DISPERSAL TOWARDS FOOD: THE SINGULAR LIMIT OF AN ALLEN-CAHN EQUATION

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ABSTRACT. The effect of dispersal under heterogeneous environment is studied in terms of the singular limit of an Allen-Cahn equation. Since biological organisms often slow down their dispersal if food is abundant, a food metric diffusion is taken to include such a phenomenon. The migration effect of the problem is approximated by a mean curvature flow after taking the singular limit which now includes an advection term produced by the spatial heterogeneity of food distribution. It is shown that the interface moves towards a local maximum of the food distribution. In other words, the dispersal taken in the paper is not a trivialization process anymore, but an aggregation one towards food.

Keywords. Fokker-Planck type diffusion, food metric, singular limit, generation and propagation of interface, perturbed motion by mean curvature

1. Introduction and biological context

The purpose of this paper is to study the singular limit, as $\varepsilon \to 0$, of the initial-boundary value problem,

$$(P^\varepsilon) \qquad \begin{cases} u_t = \nabla \cdot \left(\dfrac{1}{m} \nabla \left(\dfrac{u}{m} \right) \right) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 1)$, ν is the outward unit normal vector to the boundary $\partial\Omega$, and $\varepsilon > 0$ is a small parameter. The unknown function u is the population density, $m > 0$ is a given food (or chemical) distribution, and f is a bistable nonlinearity such as $f(u) = -u(u - \frac{1}{2})(u - 1)$. Specific assumptions on m, f, and u_0 will be given later in this section.

Migration is a key strategy for the survival of biological species and the importance of formulating a realistic dispersal theory under heterogeneous environments has been emphasized by many authors (see [17, Chapter 5], [19, 20]). Various Fokker-Planck type

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diffusion operators have been introduced to overcome the limitation that the homogeneous diffusion model has (see, for example, [5, 18]). The heterogeneous diffusion operator in (P^ε) is one of such examples, which takes the effect of heterogeneity in food distribution into account. If food is the reason for migration of a biological species, the migration distance should be related to food distribution. Food metric measures such a distance and is defined by

$$\rho(x,y) := \Big| \int_\gamma m(s)\, ds \Big|, \tag{1.1}$$

where γ is the line segment connecting the two points, x and y, and the integration is the line integral along it. This functional is really a metric in one space dimension and the diffusion operator in (P^ε) is the Laplace-Beltrami operator related to this metric (see [6, Appendix A]). For dimensions $N > 1$, the functional is not a metric if m is not constant. However, in a microscopic scale, m can be considered as constant and the functional plays the role of metric. Then, the same diffusion operator in (P^ε) gives the macroscopic level dispersal phenomenon. We refer to [7] for a microscopic scale level approach with the food metric.

The main goal of the paper is to investigate the migration effect produced by the Fokker-Planck type diffusion given in the model equation. The study of the singular limit of (P^ε) gives an excellent view to observe this effect. One might think that the reaction term would dominate the dynamics if $\varepsilon > 0$ is small. However, that is true only for the first stage of a short time period of order $O(\varepsilon^2|\ln(\varepsilon)|)$ and, as soon as an interface is generated (Theorems 1.1 and 1.2), the diffusion plays its role of migration and the interface starts to move towards food. Problem (IP) in Proposition 1.1 gives the interface motion of the singular limit. Note that, if $m = 1$, the problem turns into a classical case with homogeneous diffusion and the interface problem becomes a pure mean curvature flow. This classical case has been intensively studied (see [1, 4, 8] and references therein). However, if m is not constant, the second term $-\frac{\partial}{\partial n}(\frac{1}{m^2})$ of (IP) gives an extra dynamics. In particular, in one space dimension, the mean curvature flow part disappears and the interface moves towards a local maximum point of m because of this extra term (see Figures 1 and 2). In other words, the diffusion taken in (P^ε) is not a trivialization process, but a gathering process towards food. This new feature of interface movement indicates a method how biological organisms adapt the spatial heterogeneity of environment.

1.1. **Assumptions and main results.** The purpose of this paper is to investigate the population motion induced by the Fokker-Planck type diffusion and compare it to the classical one with homogeneous diffusion. We assume that the chemical concentration $m(x,t) : \overline{\Omega} \times [0,\infty) \to \mathbf{R}^+$ satisfies

$$\begin{cases} m > 0 \text{ in } \overline{\Omega} \times [0,\infty), \quad m \in C^{3,1}(\overline{\Omega} \times [0,\infty)), \\ \text{and } \ \dfrac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,\infty). \end{cases} \tag{1.2}$$

Furthermore, we assume that the nonlinear function f satisfies

$$\begin{cases} f \text{ is } C^2 \text{ on } \mathbf{R}, \\ f'(0) < 0,\ f'(a) > 0,\ f'(1) < 0,\ \displaystyle\int_0^1 f(u)\, du = 0, \text{ and } 0 < a < 1, \end{cases} \tag{1.3}$$

where $0, a$ and 1 are the only zeros of f. We define the 'a' level set of the initial value by

$$\Gamma_0 := \{x \in \Omega : u_0(x) = a\},$$

and suppose that

(1.4) $$u_0 \in C^2(\overline{\Omega}),$$

(1.5) $$\Gamma_0 \text{ is a } C^{2+\alpha} \text{ hypersurface without boundary,}$$

(1.6) $$\Gamma_0 \subset\subset \Omega, \quad \nabla u_0 \cdot \mathbf{n} \neq 0 \quad \text{on } \Gamma_0,$$

(1.7) $$u_0 > a \ \text{ in } \ \Omega_0^{in}, \quad u_0 < a \ \text{ in } \ \Omega_0^{ex},$$

where the exterior domain Ω_0^{ex} denotes the region placed between the boundary $\partial\Omega$ and Γ_0, the interior domain Ω_0^{in} denotes the region enclosed by the level set Γ_0, and $\mathbf{n}$ is the outward unit normal vector of the interior domain Ω_0^{in}.

It is standard that Problem (P^ε) has a unique classical solution. We will denote it by u^ε when the ε dependency of the solution needs to be clarified. In the first stage of the evolution of a solution, the effect of diffusion is negligible in comparison with that of reaction if the initial value is away from steady states. Thus, u^ε evolves according to the ordinary differential equation $u_t^\varepsilon = \frac{1}{\varepsilon^2} f(u^\varepsilon)$. Since f is a bistable nonlinearity with two stable zeros, $u = 0, 1$, the solution u^ε quickly approaches to either zero or one in most regions of Ω. Accordingly, steep transition layers (or interfaces) develop between the two regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. These transition layers emerge along the level set $\Gamma_0 = \{x \in \Omega : u_0(x) = a\}$. Note that the transition layer is located in the region where the gradient $|\nabla u^\varepsilon|$ is large and we will see that it takes a time of order $\varepsilon^2 |\ln \varepsilon|$ for such a transition layer to develop.

The first main theorem of this paper is about the generation of interface. We will use the notations:

(1.8) $$\mu_0 := f'(a) > 0, \qquad t^\varepsilon := \frac{1}{\mu_0} \varepsilon^2 |\ln \varepsilon|.$$

Theorem 1.1 (Generation of interface)**.** *Let m, f, and u_0 satisfy* (1.2)*,* (1.3)*, and* (1.4)*–*(1.7)*, respectively, and u^ε be the solution of* (P^ε)*. For any given $\eta > 0$, there exist $\varepsilon_0 > 0$ and $M > 0$ (depending only on η, f and the initial function u_0) such that, for t^ε as in* (1.8) *and $0 < \varepsilon < \varepsilon_0$,*

(1.9) $$-\eta \leq u^\varepsilon(x, t^\varepsilon) \leq 1 + \eta \qquad \textit{for} \qquad x \in \Omega,$$

(1.10) $$u^\varepsilon(x, t^\varepsilon) \geq 1 - \eta \qquad \textit{if} \qquad u_0(x) \geq a + M\varepsilon,$$

(1.11) $$u^\varepsilon(x, t^\varepsilon) \leq \eta \qquad \textit{if} \qquad u_0(x) \leq a - M\varepsilon.$$

In the second stage, after the interface has been generated, the reaction term is no longer dominant but of the same order as the diffusion. As a result, the interface starts to move slowly. We will show that the interface motion is given by

(IP) $$\begin{cases} V_n = -(N-1)\dfrac{\kappa}{m^2} - \dfrac{\partial}{\partial n}\left(\dfrac{1}{m^2}\right) & \text{on} \quad \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0, \end{cases}$$

where Γ_t is the interface at time $t > 0$, V_n is the normal velocity of the interface, and κ denotes its mean curvature. The existence and uniqueness of local-in-time solutions for Problem (IP) follows as in [3, Lemma 2.4]. We refer to [3, Theorem 2.1] for similar results of a related system to (IP); see also [10] for a study of global-in-time weak solutions.

Proposition 1.1. *Assume that Γ_0 is a $C^{2+\alpha}$ hypersurface of $\mathbf{R}^N$. Then, there exists $T > 0$ such that Problem (IP) possesses a unique solution $\Gamma_{[0,T]} := \cup_{t \in [0,T]} \Gamma_t \times \{t\} \in C^{2+\alpha, \frac{2+\alpha}{2}}$.*

Similarly, we denote by Ω_t^{in} the region enclosed by the interface Γ_t and by Ω_t^{ex} the region between $\partial\Omega$ and Γ_t. Let $\overline{d}(x,t)$ be the signed distance to Γ_t defined by

$$\overline{d}(x,t) := \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \overline{\Omega}_t^{ex}, \\ -\operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^{in}, \end{cases} \tag{1.12}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from x to Γ_t in $\mathbf{R}^N$. The second main theorem of this paper is about the convergence of the solution of Problem (P^ε) to a step function given by the solution of the interface problem.

Theorem 1.2 (Fine generation of interface and interface motion)**.** *Assume the same assumptions as in Theorem 1.1 and let $T > 0$ be as in Proposition 1.1. For any given $\eta > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$-\eta \le u^\varepsilon(x,t) \le 1+\eta \qquad \text{for} \qquad x \in \Omega, \tag{1.13}$$

$$u^\varepsilon(x,t) \ge 1-\eta \qquad \text{if} \qquad \bar{d}(x,t) \le -\varepsilon C, \tag{1.14}$$

$$u^\varepsilon(x,t) \le \eta \qquad \text{if} \qquad \bar{d}(x,t) \ge \varepsilon C, \tag{1.15}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and for all $t \in (t^\varepsilon, T]$ where t^ε is defined in (1.8)*.*

The pointwise convergence of the solution to a step function is an immediate consequence of Theorem 1.2.

