the integral constant C_0 is given by $C_0 = J_+ - cu_+$. The food distribution $m(\xi)$ is an increasing function due to (2.5) and the sign of the wave speed $c > 0$. Therefore, $m_+ > 0$. By (2.5) and the assumption (2.6), we have $u_+ = 0$. Similarly, (2.4) and assumptions (2.6)-(2.7) yield that $J_+ = 0$. Therefore, we have

$$J(\xi) - cu(\xi) = 0. \qquad (2.9)$$

Thus, after eliminating the flux J, we obtain a system of two equations,

$$\begin{aligned}
&-c^2u' + \frac{s}{m}\Big(\frac{s}{m}u\Big)' = -2c\lambda_0 u + c\frac{\kappa(m)}{m}u^2, \qquad (2.10)\\
&-cm' = -\kappa(m)u.
\end{aligned}$$

Rewrite the first equation as

$$\Big(\frac{s^2}{m^2} - c^2\Big)u' - \frac{s^2m'}{m^3}u = -2c\lambda_0 u + c\frac{\kappa(m)}{m}u^2,$$

or

$$u' - \Big(\frac{s^2}{m^2} - c^2\Big)^{-1}\Big(\frac{s^2m'}{m^3} + c\frac{\kappa(m)}{m}u\Big)u = -2c\lambda_0 u\Big(\frac{s^2}{m^2} - c^2\Big)^{-1}. \qquad (2.11)$$

The positivity of the consumption rate κ and the relation (2.5) imply that the food distribution m is a strictly monotone function. The monotonicity of m allows us to freely change the variable ξ to m in an integration formula and m is bounded by

$$0 \le m(\xi) \le m_+.$$

Now we consider the requirements of the wave speed $c \ge 0$. The infimum speed of individuals is $\frac{s}{m_+}$ and it is plausible to conjecture that the macroscopic scale wave speed cannot exceed the microscopic scale speed of the slowest individual. Indeed, we will show that there is no traveling wave solution if $c \ge \frac{s}{m_+}$ and the wave speed bound of $0 < c < \frac{s}{m_+}$ is a requirement.

Theorem 2.1. *If $c = 0$ or $c \ge \frac{s}{m_+}$, then there is no nontrivial, nonnegative, bounded traveling wave solution (u, m, J) of* (2.3)-(2.7)*. Let $0 < c < \frac{s}{m_+}$ and F be given by* (2.14)*. Then, a traveling wave solution (u, m, J) of* (2.3)-(2.7) *satisfies* (2.9)*,* (2.15) *and* (2.16)*.*

Proof. If $c = 0$, (2.5) implies

$$0 = -\kappa(m)u.$$

From the assumptions for κ in (1.3) and the positivity assumption of food $m > 0$, we conclude that

$$u \equiv 0.$$

Therefore, $c \ne 0$ for a nontrivial traveling wave solution.

If $0 < c < \frac{s}{m_+}$, the integrating factor of (2.11) is given by

$$F(\xi) := \exp\Big(\int_\xi^\infty \Big(\frac{s^2}{m(\eta)^2} - c^2 \Big)^{-1} \Big(\frac{s^2 m'(\eta)}{m(\eta)^3} + c \frac{\kappa(m(\eta))}{m(\eta)} u(\eta) \Big) d\eta \Big).$$

Multiply this integrating factor to (2.11) and obtain

$$\frac{d(u(\xi)F(\xi))}{d\xi} = -2c\lambda_0 u(\xi) \Big(\frac{s^2}{m(\xi)^2} - c^2 \Big)^{-1} F(\xi).$$

Integrating it over (ξ, ∞) gives

$$u(\xi)F(\xi) = 2c\lambda_0 \int_\xi^\infty u(\tau) \Big(\frac{s^2}{m(\tau)^2} - c^2 \Big)^{-1} F(\tau) d\tau. \tag{2.12}$$

The second equation in (2.10) is written as

$$u = c \frac{m'}{\kappa(m)}, \tag{2.13}$$

and we have

$$\log F(\xi) = \int_\xi^\infty \frac{\frac{s^2 m'(\eta)}{m(\eta)^3} + c \frac{\kappa(m(\eta))}{m(\eta)} u(\eta)}{\frac{s^2}{m(\eta)^2} - c^2} d\eta = \int_\xi^\infty \frac{\frac{s^2}{m(\eta)^3} + \frac{c^2}{m(\eta)}}{\frac{s^2}{m(\eta)^2} - c^2} m'(\eta) d\eta = \int_{m(\xi)}^{m_+} \frac{\frac{s^2}{m^3} + \frac{c^2}{m}}{\frac{s^2}{m^2} - c^2} dm,$$

or

$$F(m) = \exp\Big(\int_m^{m_+} \frac{1}{q} \frac{s^2 + c^2 q^2}{s^2 - c^2 q^2} dq \Big). \tag{2.14}$$

Here, we denote $F(\xi) = F(m(\xi))$ instead of introducing a new variable. We use this convention throughout the paper. Substituting F into (2.12) gives

$$\begin{aligned} u(\xi) &= \frac{2c\lambda_0}{F(m(\xi))}\int_\xi^\infty u(\tau)\Big(\frac{s^2}{m(\tau)^2}-c^2\Big)^{-1}F(m(\tau))d\tau \\ &= \frac{2c^2\lambda_0}{F(m(\xi))}\int_\xi^\infty \frac{m'(\tau)}{\kappa(m(\tau))}\Big(\frac{s^2}{m(\tau)^2}-c^2\Big)^{-1}F(m(\tau))d\tau \\ &= \frac{2c^2\lambda_0}{F(m(\xi))}\int_{m(\xi)}^{m_+}\frac{1}{\kappa(m)}\Big(\frac{s^2}{m^2}-c^2\Big)^{-1}F(m)dm. \end{aligned}$$

Therefore,

$$u(m) = \frac{2c^2\lambda_0}{F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq, \tag{2.15}$$

where F is given by (2.14). Finally, $m = m(\xi)$ is obtained by solving (2.5), i.e.,

$$m'(\xi) = \frac{1}{c}\kappa(m(\xi))u(m(\xi)) = \frac{2c\lambda_0\kappa(m(\xi))}{F(m(\xi))}\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq. \tag{2.16}$$

Next, let $c > \frac{s}{m_+}$. Then, there exists $\epsilon > 0$ such that, for $m \in [m_+-\epsilon, m_+)$, $s^2 - m^2c^2 < 0$. Thus, the integrating factor F in (2.14) is defined well for ξ large and we can follow the above argument and obtain

$$u(m) = \frac{2c^2\lambda_0}{F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq.$$

Here the integrand is negative and hence the total population u is negative. Thus there is no nonnegative traveling wave solution for $c > \frac{s}{m_+}$.