Corollary 1.1. *For any $0 < t \le T$, we have the following pointwise convergence:*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \begin{cases} 1, & \textit{for } \ x \in \Omega_t^{in}, \\ 0, & \textit{for } \ x \in \Omega_t^{ex}. \end{cases}$$

Remark 1.1. Let Γ_t^ε be the 'a' level set of u^ε at time t defined by $\Gamma_t^\varepsilon := \{x \in \Omega : u^\varepsilon(x,t) = a\}$. Then the interface Γ_t can be approximated by Γ_t^ε as $\varepsilon \to 0$. More precisely, by the arguments in the proof of [1, Theorem 1.5], Theorems 1.1 and 1.2 imply that there exists a constant $C > 0$ such that

$$\Gamma_t^\varepsilon \subseteq \mathcal{N}_{\varepsilon C}(\Gamma_t) \ \text{ for all } \ 0 \le t \le T,$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega : \operatorname{dist}(x,\Gamma_t) \le r\}$ is the r-neighborhood of Γ_t. Therefore, we have

$$d_{\mathcal{H}}(\Gamma_t^\varepsilon, \Gamma_t) \le \varepsilon C, \quad \text{for all } \ t \in [0,T].$$

Here $d_{\mathcal{H}}(A,B) := \max\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\}$ denotes the Hausdorff distance between two sets A and B. As a consequent, $\Gamma_t^\varepsilon \to \Gamma_t$ as $\varepsilon \to 0$, uniformly in $[0,T]$, in the sense of Hausdorff distance.

1.2. **Movement of transition layer.** Biological organisms often show a random walk type migration (diffusion) and a chemotaxis type directed migration at the same time. Chemotaxis refers to a movement induced by the gradient of a chemical substance, which has been studied by many authors. For instance, Keller and Segel equation (see [14]) is written as

$$u_t = \nabla \cdot (k(m)\nabla u - \chi(m)u\nabla m),$$

where k and χ are diffusivity and chemotactic sensitivity, respectively. The Fokker-Planck type diffusion in (P^ε) contains an advection term of chemotaxis type due to the heterogeneity of resource. This can be seen clearly if we write the diffusion term in the form of

$$\nabla \cdot \left(\frac{1}{m}\nabla\left(\frac{u}{m}\right)\right) = \nabla \cdot \left(\frac{1}{m^2}\left(\nabla u - \frac{u}{m}\nabla m\right)\right) = \nabla \cdot \left(\frac{1}{m^2}\nabla u + \frac{u}{2}\nabla\left(\frac{1}{m^2}\right)\right),$$

where the advection term gives directed movement. We can see the effect of this advection term in the movement of the transition layer in (IP). Note that it vanishes for a homogeneous environment case with constant m.

In one space dimension, $N = 1$, the movement of the transition layer (IP) is given by $V_n = -\dfrac{\partial}{\partial n}\left(\dfrac{1}{m^2}\right)$. Therefore, if m is constant, the interface becomes stationary. This phenomenon gives rise to a slow motion of the interface, namely if an initial value has a transition layer structure, the solution maintains its transition layer and interfaces move slower than any power of ε. However, if m is not constant, a transition point moves because of the drift produced by heterogeneity in m. In particular, each interface point follows the ordinary differential equation independently. For example, in the case that there are only two transition points, say ξ_L on the left and ξ_R on the right, we have the system

$$\begin{cases} \dot{\xi}_L = \dfrac{2m_x(\xi_L, t)}{m^3(\xi_L, t)} \\ \dot{\xi}_R = \dfrac{2m_x(\xi_R, t)}{m^3(\xi_R, t)}, \end{cases} \tag{1.16}$$

where $\dot{\xi}_L := d\xi_L/dt$ and $\dot{\xi}_R := d\xi_R/dt$. In particular, we will see in Section 2.2 that when m depends only on the spatial variable x, the transition points ξ_L and ξ_R approach to local maximum points of m as $t \to \infty$.

In two space dimensions, $N = 2$, if m is constant, the interface problem becomes

$$\begin{cases} V_n = -\kappa \qquad \text{on } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0. \end{cases} \tag{1.17}$$

In this case, the interface turns into a circle asymptotically [11] and eventually shrinks to a single point. On the other hand, when m is not constant, the drift term appears and the solution Γ_t of Problem (IP) with the same initial interface Γ_0 as (1.17), may lose convexity and develops very complicated patterns due to the form of food metric; see [3, Figure 1, page 1177] for numerical computations in a similar case.

Organization of the remainder of the paper: In Section 2.1, we will formally derive the interface motion equation in (IP). The derivation is based on the method of matched asymptotic expansion and a change of variable which permits to deal with the non-homogeneous structure of the diffusion term. Note that this argument has been proposed in [13] and later applied to the case of the p-Laplacian reaction-diffusion equation [15] and to the Lotka-Volterra system in a heterogeneous environment [12]. In Section 2.2, we prove that in one space dimension, the individuals have a tendency to move towards regions of rich food resources. The proofs of Theorems 1.1 and 1.2, based on the comparison principle, are presented in Sections 3 and 4.

Throughout the paper, we denote by C a generic constant, which may vary from line to line and use the following notation:

$$C_0 := \|u_0\|_{C(\overline{\Omega})} + \|\nabla u_0\|_{C(\overline{\Omega})} + \|\Delta u_0\|_{C(\overline{\Omega})} + 1. \tag{1.18}$$

2. Interface motion equation

2.1. **Formal derivation of the interface equation.** In this section, we apply the method of matched asymptotic expansions together with a change variable to Problem

(P^ε) to formally derive the interface motion equation; we refer to the book [9] and the articles [1, 16] for the method of matched asymptotic expansions for related problems.

There are two main tasks: First we need to define the interface Γ_t and then obtain the equation describing its movement. As mentioned in the introduction, the interface is located between the regions where $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$ so that it is natural to formally define Γ_t as the limit of Γ_t^ε as $\varepsilon \to 0$, where

$$\Gamma_t^\varepsilon = \{x \in \Omega : u^\varepsilon(x,t) = a\}.$$

We refer to Remark 1.1 for a justification of this convergence. We may also define Γ_t in a more explicit way by using a formal asymptotic expansion of the signed distance function to Γ_t^ε. To that purpose, we assume Γ_t^ε is a smooth hypersurface for each $t \in (0,T]$. Moreover, we define the signed distance function to Γ_t^ε by

$$\overline{d}^\varepsilon(x,t) := \begin{cases} \operatorname{dist}(x, \Gamma_t^\varepsilon) & \text{for } x \in \overline{\Omega_t^{\varepsilon,ex}}, \\ -\operatorname{dist}(x, \Gamma_t^\varepsilon) & \text{for } x \in \Omega_t^{\varepsilon,in}, \end{cases}$$

where $\Omega_t^{\varepsilon,in}$ is the region enclosed by the level set Γ_t^ε and $\Omega_t^{\varepsilon,ex}$ is the region located between $\partial\Omega$ and Γ_t^ε. Note that $\overline{d}^\varepsilon = 0$ on Γ_t^ε and $|\nabla \overline{d}^\varepsilon| = 1$ (see [2, Sections 3, 4 and 5] for further properties about the signed distance function). Suppose further that $\overline{d}^\varepsilon$ is expanded in the form

$$\overline{d}^\varepsilon(x,t) = \overline{d}_0(x,t) + \varepsilon \overline{d}_1(x,t) + \varepsilon^2 \overline{d}_2(x,t) + \cdots, \tag{2.19}$$

and define

$$\begin{aligned} \Gamma_t &:= \{x \in \Omega : \overline{d}_0(x,t) = 0\}, \\ \Omega_t^{in} &:= \{x \in \Omega : \overline{d}_0(x,t) < 0\}, \\ \Omega_t^{ex} &:= \{x \in \Omega : \overline{d}_0(x,t) > 0\}. \end{aligned}$$

Since $|\nabla \overline{d}^\varepsilon| = 1$ for all $\varepsilon > 0$ and $\overline{d}_0$ is the only term independent of ε, we have $|\nabla \overline{d}_0| = 1$. Therefore, $\overline{d}_0$ can be considered as the signed distance function to Γ_t, i.e., $\overline{d} \equiv \overline{d}_0$. Hereafter, we will use the notation $\overline{d}$ for the signed distance function to Γ_t as in (1.12).

<u>*Formal asymptotic expansions of u^ε:*</u> We assume that Γ_t is smooth for all $0 < t \leq T$ in order to formally derive the equation for interface motion Γ_t. First, we write the outer expansion of u^ε in the interior and exterior domains as

$$u^\varepsilon(x,t) = 0 + \varepsilon u_1^+(x,t) + \varepsilon^2 u_2^+(x,t) + \varepsilon^3 u_3^+(x,t) + \cdots, \quad x \in Q_T^{ex}, \tag{2.20}$$

$$u^\varepsilon(x,t) = 1 + \varepsilon u_1^-(x,t) + \varepsilon^2 u_2^-(x,t) + \varepsilon^3 u_3^-(x,t) + \cdots, \quad x \in Q_T^{in}, \tag{2.21}$$

which are valid away from the interface $\Gamma_{(0,T]} := \cup_{0<t\leq T}(\Gamma_t \times \{t\})$. Here, we note

$$Q_T^{ex} := \bigcup_{0<t\leq T} \left(\Omega_t^{ex} \times \{t\}\right), \quad Q_T^{in} := \bigcup_{0<t\leq T} \left(\Omega_t^{in} \times \{t\}\right).$$

The inner expansion of u^ε is written in the form

$$u^\varepsilon(x,t) = U_0\left(x,t,\frac{\overline{d}(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{\overline{d}(x,t)}{\varepsilon}\right) + \varepsilon^2 U_2\left(x,t,\frac{\overline{d}(x,t)}{\varepsilon}\right) + \cdots, \tag{2.22}$$

which is valid near the interface $\Gamma_{(0,T]}$. We assume that $U_j(x,t,z)$'s are smooth functions defined for $x \in \overline{\Omega}$, $t \geq 0$, and $z \in \mathbf{R}$. The stretched space variable, $\overline{d}(x,t)/\varepsilon$, has been introduced to connect the two outer expansions. Since the inner expansion (2.22) connects the two regions, $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$, the function U_0 is chosen so that

$$U_0(x,t,0) = a. \tag{2.23}$$

In order to connect the inner expansion to the two outer expansions, we need the following matching conditions:

$$U_0(x,t,+\infty)=0, \qquad U_0(x,t,-\infty)=1, \tag{2.24}$$

$$U_k(x,t,+\infty)=u_k^+, \quad U_k(x,t,-\infty)=u_k^-. \tag{2.25}$$

Motion of the interface: We will substitute the transition layer expansion (2.22) in the reaction-diffusion equation in (P^ε) and obtain equations that the formal expansion should satisfy. To that purpose, we write the reaction-diffusion equation in (P^ε) as

$$u_t=\nabla\cdot\left(\frac{1}{m^2}\nabla u\right)+\frac{1}{2}\nabla\cdot\left(u\nabla\left(\frac{1}{m^2}\right)\right)+\frac{1}{\varepsilon^2}f(u), \tag{2.26}$$

or else

$$u_t=\frac{\Delta u}{m^2}+\frac{3}{2}\nabla(\frac{1}{m^2})\cdot\nabla u+\frac{1}{2}u\Delta(\frac{1}{m^2})+\frac{1}{\varepsilon^2}f(u), \tag{2.27}$$

and compute

$$\begin{aligned}
u_t^\varepsilon &= \left[U_{0t}+U_{0z}\frac{\overline{d}_t}{\varepsilon}\right]+\varepsilon\left[U_{1t}+U_{1z}\frac{\overline{d}_t}{\varepsilon}\right]+\cdots,\\
\nabla u^\varepsilon &= \left[\nabla U_0+U_{0z}\frac{\nabla\overline{d}}{\varepsilon}\right]+\varepsilon\left[\nabla U_1+U_{1z}\frac{\nabla\overline{d}}{\varepsilon}\right]+\cdots,\\
\Delta u^\varepsilon &= \Delta U_0+\frac{2\nabla U_{0z}\cdot\nabla\overline{d}+U_{0z}\Delta\overline{d}}{\varepsilon}+\frac{U_{0zz}|\nabla\overline{d}|^2}{\varepsilon^2}\\
&\quad+\varepsilon\left[\Delta U_1+\frac{2\nabla U_{1z}\cdot\nabla\overline{d}+U_{1z}\Delta\overline{d}}{\varepsilon}+\frac{U_{1zz}|\nabla\overline{d}|^2}{\varepsilon^2}\right]+\cdots,\\
f(u^\varepsilon) &= f(U_0)+\varepsilon f'(U_0)U_1+O(\varepsilon^2),
\end{aligned}$$

where ∇,Δ stand for the gradient and Laplacian with respect to x. We substitute the above into (2.27). Then, the leading order terms are of order ε^{-2} which yield

$$0=\frac{U_{0zz}}{m^2}+f(U_0), \tag{2.28}$$

where we use the relation $|\nabla\overline{d}|=1$ near $\Gamma_{(0,T]}$.

Introduce a new variable $\tilde{z}:=zm$ and define $\Phi_0:\overline{\Omega}\times[0,T]\times\mathbf{R}\to\mathbf{R}$ by

$$\Phi_0(x,t,\tilde{z}):=U_0(x,t,\frac{\tilde{z}}{m})=U_0(x,t,z).$$

Then, we have

$$U_{0z}=m\Phi_{0\tilde{z}}, \quad U_{0zz}=m^2\Phi_{0\tilde{z}\tilde{z}}. \tag{2.29}$$

For a fixed (x,t), consider $\Phi_0(x,t,\cdot):\mathbf{R}\to\mathbf{R}$ as a function of $\tilde{z}$. It follows from (2.28), (2.29), (2.23) and (2.24) that $\Phi_0(x,t,\cdot)$ satisfies

$$\begin{cases}\Phi_{0\tilde{z}\tilde{z}}+f(\Phi_0)=0,\\ \Phi_0(-\infty)=1, \quad \Phi_0(0)=a, \quad \Phi_0(\infty)=0.\end{cases} \tag{2.30}$$

It is standard that (2.30) admits a unique solution so that Φ_0 does not depend on (x,t). Therefore, we will, hereafter, write $\Phi_0(\tilde{z})$ instead of $\Phi_0(x,t,\tilde{z})$ to stress that Φ_0 only depends on $\tilde{z}$. As a consequence, we could write U_0 in the form of $U_0(x,t,z)=\Phi_0(zm)$. Thus, we get

$$\nabla U_0=z\Phi_{0\tilde{z}}\nabla m, \quad \nabla U_{0z}=(\Phi_{0\tilde{z}}+\tilde{z}\Phi_{0\tilde{z}\tilde{z}})\nabla m. \tag{2.31}$$

Some more properties of the function Φ_0 is given in the following lemma.

Lemma 2.1 ([1, Lemma 2.1]). *There exist positive constants C and λ such that*

$$0 < 1 - \Phi_0(\tilde{z}) \le Ce^{-\lambda|\tilde{z}|} \quad \textit{for} \quad \tilde{z} \le 0,$$
$$0 < \Phi_0(\tilde{z}) \le Ce^{-\lambda|\tilde{z}|} \quad \textit{for} \quad \tilde{z} \ge 0.$$

In addition, Φ_0 decreases strictly and satisfies

$$|\Phi_{0\tilde{z}}| + |\Phi_{0\tilde{z}\tilde{z}}| \le Ce^{-\lambda|\tilde{z}|} \quad \textit{for} \quad \tilde{z} \in \mathbf{R}.$$

The next leading terms in (2.27) are of order ε^{-1} which yield

$$U_{0z}\overline{d}_t = \frac{1}{m^2}\left(2\nabla U_{0z}\cdot\nabla\overline{d} + U_{0z}\Delta\overline{d} + U_{1zz}\right) + \frac{3}{2}U_{0z}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d} + f'(U_0)U_1,$$

or

$$\frac{U_{1zz}}{m^2} + f'(U_0)U_1 = U_{0z}\left[\overline{d}_t - \frac{\Delta\overline{d}}{m^2} - \frac{3}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right] - \frac{2}{m^2}\nabla U_{0z}\cdot\nabla\overline{d}. \tag{2.32}$$

Set

$$\Phi_1(x,t,\tilde{z}) := U_1(x,t,z),$$

and rewrite the equation (2.32) in terms of Φ_0, Φ_1 using the relations in (2.29) and (2.31),

$$\begin{aligned}
&\Phi_{1\tilde{z}\tilde{z}} + f'(\Phi_0)\Phi_1 \\
&\quad = m\Phi_{0\tilde{z}}\left[\overline{d}_t - \frac{\Delta\overline{d}}{m^2} - \frac{3}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right] - \left(\Phi_{0\tilde{z}} + \tilde{z}\Phi_{0\tilde{z}\tilde{z}}\right)\frac{2}{m^2}\nabla m\cdot\nabla\overline{d}, \\
&\quad = m\Phi_{0\tilde{z}}\left[\overline{d}_t - \frac{\Delta\overline{d}}{m^2} - \frac{3}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right] + \left(\Phi_{0\tilde{z}} + \tilde{z}\Phi_{0\tilde{z}\tilde{z}}\right)m\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}, \\
&\quad = m\Phi_{0\tilde{z}}\left[\overline{d}_t - \frac{\Delta\overline{d}}{m^2} - \frac{1}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right] + m\tilde{z}\Phi_{0\tilde{z}\tilde{z}}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d} =: A(x,t;\tilde{z}).
\end{aligned}$$

Then, the equation becomes

$$\Phi_{1\tilde{z}\tilde{z}} + f'(\Phi_0)\Phi_1 = A(x,t;\tilde{z}). \tag{2.33}$$

The following lemma gives the solvability condition.

Lemma 2.2 (Solvability condition, see [1, Lemmas 2.2 and 2.3]). *Let $B(\tilde{z})$ be a bounded function on $\mathbf{R}$. Then the problem:*

$$\begin{cases} \psi_{\tilde{z}\tilde{z}} + f'(\Phi_0)\psi = B(\tilde{z}), & \tilde{z} \in \mathbf{R}, \\ \psi(0) = 0, \quad \psi \in L^\infty(\mathbf{R}), \end{cases}$$

has a solution if and only if

$$\int_{\mathbf{R}} B(\tilde{z})\Phi_{0\tilde{z}}(\tilde{z})\,d\tilde{z} = 0.$$

Furthermore:

(i) *The solution, if it exists, is unique and satisfies for a constant $C > 0$,*
$$|\psi(\tilde{z})| \le C\|B\|_{L^\infty(\mathbf{R})} \quad \textit{for all} \;\; \tilde{z} \in \mathbf{R}.$$

(ii) *If there exists a constant $\delta_1 > 0$ such that $B(\tilde{z}) = O(e^{-\delta_1|\tilde{z}|})$ as $\tilde{z} \to \pm\infty$, then*
$$|\psi_{\tilde{z}}| + |\psi_{\tilde{z}\tilde{z}}| = O(e^{-\delta_2|\tilde{z}|}) \quad \textit{as} \;\; \tilde{z} \to \pm\infty,$$
for some constant $\delta_2 > 0$.