Finally, let $c = \frac{s}{m_+}$. In this case the integrating factor F in (2.14) is not defined. Instead, for $\overline{m} \lesssim m_+$ with $m(\overline{\xi}) = \overline{m}$, we define

$$F(m,\overline{m}) := \exp\Big(\int_m^{\overline{m}}\frac{1}{q}\frac{s^2+c^2q^2}{s^2-c^2q^2}dq\Big)$$

as an integrating factor for (2.11). Then we can obtain

$$\frac{d(u(m(\xi))F(m(\xi),\overline{m})}{d\xi} = -2c\lambda_0 u(m(\xi))\Big(\frac{s^2}{m(\xi)^2}-c^2\Big)^{-1}F(m(\xi),\overline{m}).$$

Integrating it over $(\xi,\overline{\xi})$ with (2.13) gives

$$\begin{aligned} u(m(\xi))F(m(\xi),\overline{m}) - u(\overline{m}) &= 2c^2\lambda_0\int_\xi^{\overline{\xi}}\frac{m'(\tau)}{\kappa(m(\tau))}\Big(\frac{s^2}{m(\tau)^2}-c^2\Big)^{-1}F(m(\tau),\overline{m})d\tau \\ &= 2c^2\lambda_0\int_{m(\xi)}^{m(\overline{\xi})}\frac{q^2F(q,\overline{m})}{\kappa(q)(s^2-c^2q^2)}dq. \end{aligned}$$

Thus, we have

$$u(m(\xi)) = \frac{u(\overline{m})}{F(m(\xi),\overline{m})} + \frac{2c^2\lambda_0}{F(m(\xi),\overline{m})}\int_{m(\xi)}^{\overline{m}}\frac{q^2F(q,\overline{m})}{\kappa(q)(s^2-c^2q^2)}dq.$$

The left side, $u(m(\xi))$, is independent of $\overline{m}$. On the other hand, by L'hôpital's rule, the right side converges to

$$\frac{2c^2\lambda_0 m_+^3}{\kappa(m_+)(s^2+c^2m_+^2)}$$

as $\overline{m} \to m_+$. Since $u_+ = 0$, we have

$$u(m(\xi)) \equiv 0.$$

Therefore, the only traveling wave is the trivial one if $c = \frac{s}{m_+}$. □

Theorem 2.1 says that the wave speed is bounded by the speed of individuals. It is because the chemotactic traveling wave is a mass transfer phenomenon. If the wave is faster than the individuals, then individuals cannot keep up with the wave propagation even if all of them moves in the same direction. Note that a water wave is not of mass transfer phenomenon, but of water height and a water wave may propagate horizontally only with a vertical movement of water particles. On the other hand, the traveling wave speed of a Keller-Segel type equation is not bounded by a different reason. When the diffusion limit is taken, the speed of individuals $\frac{\Delta x}{\Delta t}$ diverges (see Table 6.2).

Proposition 2.2. *If there exists a nontrivial traveling wave solution of* (2.3)-(2.7) *and* $0 < c < \frac{s}{m_+}$, *then*

$$m_+ > 0, \qquad u_+ = 0, \qquad m_- = 0. \tag{2.17}$$

Proof. Let (u, J, m) be a traveling wave solution. First, the food distribution $m(\xi)$ is a strictly increasing function due to (2.5) and the positivity of the wave speed $c > 0$. Therefore, $m_+ > 0$ is a necessary condition. Second, (2.5) and conditions at infinity in (2.7) imply $u_+ = 0$.

Now we show $m_- = 0$. We assume $m_- > 0$. To derive a contradiction we employ two things. First, from (2.5), we obtain

$$\int_{-\infty}^{\infty} u(\xi)d\xi = c\int_{-\infty}^{\infty} \frac{m'(\xi)}{\kappa(m(\xi))}d\xi = c\int_{m_-}^{m_+} \frac{1}{\kappa(\eta)}d\eta < \frac{m_+ - m_-}{\kappa(m_-)}.$$

The last inequality comes from the monotonicity of κ. This result shows that the total population is finite. Since u is a continuous function, this implies

$$u_- = 0. \tag{2.18}$$

Next, since $0 < c < \frac{s}{m_+}$, we have $s > qc$ for all $q \in [m_-, m_+]$. Therefore, for $m_- < q < m_+$, we have

$$F(q) = \exp\Big(\int_q^{m_+} \frac{1}{q}\frac{s^2 + c^2q^2}{s^2 - c^2q^2}dq\Big) > 0.$$

Finally, the relations (2.14) and (2.15) give

$$\lim_{\xi \to -\infty} u(\xi) = \frac{2c^2\lambda_0}{F(m_-)}\int_{m_-}^{m_+} \frac{q^2F(q)}{\kappa(q)(s^2 - c^2q^2)}dq > 0,$$

which contradicts (2.18). Therefore, we obtained $m_- = 0$. □

Now we will focus on the decoupled equation for m:

$$\begin{aligned} &m'(\xi) = \tfrac{1}{c}\kappa(m(\xi))u(m(\xi)), \quad \xi \in \mathbf{R}, \\ &m(0) = m_0, \end{aligned} \tag{2.19}$$

where $0 < m_0 < m_+$ and $u(m)$ is given by (2.15).

Remark 2.3. *Traveling waves have translation invariance and two solutions are considered identical if one is a spatial translation of the other. By fixing the value at the origin,* $m(0) = m_0$, *we remove such an invariance and obtain the uniqueness.*

The following proposition shows that the two traveling wave solutions of the system (2.3)-(2.5) and of the decoupled problem (2.19) are identical.

Theorem 2.4. *Suppose that the consumption rate $\kappa(m)$ satisfies* (1.3) *and* $0 < m_0 < m_+$.