Applying the above lemma to the equation (2.33), we obtain

$$\int_{\mathbf{R}} A(x,t;\tilde{z})\Phi_{0\tilde{z}}(\tilde{z})\,d\tilde{z}=0.$$

It follows that

$$\int_{\mathbf{R}} m\left(\Phi_{0\tilde{z}}^2\left[\overline{d}_t-\frac{\Delta\overline{d}}{m^2}-\frac{1}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right]+\tilde{z}\Phi_{0\tilde{z}}\Phi_{0\tilde{z}\tilde{z}}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right)d\tilde{z}=0.$$

Thus, we have

$$\left[\overline{d}_t-\frac{\Delta\overline{d}}{m^2}-\frac{1}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right]\int_{\mathbf{R}}\Phi_{0\tilde{z}}^2\,d\tilde{z}+\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\int_{\mathbf{R}}\tilde{z}\Phi_{0\tilde{z}}\Phi_{0\tilde{z}\tilde{z}}\,d\tilde{z}=0.$$

Note that

$$\int_{\mathbf{R}} 2\tilde{z}\Phi_{0\tilde{z}}\Phi_{0\tilde{z}\tilde{z}}\,d\tilde{z}=\tilde{z}(\Phi_{0\tilde{z}})^2\Big|_{-\infty}^{+\infty}-\int_{\mathbf{R}}(\Phi_{0\tilde{z}})^2d\tilde{z}=-\int_{\mathbf{R}}(\Phi_{0\tilde{z}})^2d\tilde{z}, \tag{2.34}$$

where we have used an integration by parts and the fact that

$$\left|\tilde{z}(\Phi_{0\tilde{z}})^2\right|\le C|\tilde{z}|e^{-2\lambda|\tilde{z}|}\to 0 \text{ as } \tilde{z}\to\pm\infty, \quad \text{(by Lemma 2.1).}$$

Therefore,

$$\left[\overline{d}_t-\frac{\Delta\overline{d}}{m^2}-\frac{1}{2}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\right]\int_{\mathbf{R}}\Phi_{0\tilde{z}}^2\,d\tilde{z}-\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}\int_{\mathbf{R}}\frac{\Phi_{0\tilde{z}}^2}{2}\,d\tilde{z}=0,$$

which yields

$$\overline{d}_t-\frac{\Delta\overline{d}}{m^2}-\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\overline{d}=0.$$

It is well-known that $\nabla\overline{d}=n$—the outward unit normal vector on Γ_t , $-\overline{d}_t$ is equal to the normal velocity V_n of interface Γ_t, and $\Delta\overline{d}$ is equal to $(N-1)\kappa$, where κ is the mean curvature of Γ_t. Thus, we obtain

$$V_n=-(N-1)\frac{\kappa}{m^2}-\frac{\partial}{\partial n}\left(\frac{1}{m^2}\right) \quad \text{on} \quad \Gamma_t, \tag{2.35}$$

which is the interface motion equation of Γ_t as desired.

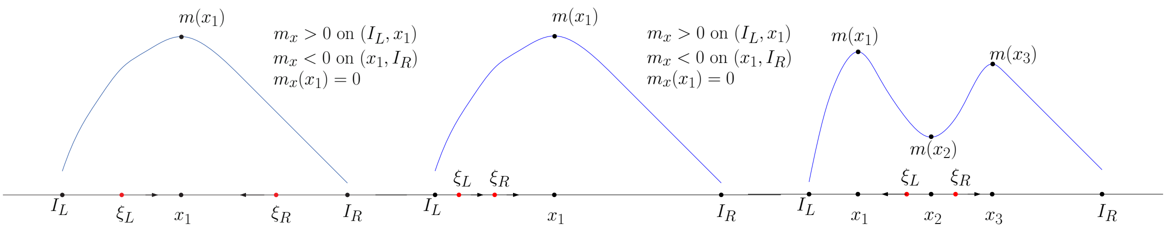
2.2. **Movement of the aggregation region in one-space dimension.** In one space dimension, the movement of an interface is as follows

$$\begin{cases}\dot{\xi}=\dfrac{2m_x(\xi)}{m^3(\xi)}\\ \xi(0)=\xi_0.\end{cases} \tag{2.36}$$

The next lemma concerns the well-posedness and the long time asymptotic behavior of solutions of (2.36) which follows from the standard theory of ordinary differential equations.

Lemma 2.3. *Let $\Omega=(I_L,I_R)\subseteq\mathbf{R}$ and $\xi_0\in\Omega$. Suppose that $m=m(x)>0$ and $m\in C^2(\overline{\Omega})$.*

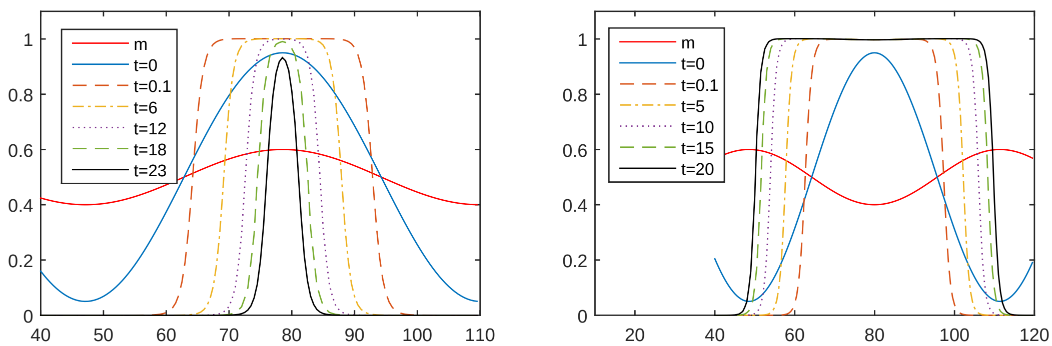
(i) *If $m_x(\xi_0)=0$, then $\xi\equiv\xi_0$ is the unique solution of (2.36).*
(ii) *If $m_x(\xi_0)<0$ and if there exists $x_*<\xi_0$ such that $m_x(x_*)=0$ and that $m_x<0$ on (x_*,ξ_0), then (2.36) has a unique solution on $[0,\infty)$. Moreover, $\lim_{t\to\infty}\xi(t)=x_*$.*
(iii) *If $m_x(\xi_0)>0$ and if there exists $x^*>\xi_0$ such that $m_x(x^*)=0$ and that $m_x>0$ on (ξ_0,x^*), then (2.36) has a unique solution on $[0,\infty)$. Moreover, $\lim_{t\to\infty}\xi(t)=x^*$.*



(a) Local maximum in Ω^{in}. (b) No critical point in Ω^{in}. (c) Local maximum in Ω^{in}.

FIGURE 1. Movement of interface governed by the interface problem (IP). Each interface approaches to a local maximum of m and escapes from a local minimum.

Lemma 2.3 explains the dynamics of the two interfaces, ξ_L and ξ_R, given in (1.16). Consider three cases. First, suppose that there is a local maximum point of m between ξ_L and ξ_R (see Figure 1(a)). Then, the both interfaces move towards the maximum point from different directions and finally meet at the maximum point for $t = +\infty$. This indicates that the interior domain $\Omega^{in} \approx \{u^\varepsilon = 1\}$, which is the region with population, disappears. Second, suppose that there is no critical point of m between ξ_L and ξ_R (see Figure 1(b)). Then, both interfaces move to the maximum point from the same direction. This is also the case that the two points will meet at a local maximum point and the interior domain $\Omega^{in} \approx \{u^\varepsilon = 1\}$ eventually disappears. Lastly, consider a case that there is a local minimum point of m between ξ_L and ξ_R (see Figure 1(c)). Then, the two interfaces move towards the two closest local maximum points to the local minimum points in the interior domain Ω^{in} from the two sides. In this case the interior domain survives asymptotically.



(a) Extinction of interior domain Ω^{in}. (b) Expansion of interior domain Ω^{in}.

FIGURE 2. Evolution of solutions of the parabolic problem (P^ε) in one space dimension. If a local maximum point of m exists Ω^{in}, the population persists. Otherwise, the population perishes.

In Figure 2 numerical simulations of the parabolic problem (P^ε) are given to compare its dynamics and the interface problem. In the simulation we use two cases using parameter values of

$$m(x) = (5 + \sin(x/10))10^{-1}, \qquad \varepsilon^2 = 0.01.$$

In the simulation we multiplied the diffusion term by the diffusivity constant $D = 10$. In Figure 2(a) the evolution of the solution is given when a local maximum point of m is in the interior domain Ω^{in}. The initial value, $t = 0$, and the chemical concentration, m, are given as in the figure with solid lines. The interface emerges as quickly as $t = 0.1$ and then moves towards the local maximum point. One may clearly observe that the interior domain Ω^{in} becomes narrower and disappears by $t = 24$ in the simulation. In Figure 2(b) a case when a local minimum point of m is in Ω^{in}. The interface emerges as quickly as $t = 0.1$ and then moves towards the two closest local maximum points from each direction. The interior domain Ω^{in} converges to the region between the two adjacent local maximum points. These simulations show that the dynamics of the interface given in (2.36) explains the evolution of the reaction diffusion equation correctly.

3. Generation of interface

In this section, we will prove Theorem 1.1. Recall that the idea of the proof is based on the comparison principle, hence we need to construct an appropriate pair of sub- and super-solutions. Since the reaction term plays an important role in the formation of interface, the form of sub-solutions and super-solutions will be based on the solution of the equation without diffusion $u_\tau = f(u) + \delta$, which is a modified equation of the ordinary differential equation corresponding to (P^ε). Here $\tau = t/\varepsilon^2$ and the parameter δ is introduced to take into account the advection term in the equation (see the form of the equation in (2.26)).

Let $\delta_0 > 0$ be small enough such that for all $\delta \in (-\delta_0, \delta_0)$, the function $f(u) + \delta$ has exactly three zeros denoted by $a_-(\delta) < a_0(\delta) < a_+(\delta)$. Set $\mu(\delta) := f'(a_0(\delta))$; then in view of (1.8) we have $\mu(0) = \mu_0$. It follows from [1, Lemma 4.1] that there exists a constant $C_1 > 0$ such that

$$|\mu(\delta) - \mu_0| \le C_1\delta \quad \text{for all} \quad \delta \in (-\delta_0, \delta_0). \tag{3.37}$$

For each $\delta \in (-\delta_0, \delta_0)$, let $Y(\tau; \xi; \delta)$ be the unique solution of the ordinary differential equation

$$(ODE) \quad \begin{cases} Y_\tau = f(Y) + \delta, & \tau \ge 0, \\ Y(0; \xi; \delta) = \xi. \end{cases} \tag{3.38}$$

We have the following lemma.

Lemma 3.1. *Then there exist positive constants δ_0, C_2, C_3 such that*

(i) $|Y| \le C_2$,

(ii) $Y_\xi > 0$ *and* $|\dfrac{Y_{\xi\xi}}{Y_\xi}| \le C_3(e^{\mu(\delta)\tau} - 1)$,

for all $(\tau; \xi; \delta) \in [0, \infty) \times (-2C_0, 2C_0) \times (-\delta_0, \delta_0)$.

Proof. (i) follows from the fact that $Y(\tau; \xi; \delta)$ monotonically converges to an equilibrium point of (ODE) as $\tau \to \infty$. For a proof of (ii), see [1, lemmas 3.2 and 4.2]. □

The next lemma is a "representation" of the formation of interface for the solution of (P^ε) by its corresponding differential equation. In other words, it is an "ODE version" of Theorem 1.1.