(i) If (u, J, m) is a solution of (2.3)-(2.5)*, then m is a solution of* (2.19) *after a spatial translation and hence m is monotone increasing for $c > 0$.*

(ii) Assume that m is a solution of (2.19)*. Let u and J be given by*

$$u(\xi) \;=\; \frac{2c^2\lambda_0}{F(m(\xi))}\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq, \tag{2.20}$$

$$J(\xi) \;=\; cu(\xi). \tag{2.21}$$

Then (u, J, m) is a solution of (2.3)-(2.5)*.*

(iii) The total population is given by

$$\int_{\mathbf{R}} u(\xi)d\xi = c\int_0^{m_+}\frac{1}{\kappa(m)}dm. \tag{2.22}$$

(iv) Suppose that $\int_0^{m_+}\frac{1}{\kappa(m)}dm < \infty$. Then, the total population of the traveling wave is uniformly bounded by

$$\int_{\mathbf{R}} u(\xi)d\xi \le \frac{s}{m_+}\int_0^{m_+}\frac{1}{\kappa(m)}dm. \tag{2.23}$$

Proof. *(i)* We have already shown in previous computations that m satisfies the differential equation in (2.19) if (u, J, m) is a solution of (2.3)-(2.5). Hence the remaining part is to obtain the initial condition $m(0) = m_0$. From the monotonicity of m given by (2.5) and the assumption $0 = m_- < m_0 < m_+$, we can find ξ_0 such that $m(\xi_0) = m_0$ in a unique way. Note that the equation is autonomous and hence a translation $m(\cdot + \xi_0)$ satisfies (2.19) including the initial condition.

(ii) Let m be a solution of (2.19) and u and J be given by (2.20) and (2.21), respectively. Equations (2.3) and (2.5) hold trivially and we show (2.4) in the rest of the proof. The formula (2.20) gives that

$$\Big(\frac{s}{m(\xi)}u(\xi)\Big)' = \Big(\frac{2c^2s\lambda_0}{m(\xi)F(m(\xi))}\Big)'\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq - \frac{2c^2s\lambda_0 m(\xi)m'(\xi)}{\kappa(m(\xi))(s^2-c^2m(\xi)^2)}$$

and

$$u'(\xi) = \Big(\frac{2c^2\lambda_0}{F(m(\xi))}\Big)'\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq - \frac{2c^2\lambda_0 m(\xi)^2m'(\xi)}{\kappa(m(\xi))(s^2-c^2m(\xi)^2)}.$$

Note that, since $F(m) = \exp(\int_m^{m_+}\frac{1}{n}\frac{s^2+c^2n^2}{s^2-c^2n^2}dn)$, we have

$$\Big(\frac{2c^2\lambda_0}{F(m(\xi))}\Big)' = \frac{2c^2\lambda_0}{F(m(\xi))}\frac{m'(\xi)}{m(\xi)}\frac{s^2+c^2m(\xi)^2}{s^2-c^2m(\xi)^2},$$

$$\Big(\frac{2c^2\lambda_0}{m(\xi)F(m(\xi))}\Big)' = \frac{2c^2\lambda_0}{m(\xi)F(m(\xi))}\frac{m'(\xi)}{m(\xi)}\frac{s^2+c^2m(\xi)^2}{s^2-c^2m(\xi)^2} - \frac{2c^2\lambda_0}{m(\xi)F(m(\xi))}\frac{m'(\xi)}{m(\xi)}.$$

From the above calculations, we have

$$\begin{aligned}-c^2u' + \frac{s}{m}\Big(\frac{s}{m}u\Big)' &= \frac{2c^4\lambda_0 m'(\xi)}{m(\xi)F(m(\xi))}\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq - \frac{2c^2\lambda_0 m'(\xi)}{\kappa(m(\xi))}\\ &= c\frac{\kappa(m)}{m}u^2 - 2c\lambda_0 u,\end{aligned}$$

which completes (2.4).

(iii,iv) By (2.5), we have

$$\int_{\mathbf{R}} u(\xi)d\xi = c\int_{\mathbf{R}} \frac{m'(\xi)}{\kappa(m(\xi))}d\xi = c\int_0^{m_+}\frac{1}{\kappa(m)}dm.$$

Therefore, if $\int_0^{m_+}\frac{1}{\kappa(m)}dm<\infty$, then the total population of the traveling wave solution is finite. Since $c<\frac{s}{m_+}$, the total population is uniformly bounded by (2.23). □

The three equations in (2.3)–(2.5) are decoupled and we obtained a single equation (2.19) for food distribution m only, which indicates that the whole system is determined by the food distribution. Such a decoupling is a rare case which cannot be expected usually. Note that similar decoupling appears in [9] for the macroscopic scale diffusion model (1.2). The following analysis is based on this decoupled single equation which makes the full analysis of the traveling wave phenomenon accessible in a simpler way.

Remark 2.5. *We return to traveling wave solutions of* (1.4)*,*

$$u^r(x,t)=u^r(\xi),\quad u^\ell(x,t)=u^\ell(\xi),\quad m(x,t)=m(\xi).$$

From (2.9) *and the definitions of* $u(\xi)$ *and* $J(\xi)$ *in* (2.1)*, it follows that*

$$\begin{aligned} u(\xi) &= u^r(\xi)+u^\ell(\xi),\\ cu(\xi) &= \frac{s}{m(\xi)}u^r(\xi)-\frac{s}{m(\xi)}u^\ell(\xi). \end{aligned}$$

Thus, u^r *and* u^ℓ *are obtained from* J *and* u *by solving the linear system, i.e.,*

$$u^\ell(\xi)=\frac{u(\xi)(s-cm(\xi))}{2s},\quad u^r(\xi)=\frac{u(\xi)(s+cm(\xi))}{2s}. \tag{2.24}$$

Remark 2.6. *The connection between the diffusion equation* (1.1) *and discrete velocity model* (2.2) *is obtained by taking a diffusion limit. Differentiate the first and the second equations in* (2.2) *with respect to* t *and* x*, respectively, and combine them to obtain a telegraphy equation,*

$$u_{tt}+2\lambda_0u_t=\Big(\frac{s}{m}\Big(\frac{s}{m}u\Big)_x+\frac{m_t}{m}J\Big)_x.$$

After a rescaling it using a diffusion scale, the terms in the diffusion equation (1.1) *are the leading order ones with*

$$D=\frac{s^2}{2\lambda_0}. \tag{2.25}$$

3. Existence and uniqueness of decoupled problem

In this section we show the existence of a traveling wave solution (u,J,m) of (2.3)-(2.5) that satisfies the conditions at infinity given in (2.6)-(2.7). It is sufficient to show the existence of the decoupled equations (2.19)-(2.21) by Theorem 2.4.

Lemma 3.1. *Let* $0<c<\frac{s}{m_+}$ *and* $F(m)$ *be given by*(2.14)*. Then, for* $m\in(0,m_+)$*,*

$$\frac{m_+}{m}<F(m)<\Big(\frac{m_+}{m}\Big)^{C_0},\qquad C_0=\frac{s^2+c^2m_+^2}{s^2-c^2m_+^2}. \tag{3.1}$$

Proof. Since the ratio $\frac{s^2+c^2m^2}{s^2-c^2m^2}$ increases as $m\uparrow m_+$, we have

$$\log F(m)=\int_m^{m_+}\frac{1}{q}\frac{s^2+c^2q^2}{s^2-c^2q^2}dq<\frac{s^2+c^2m_+^2}{s^2-c^2m_+^2}\int_m^{m_+}\frac{1}{q}dq=C_0\log\frac{m_+}{m}=\log\Big(\frac{m_+}{m}\Big)^{C_0}.$$

Similarly, since $\frac{s^2+c^2m^2}{s^2-c^2m^2} > 1$, we have

$$\log F(m) = \int_m^{m_+} \frac{1}{q}\frac{s^2+c^2q^2}{s^2-c^2q^2}dq > \int_m^{m_+} \frac{1}{q}dq = \log\frac{m_+}{m}.$$

The comparison relation (3.1) holds since the logarithmic function is an increasing function. $\square$

Proposition 3.2. *Let* $0 < c < \frac{s}{m_+}$ *and* $\kappa(m)$ *satisfy* (1.3). *Then,*

$$\lim_{m\to m_+} \kappa(m)u(m) = 0, \tag{3.2}$$

$$\lim_{m\to 0+} \kappa(m)u(m) = 0, \tag{3.3}$$

$$\lim_{m\to m_+} \frac{\kappa(m)u(m)}{m-m_+} = -\frac{2c^2\lambda_0 m_+^2}{(s^2-c^2m_+^2)}, \tag{3.4}$$

where $u(m)$ *is given by* (2.15).

Proof. Since $m(\xi)$ is monotone by (2.5) and we assume that $m(\xi)$ satisfies (2.5) and (2.6), the convergence in (3.2)-(3.3) hold.

Note that

$$\kappa(m)u(m) = \frac{2c^2\lambda_0\kappa(m)}{F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq,$$

where

$$F(m) = \exp\Big(\int_m^{m_+}\frac{1}{q}\frac{s^2+c^2q^2}{s^2-c^2q^2}dq\Big).$$

Since $0 < c < \frac{s}{m_+}$, the above integrals are well-defined. From the definition of $F(m)$, we have

$$\lim_{m\to m_+} F(m) = 1. \tag{3.5}$$

Next, using L'hôpital's rule, we deduce

$$\begin{aligned}
\lim_{m\to m_+}\frac{\kappa(m)u(m)}{m-m_+} &= \lim_{m\to m_+}\frac{2c^2\lambda_0\kappa(m)}{(m-m_+)F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq\\
&= \lim_{m\to m_+}\Big[\frac{2c^2\lambda_0\kappa'(m)}{(m-m_+)F'(m)+F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq\\
&\qquad -\frac{2c^2\lambda_0\kappa(m)}{(m-m_+)F'(m)+F(m)}\frac{m^2F(m)}{\kappa(m)(s^2-c^2m^2)}\Big]\\
&= \lim_{m\to m_+}\Big[\frac{2c^2\lambda_0\kappa'(m)}{(m-m_+)F'(m)+F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq\\
&\qquad -\frac{2c^2\lambda_0}{(m-m_+)F'(m)+F(m)}\frac{m^2F(m)}{(s^2-c^2m^2)}\Big].
\end{aligned}$$

Note that

$$F'(m) = -F(m)\frac{1}{m}\frac{s^2+c^2m^2}{s^2-c^2m^2} \quad\text{and}\quad \lim_{m\to m_+}F'(m) = -\frac{1}{m_+}\frac{s^2+c^2m_+^2}{s^2-c^2m_+^2}. \tag{3.6}$$

Therefore, from (3.5)-(3.6) and the hypotheses (1.3) for κ, the first part of the limit becomes

$$\lim_{m\to m_+} \frac{2c^2\lambda_0\kappa'(m)}{(m-m_+)F'(m)+F(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq = 0.$$

By a direct calculation, the other part becomes

$$\lim_{m\to m_+} \frac{2c^2\lambda_0}{(m-m_+)F'(m)+F(m)}\frac{m^2F(m)}{(s^2-c^2m^2)} = \frac{2c^2\lambda_0 m_+^2}{(s^2-c^2m_+^2)},$$

which completes (3.4). □

Now we show the existence and the uniqueness of the solution of the decoupled problem (2.19).

Lemma 3.3 (uniqueness and local existence)**.** *Let $\kappa(m)$ satisfy* (1.3) *and $0 < m_0 < m_+$. Then, there exists $\epsilon_0 > 0$ such that the solution of* (2.19) *exists and is unique on $|\xi| < \epsilon$ for all $\epsilon < \epsilon_0$.*

Proof. Since

$$\frac{d(\kappa(m)u(m))}{dm} = -\frac{2c^2\lambda_0 m^2}{s^2-c^2m^2} + \frac{2c^2\lambda_0(\kappa'(m)F(m)-\kappa(m)F'(m))}{F^2(m)}\int_m^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq$$

is bounded locally at $m = m_0$ for any $0 < m_0 < m_+$, the product $\kappa(m)u(m)$ is Lipschitz continuous. Since the right side of (2.19) is Lipschitz, the Picard iteration method gives the local existence and uniqueness of the solution. □

The detail of the proof is omitted which is almost identical to the macroscopic scale model (see [9, Lemma 4.1]). Now we show the global existence of the decoupled problem using the lemma.