Lemma 3.2 ([1, Lemma 4.7])**.** *Let $\eta > 0$ be arbitrarily small. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(\eta), C_4$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\xi \in (-2C_0, 2C_0)$, we have*

(i) *for all* $\xi \in (-2C_0, 2C_0)$,

$$-\eta \le Y(\mu_0^{-1}|\ln\varepsilon|; \xi; \pm\varepsilon) \le 1 + \eta,$$

(ii) *for all* $\xi \in (-2C_0, 2C_0)$ *satisfying* $|\xi - a| \ge C_4\varepsilon$, *we have*

$$\text{if } \xi \ge a + C_4\varepsilon, \text{ then } Y(\mu_0^{-1}|\ln\varepsilon|; \xi; \pm\varepsilon) \ge 1 - \eta,$$

$$\text{if } \xi \le a - C_4\varepsilon, \text{ then } Y(\mu_0^{-1}|\ln\varepsilon|; \xi; \pm\varepsilon) \le \eta.$$

3.1. **Constructing a pair of sub- and super-solutions: The case $\dfrac{\partial u_0}{\partial \nu} = 0$.** We are now ready to define a pair of sub- and super-solutions. Our pair of sub- and super-solutions has the form of

$$w^{\varepsilon,\pm}(x,t) := Y(\frac{t}{\varepsilon^2}; u_0(x) \pm C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1); \pm\varepsilon),$$

where C_5 is a positive constant which will be selected later.

Remark 3.1. In view of (3.37) and (1.8), we can prove that as $\varepsilon \to 0$,

$$\varepsilon e^{\mu(\varepsilon)\frac{t^\varepsilon}{\varepsilon^2}} - \varepsilon = e^{(\frac{\mu(\varepsilon)}{\mu_0}-1)|\ln\varepsilon|} - \varepsilon \to 1.$$

Thus, the monotonicity of the exponential function $e^{(\cdot)}$ implies, for ε_0 small enough,

$$0 < \varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1) \le 2\varepsilon \quad \text{for all} \quad t \in [0, t^\varepsilon] \text{ and every } \varepsilon \in (0, \varepsilon_0). \tag{3.39}$$

Hence the perturbation $C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1)$ is of order ε.

Lemma 3.3. *Assume that $\dfrac{\partial u_0}{\partial \nu} = 0$. Then there exist $\varepsilon_0 > 0$ and $C_5 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $w^{\varepsilon,\pm}$ is a pair of sub- and super-solutions of (P^ε) in the domain $\overline{\Omega} \times [0, t^\varepsilon]$.*

Proof. It is easy to see that $u_0 = w^{\varepsilon,+}(0)$ and that $\dfrac{\partial w^{\varepsilon,+}}{\partial \nu} = 0$. Next we claim that $\mathcal{L}(w^{\varepsilon,+}) \ge 0$ in $\overline{\Omega} \times [0, t^\varepsilon]$, where $\mathcal{L}$ is defined by

$$\mathcal{L}(u) := u_t - \frac{\Delta u}{m^2} - \frac{3}{2}\nabla(\frac{1}{m^2}) \cdot \nabla u - \frac{1}{2}u\Delta(\frac{1}{m^2}) - \frac{1}{\varepsilon^2}f(u). \tag{3.40}$$

Note that

$$w_t^{\varepsilon,+} = \frac{1}{\varepsilon^2}Y_\tau + C_5\mu(\varepsilon)e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}}Y_\xi,$$
$$\nabla w^{\varepsilon,+} = Y_\xi \nabla u_0, \qquad \Delta w^{\varepsilon,+} = Y_{\xi\xi}|\nabla u_0|^2 + Y_\xi \Delta u_0.$$

Thus

$$\mathcal{L}(w^{\varepsilon,+}) = \frac{1}{\varepsilon^2}\Big[Y_\tau - f(Y) - \frac{\varepsilon^2}{2}Y\Delta(\frac{1}{m^2})\Big] + Y_\xi\left[C_5\mu(\varepsilon)e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - \frac{|\nabla u_0|^2}{m^2}\frac{Y_{\xi\xi}}{Y_\xi} - \frac{\Delta u_0}{m^2} - \frac{3}{2}\nabla(\frac{1}{m^2})\nabla u_0\right].$$

Set

$$\widetilde{C} := \max_{\overline{\Omega}\times[0,1]}\left\{\frac{1}{2}\left|\Delta(\frac{1}{m^2})\right| + \frac{|\nabla u_0|^2}{m^2} + \frac{|\Delta u_0|}{m^2} + \frac{3}{2}\left|\nabla(\frac{1}{m^2})\nabla u_0\right|\right\}.$$

Then for ε_0 small enough such that $t^\varepsilon \le 1$ for every $\varepsilon \in (0, \varepsilon_0)$, we have

$$\mathcal{L}(w^{\varepsilon,+}) \ge \frac{1}{\varepsilon}\Big[1 - \varepsilon\widetilde{C}\,|Y|\Big] + Y_\xi\left[C_5\mu(\varepsilon)e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - \widetilde{C}\left|\frac{Y_{\xi\xi}}{Y_\xi}\right| - \widetilde{C}\right].$$

In view of the inequalities (3.39) and (1.18), we may again choose ε_0 small enough such that

$$|u_0(x) \pm C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1)| \le 2C_0 \text{ for all } t \in [0, t^\varepsilon], \text{ and every } \varepsilon \in (0, \varepsilon_0).$$

Therefore, by Lemma 3.1, we obtain

$$\mathcal{L}(w^{\varepsilon,+}) \ge \frac{1}{\varepsilon}\Big[1 - \varepsilon\widetilde{C}C_2\Big] + Y_\xi\left[C_5\mu(\varepsilon)e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - \widetilde{C}C_3e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - \widetilde{C}\right],$$

in the domain $\overline{\Omega} \times [0, t^\varepsilon]$. Therefore, we have

$$\begin{aligned}\mathcal{L}(w^{\varepsilon,+}) &\geq \frac{1}{\varepsilon}[1 - \varepsilon\widetilde{C}C_2] + Y_\xi \left[(\frac{C_5}{2}\mu_0 - \widetilde{C}C_3)e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - \widetilde{C} \right] \qquad \text{(by (3.37))}, \\ &\geq 0\end{aligned}$$

provided that ε_0 is small enough and that $C_5 \geq 2\mu_0^{-1}(C_3 + 1)\widetilde{C}$. Similarly, we can show that $w^{\varepsilon,-}$ is a sub-solution of (P^ε) which completes the proof. □

3.2. **Constructing a pair of sub- and super-solutions: The general case.** We will construct a pair of sub and super-solution of the form:

$$\hat{w}^{\varepsilon,\pm}(x,t) := Y(\frac{t}{\varepsilon^2}; u_0^\pm(x) \pm \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1); \pm\varepsilon),$$

where $u_0^\pm$ is a modification of u_0 such that $u_0^- \leq u_0 \leq u_0^+$ and $\partial_\nu u_0^\pm = 0$. Another essential property of $u_0^\pm$ which will be used in next section is that $u_0^\pm$ must have Γ_0 as their a-level sets. More precisely, we need the following properties:

(3.41)

$$\Gamma_0 = \{x \in \Omega : u_0^\pm(x) = a\}, \quad \Omega_0^{in} = \{x \in \Omega : u_0^\pm(x) > a\}, \quad \Omega_0^{ex} = \{x \in \Omega : u_0^\pm(x) < a\}.$$

To that purpose, let $d_0 > 0$ be small enough such that the function $\operatorname{dist}(x, \partial\Omega)$ is smooth in the set $\{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < 2d_0\}$ and that $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < 2d_0\} \cap \Gamma_0 = \emptyset$. We remark that the maximum of u_0 on the compact set $\{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq d_0\}$ is smaller than a and denote it by $a - \varrho$ for some constant $\varrho > 0$. Let $\chi : [0, \infty) \to \mathbf{R}$ is a smooth function satisfying $\chi(0) = \chi'(0) = 0$, $0 \leq \chi \leq 1$ and $\chi = 1$ on $[d_0, \infty)$. Set

$$\begin{aligned}u_0^+(x,t) &:= \chi(\operatorname{dist}(x, \partial\Omega))u_0 + [1 - \chi(\operatorname{dist}(x, \partial\Omega))](a - \varrho), \\ u_0^-(x,t) &:= \chi(\operatorname{dist}(x, \partial\Omega))u_0 + [1 - \chi(\operatorname{dist}(x, \partial\Omega))] \min_{\overline{\Omega}} u_0.\end{aligned}$$

Then $u^+ = u^- = u_0$ in the set $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq d_0\} \supset \Omega_0^{in}$. Moreover, $u_0^- \leq u_0 \leq u_0^+$ on $\overline{\Omega}$ and $u_0^\pm$ satisfies the homogeneous Neumann boundary condition due to the relations $\chi(0) = \chi'(0) = 0$. Thus, a similar argument as in Lemma 3.3 shows that $\hat{w}^{\varepsilon,\pm}$ is a pair sub- and super-solutions to (P^ε) in the domain $\overline{\Omega} \times [0, t^\varepsilon]$.

3.3. **Proof of Theorem 1.1.** First we prove inequalities involving u_0 and $u_0^\pm$.

Lemma 3.4. *Let M, ε be real positive numbers. Then*

(i) $u_0^-(x) \geq a + M\varepsilon$ *iff* $u_0(x) \geq a + M\varepsilon$;
(ii) *for all* $\varepsilon \in (0, \frac{\varrho}{M})$, $u_0^+(x) \leq a - M\varepsilon$ *iff* $u_0(x) \leq a - M\varepsilon$. *Here ϱ is given in Section 3.1.*

Proof. (i) If one of the two inequalities in (i) holds, then we have $x \in \Omega_0^{in}$. Thus (i) is trivial since $u_0^- = u_0$ in Ω_0^{in}. (ii) If $u_0(x) \leq a - M\varepsilon$, then, in view of the expression of u_0^+, we have

$$u_0^+(x) \leq \max\{u_0(x), a - \varrho\} \leq \max\{a - M\varepsilon, a - \varrho\} = a - M\varepsilon.$$

The inverse implication is trivial since $u_0 \leq u_0^+$. □

Now we turn to the proof of Theorem 1.1. Since $\hat{w}^{\varepsilon,\pm}$ is a pair of sub- and super-solutions of (P^ε), we have

(3.42)

$$Y(\frac{t}{\varepsilon^2}; u_0^-(x) - \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1); -\varepsilon) \leq u^\varepsilon(x,t) \leq Y(\frac{t}{\varepsilon^2}; u_0^+(x) + \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1); \varepsilon).$$

Note that $|u_0^{\pm}|_{C(\overline{\Omega})} \leq C_0$, so that by virtue of (3.39), we have for ε_0 small enough,

$$|u_0^{\pm}(x) \pm \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1)| \leq 2C_0 \quad \text{for all} \quad t \in [0, t^\varepsilon] \text{ and every } \varepsilon \in (0, \varepsilon_0).$$

Thus the first assertion (1.9) of Theorem 1.1 follows from Lemma 3.2 (i) and (3.42).