Theorem 3.4 (Global Existence)**.** *Let $\kappa(m)$ satisfy* (1.3) *and $0 < m_0 < m_+$. Then, there exists a unique solution $m = m(\xi)$ of* (2.19) *defined for all $\xi \in \mathbf{R}$ and*

$$m(\xi) \to 0 \quad as \quad \xi \to -\infty, \quad and \quad m(\xi) \to m_+ \quad as \quad \xi \to \infty.$$

Proof. The local existence in Lemma 3.3 and classical phase plane analysis method give the global existence of the solution of the decoupled problem (2.19). We only provide a sketch of the proof and one may find a rigorous proof from [9, Section 4.1] in a similar situation.

Let (T_-, T_+) be the maximal interval for the existence of the solution. We will show that $T_+ = \infty$ only and the other part is similar. Suppose that there exists a $\xi_0 > 0$ such that $m(\xi_0) = m_+$. Then, (2.19) implies

$$\xi_0 = \int_0^{\xi_0} 1\,d\xi = \int_0^{\xi_0}\frac{m'(\xi)}{\kappa(m(\xi))u(m(\xi))}\,d\xi = \int_{m_0}^{m_+}\frac{1}{\kappa(m)u(m)dm}. \tag{3.7}$$

However, (3.4) in Proposition 3.2 implies that the integral in the right side of (3.7) is not finite. Therefore,

$$m(\xi) < m_+ \quad \text{for all} \quad \xi > 0. \tag{3.8}$$

We claim that $T_+ = \infty$ and $m(\xi) \to m_+$ as $\xi \to \infty$. If not, then we have three possibilities:

$$\begin{aligned} &A. \quad T_+ < \infty,\ m(T_+) < m_+,\\ &B. \quad T_+ < \infty,\ m(T_+) = m_+,\\ &C. \quad T_+ = \infty,\ m(\xi) \to m_\infty < m_+ \text{ as } \xi \to \infty. \end{aligned}$$

Suppose that $T_+ < \infty$ and $m(T_+) < m_+$. Then, by taking $m_0 = m(T_+) < m_+$ in Lemma 3.3, the interval of existence is extended and hence the domain (T_-, T_+) is not the maximal. Hence Case A is not allowed. Case B is not allowed neither by (3.8). Lastly, suppose that $T_+ = \infty$ and $m(\xi) \to m_\infty < m_+$ as $\xi \to \infty$. Recall (3.7), i.e.,

$$\xi = \int_{m_0}^{m(\xi)} \frac{1}{\kappa(m)u(m)} dm, \quad \xi \in \mathbf{R}. \tag{3.9}$$

Then the right side converges to $\int_{m_0}^{m_\infty} \frac{1}{\kappa(m)u(m)} dm < \infty$ as the left side diverges $\xi \to \infty$. Hence, the last case also deduces a contradiction. Therefore, we conclude that $T_+ = \infty$ and $m(\xi) \to m_+$ as $\xi \to \infty$ as claimed. □

4. Asymptotic limits and existence of traveling waves

In this section we finally show that the two conditions (1) and (2) in Theorem 1.1 are equivalent. Instead of directly considering the system (1.6)-(1.7) for (u^r, u^ℓ, m), we will work with an equivalent system (2.3)-(2.7) for (u, J, m). The asymptotic limits in Theorem 4.1(2) are obtained in Proposition 2.2 except for u_-, which will be obtained in this section.

Theorem 4.1 (Theorem 1.1 in terms of (u, J, m))**.** *Let $0 < c < \frac{s}{m_+}$ and $\kappa(m)$ satisfy* (1.3)*. Then, the traveling wave solution of* (2.2) *is unique up to a translation and the following two are equivalent:*

(1) *There exists a traveling wave solution (u, J, m) of* (2.3)-(2.5) *with the wave speed c and the asymptotic limits in* (2.6)-(2.7)*.*
(2) *The asymptotic limits satisfy $0 = m_- < m_+ < \infty$, $u_+ = 0$, $J_\pm = cu_\pm$, and*

$$u_- = \frac{c^2}{D} \lim_{m \to 0+} \frac{m^3}{\kappa(m)} < \infty, \quad D = \frac{s^2}{2\lambda_0}. \tag{4.1}$$

Proof. The uniqueness of food distribution m comes from Theorem 3.4. The uniqueness of the other two are from Theorem 2.4*(ii)*, where $u(\xi)$ is given by the composition $u(\xi) = u(m(\xi))$ and $J(\xi) = cu(\xi)$. Therefore, the traveling wave solution (u, J, m) of (2.2) is decided in a unique way up to a translation. Next, we show that the two statements in the theorem are equivalent.

($\Rightarrow$) Let (u, J, m) be a traveling wave solution of (2.3)-(2.7). Then, $0 = m_- < m_+ < \infty$ and $u_+ = 0$ by Proposition 2.2. Hence, it is left to show the other asymptotic limit u_-. Remember that

$$F(m) = \exp\Big(\int_m^{m_+} \frac{1}{q}\frac{s^2 + c^2q^2}{s^2 - c^2q^2} dq\Big), \quad F'(m) = -F(m)\frac{1}{m}\frac{s^2 + c^2m^2}{s^2 - c^2m^2}.$$

We divide the proof into two cases. First, consider a case that

$$\int_0^{m_+} \frac{q^2F(q)}{\kappa(q)(s^2 - c^2q^2)} dq = \infty.$$

For this case, it follows from the relation (2.15), Lemma 3.1, and L'hôpital's rule that

$$\begin{aligned}\lim_{\xi\to-\infty} u(\xi) &= \lim_{m\to 0+} \frac{2c^2\lambda_0}{F(m)} \int_m^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq \\ &= \lim_{m\to 0+} \frac{2c^2\lambda_0}{F'(m)} \frac{-m^2F(m)}{\kappa(m)(s^2-c^2m^2)} \\ &= \lim_{m\to 0+} \frac{2c^2\lambda_0}{s^2+c^2m^2} \frac{m^3}{\kappa(m)} \\ &= \frac{2c^2\lambda_0}{s^2} \lim_{m\to 0+} \frac{m^3}{\kappa(m)}.\end{aligned}$$

Therefore, (4.1) holds.

Second, consider the other case

$$\int_0^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq < \infty.$$

By Lemma 3.1, we have

$$\frac{m_+}{m} \le F(m) \le \Big(\frac{m_+}{m}\Big)^{C_0}, \quad C_0 = \frac{s^2+c^2m_+^2}{s^2-c^2m_+^2}.$$

Then,

$$\lim_{\xi\to-\infty} u(\xi) = \lim_{m\to 0+} \frac{2c^2\lambda_0}{F(m)} \int_m^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq = 0.$$

To conclude (4.1), we must show $\lim_{m\to 0+} \frac{m^3}{\kappa(m)} = 0$. Suppose that there exists $\epsilon > 0$ such that $\lim_{m\to 0+} \frac{m^3}{\kappa(m)} \ge \epsilon$. Then,

$$\int_m^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq \ge c_0 \int_m^{m_+} \epsilon q^{-1}dq \to \infty \quad \text{as} \quad m \to 0+,$$

which contradicts the assumption of this second case. Here, $c_0 > 0$ is a positive lower bound of $\frac{F(q)}{s^2-c^2q^2}$ on the interval $[0, m_+]$, which comes from the relation $F(0) = 1$ and $0 < c < \frac{s}{m_+}$. Therefore, there is no such constant $\epsilon > 0$ and hence $\lim_{m\to 0+} \frac{m^3}{\kappa(m)} = 0$. The other asymptotic limits for J are automatic by the relation $J = cu$.

($\Leftarrow$) Let m be the unique solution of (2.19) given in Theorem 3.4 and

$$\lim_{\xi\to-\infty} m(\xi) = m_-, \quad \lim_{\xi\to\infty} m(\xi) = m_+.$$

Define

$$u(\xi) = \frac{2c^2\lambda_0}{F(m(\xi))} \int_{m(\xi)}^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq \quad \text{and} \quad J(\xi) = cu(\xi) \quad \text{for} \quad \xi \in \mathbf{R}.$$

Then, (u, J, m) is a traveling wave solution of (2.3)-(2.5) with the wave speed c. Now we check if this solution satisfies the other asymptotic limits. For the total population u, we have

$$\lim_{\xi\to\infty} u(\xi) = \lim_{\xi\to\infty} \frac{2c^2\lambda_0}{F(m(\xi))} \int_{m(\xi)}^{m_+} \frac{q^2 F(q)}{\kappa(q)(s^2-c^2q^2)}dq = 0.$$

By the definition of u and L'hôpital's rule, we have

$$\begin{aligned}\lim_{\xi\to-\infty} u(\xi) &= \lim_{m\to 0+} \frac{2c^2\lambda_0}{F(m)} \int_m^{m_+} \frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq = \lim_{m\to 0+} \frac{2c^2\lambda_0}{F'(m)} \frac{-m^2F(m)}{\kappa(m)(s^2-c^2m^2)} \\ &= \lim_{m\to 0+} \frac{2c^2\lambda_0}{s^2+c^2m^2}\frac{m^3}{\kappa(m)} = \frac{2c^2\lambda_0}{s^2}\lim_{m\to 0+}\frac{m^3}{\kappa(m)}.\end{aligned}$$

Therefore, u satisfies the asymptotic limits and hence so is J. □

Remark 4.2. *To complete the proof of Theorem 1.1 it is left to show that the conditions for the asymptotic limits in Theorem 1.1(2) and Theorem 4.1(2) are equivalent. The condition $u_+ = 0$ is equivalent to the condition $u^r_+ = u^\ell_+ = 0$ since the population densities are nonnegative, i.e., $u^r_+, u^\ell_+ \geq 0$. Since $m_- = 0$, the relations in* (2.24) *imply that*

$$u^\ell_- = u^r_- = \frac{u_-}{2}.$$

Therefore, the conditions (1.8) *and* (4.1) *are equivalent to each other.*

5. Numerical simulations

In this section, we compare the shape of the traveling wave solutions for u^r, u^ℓ and m numerically. We assume that the consumption rate κ satisfies a power law:

$$\kappa(m) = m^p, \quad p > 0.$$

We will construct solutions numerically with four different powers $p > 0$. The first two cases are with $p = 0.5$ and 2, which give pulse type traveling waves with finite and infinite population, respectively. The other two cases are with $p = 3$ and 4, which give front type traveling wave and unbounded function, respectively. In order to compute the traveling wave solutions, we first take the decoupled ordinary differential equation (2.16) and construct a numeric solution of $m(\xi)$ with the given parameters. We recall the decoupled equation for m with an initial value:

$$\begin{aligned} m'(\xi) &= \frac{1}{c}\kappa(m(\xi))u(m(\xi)) = \frac{2c\lambda_0\kappa(m(\xi))}{F(m(\xi))}\int_{m(\xi)}^{m_+}\frac{q^2F(q)}{\kappa(q)(s^2-c^2q^2)}dq, \\ m(0) &= m_0, \end{aligned} \tag{5.1}$$

where

$$F(m) = \exp\Big(\int_m^{m_+} \frac{1}{q}\frac{s^2+c^2q^2}{s^2-c^2q^2}dq\Big).$$

We set the parameters in (5.1) as

$$m_0 = 0.1, \quad m_+ = 1, \quad \lambda_0 = 1, \quad \text{and} \quad s = \sqrt{2}.$$

In particular, the corresponding diffusivity constant (2.25) becomes $D = \frac{s^2}{2\lambda_0} = 1$. We have chosen these parameters for a easier comparison with traveling waves for diffusion equation cases in [9, Section 5].

The wave speed is bounded by $0 < c < \frac{s}{m_+}$, which gives an upper bound of wave speed $c < \sqrt{2}$ under the given parameters. In the numerical simulation we consider two wave speeds,

$$c = 1.35 \quad \text{and} \quad \sqrt{0.5},$$

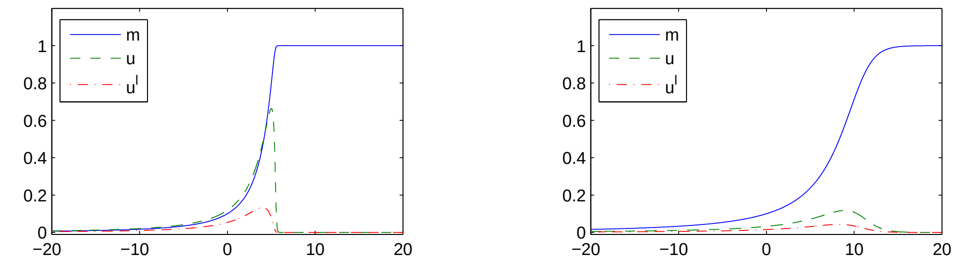
which represent the two cases when the wave speed is close to or away from the speed limit $\frac{s}{m_+}$, respectively. For these cases, the necessary and sufficient conditions to have a traveling wave solution are

$$u_+ = 0, \quad m_- = 0, \quad u_- = c^2 \lim_{m\to 0+} \frac{m^3}{\kappa(m)}.$$