Next we prove the assertion (1.10). By (3.39), we have for ε_0 small enough,

$$u_0^-(x) - \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1) \geq u_0^-(x) - 2\hat{C}_5\varepsilon \quad \text{for all} \quad t \in [0, t^\varepsilon] \text{ and every } \varepsilon \in (0, \varepsilon_0).$$

Thus, if $u_0^-(x) \geq a + M\varepsilon$ with $M \geq C_4 + 2\hat{C}_5$, then

$$u_0^-(x) - \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon^2}} - 1) \geq a + C_4\varepsilon \quad \text{for all} \quad t \in [0, t^\varepsilon] \text{ and every } \varepsilon \in (0, \varepsilon_0).$$

Consequently, by Lemma 3.2 (ii), if $u_0^-(x) \geq a + M\varepsilon$ with $M \geq C_4 + 2\hat{C}_5$, then

$$Y(\frac{t^\varepsilon}{\varepsilon^2}; u_0^{\pm}(x) - \hat{C}_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t^\varepsilon}{\varepsilon^2}} - 1); -\varepsilon) \geq 1 - \eta,$$

and hence $u^\varepsilon(x, t^\varepsilon) \geq 1 - \eta$ due to (3.42). On the other hand, by Lemma 3.4, $u_0^-(x) \geq a + M\varepsilon$ is equivalent to $u_0(x) \geq a + M\varepsilon$ so that (1.10) follows. The assertion (1.11) can be treated by the same way. This completes the proof of Theorem 1.1.

4. Motion of interface

This section is devoted to the proof of Theorem 1.2. The idea of the proof is based on the observation given in Section 2.1, that the interface moves according to motion by mean curvature with drift and that the profile of the transition layer is well approximated by the expansion (2.22). Therefore, we will construct a pair of sub- and super-solutions $u^{\varepsilon,\pm}$ of the form

$$u^{\varepsilon,\pm}(x,t) \approx U_0\left(x, t, \frac{\bar{d}(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x, t, \frac{\bar{d}(x,t)}{\varepsilon}\right).$$

The construction of this pair of sub- and super-solution consists of several steps and will be presented in next subsections. More precisely, in Subsections 4.1 and 4.2, as a preparation, we first introduce the cut-off distance function $\hat{d}(x,t)$, and a corresponding function $\hat{U}_1$ and estimate them. Then in Subsection 4.3, we give the explicit form for sub-solutions, super-solutions and state the key lemma 4.7. The proof of Lemma 4.7 is presented in Subsection 4.4. Finally, we apply the key lemma Lemma 4.7 to prove Theorem 1.2 in Subsection 4.5.

4.1. **Modification of the signed distance function.** Next we introduce the cut-off function $\hat{d}(x,t)$, which coincides with $\bar{d}(x,t)$ near Γ_t and is constant near $\partial\Omega$ in order to take into account the Neuman boundary condition (4.57). Since the interface $\Gamma_{[0,T]}$ is $C^{2+\alpha,\frac{2+\alpha}{2}}$, it follows from [2, Theorems 1 and 2] that there exists $d_1 > 0$ small enough such that $\mathrm{dist}(\Gamma_t, \partial\Omega) \geq 3d_1$ and that $\overline{d}$ is $C^{2+\alpha,\frac{2+\alpha}{2}}$ in the tubular neighborhood $\{(x,t) \in \Omega \times [0,T] : \overline{d}(x,t) < 3d_1\}$ of $\Gamma_{[0,T]}$. Let $\zeta(s)$ be a smooth increasing function on $\mathbf{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq d_1, \\ -2d_1 & \text{if } s \leq -2d_1, \\ 2d_1 & \text{if } s \geq 2d_1. \end{cases} \tag{4.43}$$

We define the modified signed distance $\hat{d}$ by

$$\hat{d}(x,t) = \zeta(\overline{d}(x,t)).$$

Note that

$$\{(x,t)\in\Omega\times[0,T]:|\overline{d}(x,t)|<d_1\}=\{(x,t)\in\Omega\times[0,T]:|\hat{d}(x,t)|<d_1\},$$

and that $\hat{d}$ coincides with $\bar{d}$ in that region. As a consequence, we have

$$\hat{d}_t-\frac{\Delta\hat{d}}{m^2}-\nabla\hat{d}\cdot\nabla(\frac{1}{m^2})=0\quad\text{on}\quad\Gamma_t. \tag{4.44}$$

Moreover, $\hat{d}$ is constant near $\partial\Omega$ and the following properties hold.

Lemma 4.1. *There exists a constant $C_d>0$ such that*

(i) $|\hat{d}|+|\nabla\hat{d}|+|\Delta\hat{d}|\le C_d$,

(ii) $|\hat{d}_t-\dfrac{\Delta\hat{d}}{m^2}-\nabla\hat{d}\cdot\nabla(\dfrac{1}{m^2})|\le C_d|\hat{d}|$,

in $\overline{\Omega}\times[0,T]$.

Proof. (i) Since $|\hat{d}|$, $|\nabla\hat{d}|$, $|\Delta\hat{d}|$ are continuous in the compact set $\overline{\Omega}\times[0,T]$, they are uniformly bounded. Thus the first inequality follows.

(ii) In the region $\{|\overline{d}|<d_1\}$, the inequality follows from (4.44), the mean value theorem and the fact that $\hat{d}(x,t)=\bar{d}(x,t)\in C^{2+\alpha,\frac{2+\alpha}{2}}$ in this region. Next, in the region $\{|\overline{d}|\ge d_1\}$, we have $|\hat{d}|\ge d_1$ since ζ is increasing. Thus (ii) follows from the uniform boundedness of $|\hat{d}_t-\frac{\Delta\hat{d}}{m^2}-\nabla\hat{d}\cdot\nabla(\frac{1}{m^2})|(|\hat{d}|)^{-1}$ in that region. □

Lemma 4.2. *Given a positive real number $M'>0$. Then there exits $\varepsilon_0=\varepsilon_0(M')>0$ small enough such that the followings hold for every $\varepsilon\in(0,\varepsilon_0)$,*

$$\bar{d}(x,t)\ge M'\varepsilon \qquad\Leftrightarrow\qquad \hat{d}(x,t)\ge M'\varepsilon,$$
$$\bar{d}(x,t)\le -M'\varepsilon \qquad\Leftrightarrow\qquad \hat{d}(x,t)\le -M'\varepsilon.$$

Proof. The lemma follows from the fact that $\bar{d}=\hat{d}$ in the region $\{|\bar{d}|\le d_1\}=\{|\hat{d}|\le d_1\}$. We omit the details of the proof. □

4.2. **Estimates for the functions $U_0,\hat{\Phi}_1,\hat{U}_1$.** Recall that the function $U_0:\overline{\Omega}\times[0,T]\times\mathbf{R}\to\mathbf{R}$ is defined by

$$U_0(x,t,z)=\Phi_0(m(x,t)z).$$

We have the following lemma.

Lemma 4.3. *There exists a constant $\hat{C}_1>0$ such that*

(i) $|U_0|+|U_{0t}|+|\nabla U_0|+|\Delta U_0|\le\hat{C}_1$,

(ii) $|U_{0z}|+|U_{0zz}|\le\hat{C}_1\exp(-\lambda_1|z|)$,

(iii) $\nabla U_0\cdot\nu=0$ *on* $\partial\Omega\times[0,T]\times\mathbf{R}$,

for all $(x,t,z)\in\overline{\Omega}\times[0,T]\times\mathbf{R}$.

Proof. (i) and (ii) Let us compute the derivatives of U_0 with respect to t,x and z. We obtain

$$U_{0t}=m_tz\Phi_{0\tilde{z}},\quad \nabla U_0=z\Phi_{0\tilde{z}}\nabla m,\quad \Delta U_0=z\Phi_{0\tilde{z}}\Delta m+z^2\Phi_{0\tilde{z}\tilde{z}}|\nabla m|^2,$$
$$U_{0z}=m\Phi_{0\tilde{z}},\qquad U_{0zz}=m^2\Phi_{0\tilde{z}\tilde{z}}.$$

The above identities, the estimates in Lemma 2.1 and the uniform boundedness of m, $|\nabla m|$, Δm in $\overline{\Omega}\times[0,T]$ imply (i) and (ii).

(iii) The hypothesis $\dfrac{\partial m}{\partial\nu}=0$ implies

$$\nabla U_0\cdot\nu=z\Phi_{0\tilde{z}}\nabla m\cdot\nu=0,$$

which completes the proof of (iii). □

Consider the function $\hat{\Phi}_1 : \overline{\Omega} \times [0,T] \times \mathbf{R} \to \mathbf{R}$, which satisfies

$$\begin{cases} \hat{\Phi}_{1\tilde{z}\tilde{z}} + f'(\Phi_0)\hat{\Phi}_1 = m\left(\frac{1}{2}\Phi_{0\tilde{z}} + \tilde{z}\Phi_{0\tilde{z}\tilde{z}}\right)\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\hat{d}, \\ \hat{\Phi}_1(x,t,0) = 0. \end{cases} \tag{4.45}$$

For each $(x,t) \in \overline{\Omega} \times [0,T]$, the problem is solvable due to Lemma 2.1 and the identity (2.34).

Lemma 4.4. *For all* $(x,t,z) \in \overline{\Omega} \times [0,T] \times \mathbf{R}$*, we have*

(i) $\left|\hat{\Phi}_1\right| + \left|\hat{\Phi}_{1t}\right| + \left|\nabla\hat{\Phi}_1\right| + \left|\Delta\hat{\Phi}_1\right| \le \hat{C}_2$,

(ii) $\left|\hat{\Phi}_{1\tilde{z}}\right| + \left|\hat{\Phi}_{1\tilde{z}\tilde{z}}\right| + \left|\tilde{z}\hat{\Phi}_{1\tilde{z}\tilde{z}}\right| \le \hat{C}_2\exp(-\lambda_2|\tilde{z}|)$,

(iii) $\hat{\Phi}_1 = 0$ *and* $\nabla\hat{\Phi}_1 = 0$ *on* $\partial\Omega \times [0,T] \times \mathbf{R}$.

Proof. (i) According to Lemma 2.1, the functions $\Phi_{0\tilde{z}}$ and $\tilde{z}\Phi_{0\tilde{z}\tilde{z}}$ are uniformly bounded in $\mathbf{R}$. Furthermore, $\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\hat{d}$ is continuous in the compact set $\overline{\Omega}\times[0,T]$, hence uniformly bounded. Therefore, the right-hand-side in the first equation of (4.45) is uniformly bounded in $\overline{\Omega}\times[0,T]\times\mathbf{R}$. Thus, we deduce from Lemma 2.2 that $\hat{\Phi}_1$ is uniformly bounded.

Next we prove the boundedness of $\hat{\Phi}_{1t}$. Since Φ_0 does not depend on t, taking the derivative with respect to t in the first equation of (4.45) yields

$$\hat{\Phi}_{1t\tilde{z}\tilde{z}} + f'(\Phi_0)\hat{\Phi}_{1t} = \left(\frac{1}{2}\Phi_{0\tilde{z}} + \tilde{z}\Phi_{0\tilde{z}\tilde{z}}\right)\frac{\partial}{\partial t}\left(m\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\hat{d}\right), \tag{4.46}$$

so that applying the above arguments and Lemma 2.2 to the function $\hat{\Phi}_{1t}$, we deduce the uniform boundedness of $\hat{\Phi}_{1t}$ in $\overline{\Omega}\times[0,T]\times\mathbf{R}$. Similar results for $\nabla\hat{\Phi}_1$ and $\Delta\hat{\Phi}_1$ can be proven by the same way.

(ii) Assertion (ii) follows from Lemma 2.2 (ii).

(iii) Since $\hat{d}(x,t)$ is constant for all x near $\partial\Omega$ and all $t\in[0,T]$, it follows that

$$\hat{\Phi}_{1\tilde{z}\tilde{z}} + f'(\Phi_0)\hat{\Phi}_1 = 0, \tag{4.47}$$

for all x near $\partial\Omega$ and all $t \in [0,T], \tilde{z} \in \mathbf{R}$. Therefore, $\hat{\Phi}_1(x,t,\tilde{z})$ is identically zero for all x near $\partial\Omega$ and all $t\in[0,T], \tilde{z}\in\mathbf{R}$; hence (iii) follows. □

Set $\hat{U}_1(x,t,z) := \hat{\Phi}_1(x,t,mz)$; then, in view of (4.45), it is easy to check that

$$\frac{\hat{U}_{1zz}}{m^2} + f'(U_0)\hat{U}_1 = -\frac{1}{2}U_{0z}\nabla\left(\frac{1}{m^2}\right)\cdot\nabla\hat{d} - \frac{2}{m^2}\nabla U_{0z}\cdot\nabla\hat{d}. \tag{4.48}$$

The next lemma gives estimates for $\hat{U}_1$ and its derivatives.

Lemma 4.5. *For all* $(x,t,z)\in\overline{\Omega}\times[0,T]\times\mathbf{R}$*, we have*

(i) $\left|\hat{U}_1\right| + \left|\hat{U}_{1t}\right| + \left|\nabla\hat{U}_1\right| + \left|\Delta\hat{U}_1\right| \le \hat{C}_3$,

(ii) $\left|\hat{U}_{1z}\right| + \left|\hat{U}_{1zz}\right| \le \hat{C}_3\exp(-\lambda_3|z|)$,

(iii) $\nabla\hat{U}_1\cdot\nu = 0$ *on* $\partial\Omega\times[0,T]\times\mathbf{R}$.

The above lemma follows from a similar argument in the proof of Lemma 4.3; we omit its proof.

4.3. **The form of the pair sub- and super-solutions.** In order to define a pair of sub- and super-solutions, we first introduce a constant β which only depends on f. Fix $b > 0$ small enough such that $f'(s) < 0$ on $[0, b] \cup [1 - b, 1]$; set

$$-\beta := \sup \left\{ \frac{f'(s)}{3} : s \in [0, b] \cup [1 - b, 1] \right\}. \tag{4.49}$$

We define two positive constants $\underline{m}, \overline{m}$ by

$$\underline{m} := \inf_{\overline{\Omega} \times [0,T]} m, \qquad \overline{m} := \sup_{\overline{\Omega} \times [0,T]} m. \tag{4.50}$$

The following result plays an important role in the proof of Lemma 4.7 below.

Lemma 4.6. *Let β be given by* (4.49)*. Then there exists a constant σ_0 small enough such that for every $0 \le \sigma < \sigma_0$, we have*

$$U_{0z} + \sigma f'(U_0) \le -3\sigma\beta \quad \text{in } \overline{\Omega} \times [0, T] \times \mathbf{R}.$$

Proof. The assertion is trivial when $\sigma = 0$. Next consider the case $\sigma > 0$. Since

$$U_{0z} + \sigma f'(U_0) = m\Phi_{0\tilde{z}} + \sigma f'(\Phi_0) \le \underline{m}\Phi_{0\tilde{z}} + \sigma f'(\Phi_0),$$

it is sufficient to show that there exists $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$,

$$\frac{\underline{m}\Phi_{0\tilde{z}}(\tilde{z})}{\sigma} + f'(\Phi_0(\tilde{z})) \le -3\beta \quad \text{for all} \quad \tilde{z} \in \mathbf{R}. \tag{4.51}$$

Note that since $0 \le \Phi_0(\tilde{z}) \le 1$ for all $\tilde{z} \in \mathbf{R}$, we can write $\mathbf{R}$ as $\mathbf{R} = J_1 \cup J_2$ with

$$J_1 := \{\tilde{z} : \Phi_0(\tilde{z}) \in [0, b] \cup [1 - b, 1]\}, \quad \text{and} \quad J_2 := \{\tilde{z} : \Phi(z) \in [b, 1 - b]\}.$$

Hence we need to prove (4.51) for $\tilde{z} \in J_1$ and $\tilde{z} \in J_2$:

The case that $\tilde{z} \in J_1$: On the set $J_1 := \{\tilde{z} : \Phi_0(\tilde{z}) \in [0, b] \cup [1 - b, 1]\}$, we have that by (4.49), for all $\sigma > 0$,

$$\sup_{\tilde{z} \in J_1} \left(\frac{\underline{m}\Phi_{0\tilde{z}}}{\sigma} + f'(\Phi_0) \right) \le \sup_{\tilde{z} \in J_1} f'(\Phi_0) = -3\beta,$$

where we have used the property $\Phi_{0\tilde{z}} < 0$.

The case that $\tilde{z} \in J_2$: On the compact set $J_2 := \{\tilde{z} : \Phi(z) \in [b, 1 - b]\}$, we have

$$\sup_{\tilde{z} \in J_2} \left(\frac{\underline{m}\Phi_{0\tilde{z}}}{\sigma} + f'(\Phi_0) \right) \le \frac{\underline{m} \sup_{\tilde{z} \in J_2} \Phi_{0\tilde{z}}}{\sigma} + \sup_{s \in [b, 1-b]} f'(s),$$

so that

$$\lim_{\sigma \to 0^+} \sup_{\tilde{z} \in J_2} \left(\frac{\underline{m}\Phi_{0\tilde{z}}}{\sigma} + f'(\Phi_0) \right) \le \lim_{\sigma \to 0^+} \left(\frac{\underline{m} \sup_{\tilde{z} \in J_2} \Phi_{0\tilde{z}}}{\sigma} + \sup_{s \in [b, 1-b]} f'(s) \right) = -\infty,$$

which implies (4.51). Thus we complete the proof of Lemma 4.6. □

We define $u^{\varepsilon,\pm}$ as follows

$$u^{\varepsilon,\pm}(x,t) = U_0 \left(x, t, \frac{\hat{d}(x,t) \pm \varepsilon p(t)}{\varepsilon} \right) + \varepsilon \hat{U}_1 \left(x, t, \frac{\hat{d}(x,t) \pm \varepsilon p(t)}{\varepsilon} \right) \pm q(t), \tag{4.52}$$

where

$$\begin{aligned} p(t) &= e^{-\frac{\beta t}{\varepsilon^2}} - e^{Lt} - K, \\ q(t) &= \sigma(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}). \end{aligned} \tag{4.53}$$

Here β is defined in (4.49) and the positive constants L, K, σ will selected later. Note that

$$p_t = -\frac{q}{\sigma\varepsilon^2}, \quad |p(t)| \le 1 + e^{LT} + K \ \text{ for } \ t \in [0,T]. \tag{4.54}$$

Let define two constants σ_1, σ_2 by

$$\sigma_1 := \frac{1}{2(\beta+1)}, \qquad \sigma_2 := \frac{2}{(\beta+1)\sup_{s\in[-1,2]}|f''(s)|} \tag{4.55}$$

and recall the definition of operator $\mathcal{L}$ in (3.40):

$$\mathcal{L}(u) := u_t - \frac{\Delta u}{m^2} - \frac{3}{2}\nabla(\frac{1}{m^2}) \cdot \nabla u - \frac{1}{2}u\Delta(\frac{1}{m^2}) - \frac{1}{\varepsilon^2}f(u).$$

Lemma 4.7. *Let β be given by (4.49) and fix $\eta \in (0, \eta_0)$ and $0 < \sigma < \min\{\sigma_0, \sigma_1, \sigma_2\}$. Then for each $K > 0$, there exist $L > 0$ large enough and ε_0 small enough such that*

$$\mathcal{L}(u^{\varepsilon,-}) \le 0 \le \mathcal{L}(u^{\varepsilon,+}) \qquad in \quad \overline{\Omega} \times [0,T], \tag{4.56}$$

$$\frac{\partial u^{\varepsilon,-}}{\partial \nu} = \frac{\partial u^{\varepsilon,+}}{\partial \nu} = 0 \qquad on \quad \partial\overline{\Omega} \times [0,T], \tag{4.57}$$

for every $\varepsilon \in (0, \varepsilon_0)$.