∘ (Case 1. $0 < p < 1$.) This case is biologically more meaningful. If there is more food, the consumption rate increases, but the increase is sublinear in general. In this case, the relation (2.22) gives an upper bound to the total population,

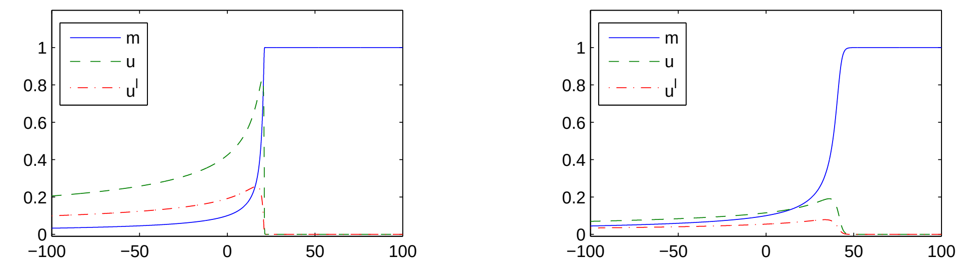
$$\int u(\xi)d\xi = c\int_0^1 m^{-p}dm = \frac{c}{1-p} < \frac{\sqrt{2}}{1-p}.$$

If the population is finite, the corresponding traveling waves is a pulse type. For numerical computation we took a case with $\kappa(m) = m^{1/2}$. Two examples are given in Figure 5.1 with the two wave speeds $c = 1.35$ and $\sqrt{0.5}$. In the figures, the graph of m, u and u^ℓ are given where the graph of $u^r = u - u^\ell$ is omitted. We observe from the numerical solution that the traveling waves u and u^ℓ are of pulse type. The solutions are uniformly bounded for all $c < \sqrt{2}$. There are two observations from waves in Figure 5.1. First, the wave front is steeper than its back. This is because of the diffusivity decreases when food is abundant. Second, the wave front becomes steeper as the wave speed increases to the maximum speed, i.e., as $c \uparrow \sqrt{2}$. Note that to keep up with a speed close to the maximum speed, most of the individuals should move one direction. In the case, the advection phenomenon is dominant and a Burgers equation type shock wave appears in the front. We may observe the same phenomenon in the rest of cases.



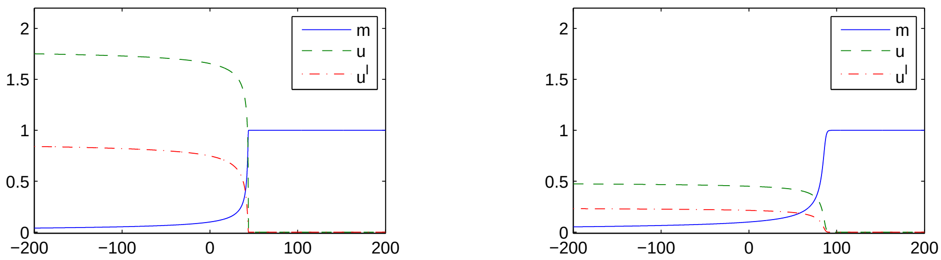
(a) $c = 1.35$ (b) $c = \sqrt{0.5}$

FIGURE 5.1. A pulse type traveling wave of finite mass is obtained when $\kappa(m) = m^{1/2}$.



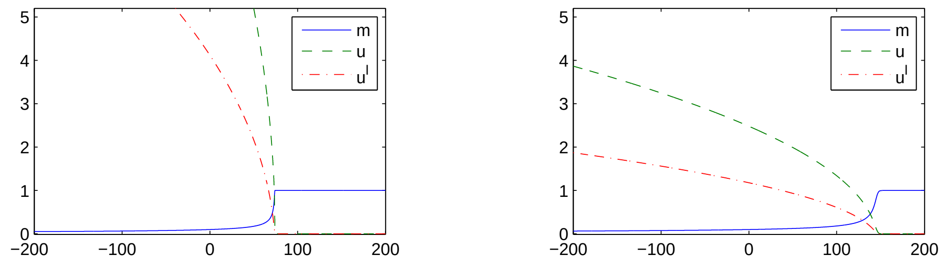
(a) $c = 1.35$ (b) $c = \sqrt{0.5}$

FIGURE 5.2. A pulse type traveling wave of infinite mass is obtained when $\kappa(m) = m^2$.



(a) $c = 1.35$ (b) $c = \sqrt{0.5}$

FIGURE 5.3. A front type traveling wave is obtained when $\kappa(m) = m^3$.



(a) $c = 1.35$ (b) $c = \sqrt{0.5}$

FIGURE 5.4. There is no bounded traveling wave when $\kappa(m) = m^4$.

∘ (Case 2. $1 \le p < 3$.) If $p \ge 1$, the total population of a traveling wave solution is infinite, i.e.,

$$\int u(\xi)d\xi = c\int_0^1 m^{-p}dm = \infty.$$

However, if $p < 3$, the limit of the population density at $-\infty$ is zero, i.e.,

$$u_- = \frac{c^2}{D}\lim_{m\to 0+}\frac{m^3}{\kappa(m)} = 0.$$

Therefore, traveling waves are pulse type with unbounded total population. For numerical computations we took a case with $\kappa(m) = m^2$. Two examples of this case are given in Figure 5.2. We observe from the numerical solutions that traveling waves u and u^ℓ are of pulse type, but with thick tails.

∘ (Case 3. $p = 3$.) Now we consider a critical case $\kappa(m) = m^3$. If $p = 3$,we have

$$u_- = \frac{c^2}{D}\lim_{m\to 0+}\frac{m^3}{\kappa(m)} = c^2.$$

Therefore, the total population u is a front type traveling wave. Two examples of this case are given in Figure 5.3. We confirm that the simulated asymptotic limits in the figures are same as the exact limits computed above. The population is uniformly bounded by $u < 2$ for all $c < \sqrt{2}$. However, the derivatives of the solutions blow up as the speed approaches to its upper bound, $c \uparrow \sqrt{2}$.

∘ (Case 4. $p > 3$): If $p > 3$, we have

$$u_- = \frac{c^2}{D}\lim_{m\to 0+}\frac{m^3}{\kappa(m)} = \infty.$$

In other words, for any limit u_-, there is no corresponding traveling wave solution. Two examples of the numerical solution are given in Figure 5.4.

6. Conclusions

Chemotaxis without gradient sensing. Most bacterial chemotaxis models, if not all, assumes that microscopic scale organisms sense the macroscopic scale chemical gradient. For example, the existence of traveling wave solutions to a discrete velocity kinetic model,

$$
\begin{aligned}
\partial_t u^r + s\partial_x u^r &= -\lambda^+ u^r + \lambda^- u^\ell, \\
\partial_t u^\ell - s\partial_x u^\ell &= +\lambda^+ u^r - \lambda^- u^\ell, \\
\partial_t m &= D\partial_x^2 m + g(u, m).
\end{aligned} \tag{6.1}
$$

has been shown in [28]. This is a kinetic model with a discrete velocity field, $\{-s, s\}$ (see [6, 7, 13, 30] for more examples), and $\lambda^\pm$ are turning frequencies given by

$$\lambda^\pm = \lambda_0 \mp sk\partial_x \Phi(m).$$

In this model organisms sense the gradient $\partial_x \Phi(m)$ and take different turning rates, $\lambda^- \neq \lambda^+$, depending on the gradient of a chemical potential.

The work presented in this paper is a part of efforts to obtain chemotaxis phenomenon without sensing the gradient (see [9, 36, 37]). The discrete velocity kinetic model (1.4) introduced in this paper takes a constant turning frequency λ_0, but not the usual gradient sensing assumption. However, the speed is constant with respect to food metric and hence is reversely proportional to food distribution in the Euclidean space. The results presented in this paper show that traveling wave phenomena is obtained without the gradient sensing assumption.

A comparison to Keller-Segel logarithmic model. The traveling wave phenomenon for a Keller-Segel type logarithmic model has been documented well. For a comparison with (1.2), consider

$$
\begin{aligned}
u_t &= \Big(du_x - \chi \frac{u}{m} m_x\Big)_x, \quad x \in \mathbf{R},\ t > 0, \\
m_t &= -\kappa(m) u,
\end{aligned} \tag{6.2}
$$

where $d, \chi > 0$ are diffusivity and chemosensitivity, respectively. The model equation (1.2) is a case that $d = \chi = m^{-2}$ and hence $\frac{\chi}{d} = 1$. For comparison, we discuss for the case with $\frac{\chi}{d} = 1$. After integrating the traveling wave equation for u, one may obtain

$$du' + cu - \chi m^{-1} m' u = C_0, \tag{6.3}$$

where c is a wave speed and C_0 is an integration constant. The classical traveling wave theories consider two cases, $C_0 = 0$ and $C_0 \neq 0$ (see [34, Sections 3 and 4]). However, the corresponding constant in (2.8) is zero and the case of $C_0 \neq 0$ is deleted. This simplicity comes from the fact that the advection and diffusion terms in (1.2) are from a single Fokker-Planck type diffusion.

Another difference is in the relation between traveling wave types and the nonlinearity of the consumption rate $\kappa(m)$ (see Table 6.1). We have started our discussion with a general consumption rate satisfying (1.3). For an explicit discussion, $\kappa(m) = m^p$ is often considered as we did in numerical computations. Biologically more meaningful regime is when the rate has a sublinear growth, i.e., $0 < p < 1$. If so, the total population of a traveling wave solution of the kinetic model is uniformly bounded by (2.23), i.e., $\int_{\mathbf{R}} u(\xi)d\xi \leq \frac{s}{m_+}\int_0^{m_+} \frac{1}{\kappa(m)}dm$. Critical differences come when $p \geq 1$. The logarithmic model (6.2) has front type traveling wave for population u if $p = 1$ and there is no bounded traveling wave if $p > 1$. However, the food metric models (1.2) and (1.4) have infinite mass traveling pulse if $1 \leq p < 3$, which does not exist for classical Keller-Segel type equations. This thick tail traveling wave is a new addition in the traveling wave theory as far as the authors know. For the logarithmic model case, there is no infinite population pulse type traveling wave and traveling wave type jumps from finite population pulse to front ones. The food metric model allows infinite population pulse when $1 \le p < 3$. To fill the gap between the two cases, one may consider a diffusion operator such as $u_t = (m^\alpha (m^\beta u)_x)_x$, where we have considered a case with $\alpha = \beta = -1$.

TABLE 6.1. Traveling wave comparison. Food distribution m is monotone for both cases. Population distribution depends on the nonlinearity $p > 0$.

type of u	logarithmic model (6.2), $\frac{\chi}{d} = 1$	food metric model (1.2), (1.4)
pulse (finite mass)	$0 < p < 1$	$0 < p < 1$
pulse (infinite mass)	none	$1 \le p < 3$
front	$p = 1$	$p = 3$
unbounded	$p > 1$	$p > 3$

Macrocopic scale versus mesocopic scale models. The traveling wave phenomena in (1.2) and in (1.4) are similar. First, the shapes of traveling waves are similar as shown in Figures 5.1–5.4. The two cases are compared in Table 6.2. A main difference is that the wave speed of the discrete kinetic model is bounded above, but the one of diffusion model has no upper bound. The first order derivative of traveling wave solutions of discrete velocity model blows up as the speed approaches to the speed limit. Necessary and sufficient conditions for existence of a traveling wave solution are identical for both cases. These observations suggest that the solution of discrete velocity model converges to the diffusion model in a diffusion limit scale. A formal derivation of the diffusion model from the discrete velocity model is given in Remark 2.6. However, a rigorous convergence proof is not done yet.

TABLE 6.2. Chemotactic traveling waves of discrete velocity and diffusion models

	diffusion model (1.2) (from [9])	discrete velocity model (1.4)
range of wave speed	$0 < c$	$0 < c < \frac{s}{m_+}$
conditions for asymptotic limits	$0 < m_- < m_+, u_+ = 0,$ $u_- = \frac{c^2}{D} \lim_{m \to 0+} \frac{m^3}{\kappa(m)} < \infty$	$0 < m_- < m_+, u_+ = 0,$ $u_- = \frac{2c^2\lambda_0}{s^2} \lim_{m \to 0+} \frac{m^3}{\kappa(m)} < \infty$

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