The proof of the lemma will be presented in next subsection. We first make some remarks. We will proceed with the proof under the assumptions that

$$\varepsilon_0^2 L e^{LT} \le 1, \qquad \varepsilon_0 \hat{C}_3 \le \frac{1}{2}, \tag{4.58}$$

where $\hat{C}_3$ is given in Lemma 4.5. Therefore, it follows that

$$0 < q(t) \le \sigma(\beta+1) \ \text{ in } \ [0,T]. \tag{4.59}$$

The assumption $\sigma < \sigma_1$ implies that $|q(t)| < \frac{1}{2}$. Consequently, we have

$$-1 \le u^{\varepsilon,\pm} \le 2 \ \text{ in } \ \overline{\Omega} \times [0,T]. \tag{4.60}$$

4.4. **Proof of Lemma 4.7.** First we prove (4.57). Since $\hat{d}$ is constant near $\partial\Omega$, we have $\frac{\partial \hat{d}}{\partial \nu} = 0$ on $\partial\Omega \times [0,T]$. On the other hand, by Lemmas 4.3 (iii) and 4.5 (iii),

$$\frac{\partial U_0}{\partial \nu} = \frac{\partial U_1}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \times (0,T).$$

Therefore

$$\frac{\partial u^{\varepsilon,+}}{\partial \nu} = \nabla u^{\varepsilon,+} \cdot \nu = \nabla U_0 \cdot \nu + U_{0z}\frac{\nabla \hat{d}}{\varepsilon} \cdot \nu + \varepsilon \nabla \hat{U}_1 \cdot \nu + \hat{U}_{1z}\nabla \hat{d} \cdot \nu = 0 \ \text{ on } \ \partial\Omega \times (0,T).$$

Similarly, we have $\frac{\partial u^{\varepsilon,-}}{\partial \nu} = 0$ on $\partial\Omega \times [0,T]$.

In the remaining part of this subsection, we will prove the inequality $\mathcal{L}(u^{\varepsilon,+}) \ge 0$. The inequality $\mathcal{L}(u^{\varepsilon,-}) \le 0$ can be shown by the same argument. The main idea of the proof is to expand $\mathcal{L}(u^{\varepsilon,+})$ using the form of $u^{\varepsilon,+}$ in (4.52), collect terms of the same order (e.g., $\varepsilon^{-2}, \varepsilon^{-1}$), and then estimate them. First we have

$$\begin{aligned} u_t^{\varepsilon,+} &= U_{0t} + U_{0z}(\frac{\hat{d}_t}{\varepsilon} + p_t) + \varepsilon\hat{U}_{1t} + \hat{U}_{1z}(\hat{d}_t + \varepsilon p_t) + q_t \\ &= \Big[U_{0z}p_t + q_t\Big] + \Big[U_{0z}\frac{\hat{d}_t}{\varepsilon} + \hat{U}_{1z}\varepsilon p_t\Big] + \Big[U_{0t} + \hat{U}_{1z}\hat{d}_t + \varepsilon\hat{U}_{1t}\Big], \end{aligned}$$

where p_t, q_t are terms of order ε^{-2} (see the relation (4.54)). A straight forward computation yields

$$\begin{aligned}
\nabla u^{\varepsilon,+} &= \nabla U_0 + U_{0z}\frac{\nabla \hat{d}}{\varepsilon} + \varepsilon\nabla \hat{U}_1 + \hat{U}_{1z}\nabla \hat{d},\\
\Delta u^{\varepsilon,+} &= \Delta U_0 + \frac{2\nabla U_{0z}\cdot\nabla \hat{d} + U_{0z}\Delta \hat{d}}{\varepsilon} + \frac{U_{0zz}|\nabla \hat{d}|^2}{\varepsilon^2}\\
&\qquad + \varepsilon\Delta\hat{U}_1 + 2\nabla\hat{U}_{1z}\cdot\nabla\hat{d} + \hat{U}_{1z}\Delta\hat{d} + \frac{\hat{U}_{1zz}|\nabla\hat{d}|^2}{\varepsilon},\\
f(u^{\varepsilon,+}) &= f(U_0) + (\varepsilon\hat{U}_1 + q)f'(U_0) + \frac{1}{2}(\varepsilon\hat{U}_1+q)^2 f''(\theta)\\
&= \Big[f(U_0) + qf'(U_0) + \frac{q^2}{2}f''(\theta)\Big] + \Big[\varepsilon\hat{U}_1 f'(U_0) + \varepsilon\hat{U}_1 q f''(\theta)\Big] + \frac{\varepsilon^2(\hat{U}_1)^2 f''(\theta)}{2}.
\end{aligned}$$

Here we have used the Taylor expansion in the expression of $f(u^{\varepsilon,+})$, and we remark that in view of (4.60), $\theta = \theta(x,t) \in (-1,2)$. Next we write $\mathcal{L}(u^{\varepsilon,+})$ in the form

$$\mathcal{L}(u^{\varepsilon,+}) = S_1 + R_1 + S_2 + R_2 + R_3 + R_4,$$

where S_1, S_2, inspired from Section 2, are of order $\varepsilon^{-2}, \varepsilon^{-1}$, respectively; R_1, R_2 are the remaining terms of order $\varepsilon^{-2}, \varepsilon^{-1}$ which were neglected in the formal computations in Section 2 and R_3, R_4 are terms of order 1. More precisely,

$$\begin{aligned}
S_1 &:= \frac{1}{\varepsilon^2}\left[-\frac{U_{0zz}}{m^2}|\nabla\hat{d}|^2 - f(U_0)\right],\\
R_1 &:= \Big[U_{0z}p_t + q_t\Big] - \frac{1}{\varepsilon^2}\Big[qf'(U_0) + \frac{q^2}{2}f''(\theta)\Big],\\
S_2 &:= U_{0z}\frac{\hat{d}_t}{\varepsilon} - \frac{2\nabla U_{0z}\cdot\nabla\hat{d} + U_{0z}\Delta\hat{d}}{m^2\varepsilon} - \frac{\hat{U}_{1zz}|\nabla\hat{d}|^2}{m^2\varepsilon} - \frac{3}{2}U_{0z}\frac{\nabla\hat{d}}{\varepsilon}\cdot\nabla\left(\frac{1}{m^2}\right) - \frac{1}{\varepsilon}\hat{U}_1 f'(\hat{U}_0),\\
R_2 &:= \hat{U}_{1z}\varepsilon p_t - \frac{1}{\varepsilon}\hat{U}_1 q f''(\theta),\\
R_3 &:= \Big[U_{0t} + \hat{U}_{1z}\hat{d}_t + \varepsilon\hat{U}_{1t}\Big] - \frac{1}{m^2}\Big[\Delta U_0 + \varepsilon\Delta U_1 + 2\nabla\hat{U}_{1z}\cdot\nabla\hat{d} + \hat{U}_{1z}\Delta\hat{d}\Big],\\
R_4 &:= -\frac{3}{2}\Big[\nabla U_0 + \varepsilon\nabla\hat{U}_1 + \hat{U}_{1z}\nabla\hat{d}\Big]\cdot\nabla\left(\frac{1}{m^2}\right) - \frac{1}{2}(u^{\varepsilon,+})\Delta\left(\frac{1}{m^2}\right) - \frac{(\hat{U}_1)^2 f''(\theta)}{2},
\end{aligned}$$

In what follows, we will estimate the terms above in the domain $\overline{\Omega}\times[0,T]$. Note that R_1 and R_2 play an important role to deduce the sign for $\mathcal{L}(u^{\varepsilon,+})$ and that the values of $U_0, \hat{U}_1$ are computed at the point $(x, t, \frac{\hat{d}(x,t)+\varepsilon p(t)}{\varepsilon})$.

Estimate of the term $\mathbf{S_1}$

Using (2.28), we write S_1 in the form

$$S_1 = -\frac{1}{\varepsilon^2}\frac{U_{0zz}}{m^2}\left(|\nabla\hat{d}|^2 - 1\right).$$

In the region where $|\hat{d}| \le d_1$, we have $|\nabla\hat{d}| = 1$ so that $S_1 = 0$. On the other hand, in the region where $|\hat{d}| \ge d_1$, we have (cf. Lemma 4.3 (ii) and (4.54)),

$$\frac{|U_{0zz}|}{\varepsilon^2} \le \frac{\hat{C}_1}{\varepsilon^2}e^{-\lambda_1\left|\frac{\hat{d}}{\varepsilon}+p(t)\right|} \le \frac{\hat{C}_1}{\varepsilon^2}e^{-\lambda_1[\frac{d_1}{\varepsilon}-|p(t)|]} \le \frac{\hat{C}_1}{\varepsilon^2}e^{-\lambda_1[\frac{d_1}{\varepsilon}-(1+e^{LT}+K)]}.$$

Choosing ε_0 small enough such that

$$\frac{d_1}{2\varepsilon_0} - (1 + e^{LT} + K) \geq 0, \tag{4.61}$$

we deduce that

$$\frac{|U_{0zz}|}{\varepsilon^2} \leq \frac{\hat{C}_1}{\varepsilon^2} e^{-\lambda_1 \frac{d_1}{2\varepsilon}} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus, for ε_0 small enough, $\dfrac{|U_{0zz}|}{\varepsilon^2}$ is uniformly bounded in the region $\{|\hat{d}| \geq d_1\}$, hence so is S_1. Consequently, there exists a constant $\tilde{C}_1$ independent of ε, L such that

$$|S_1| \leq \tilde{C}_1 \quad \text{in} \quad \overline{\Omega} \times [0, T] \tag{4.62}$$

provide that ε_0 satisfies (4.61).

<u>Estimate of the term $\mathbf{R_1}$</u>

Substituting $p_t = -\dfrac{q}{\sigma\varepsilon^2}$ and then replacing q by its explicit form (4.53), we obtain

$$\begin{aligned} R_1 &= \frac{q}{\sigma\varepsilon^2}\Big[- U_{0z} - \sigma f'(U_0) - \frac{q\sigma}{2} f''(\theta)\Big] + q_t \\ &= \frac{1}{\varepsilon^2}(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt})\Big[- U_{0z} - \sigma f'(U_0) - \frac{\sigma^2}{2}(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}) f''(\theta)\Big] - \frac{1}{\varepsilon^2}\sigma\beta^2 e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 \sigma L^2 e^{Lt} \\ &= \frac{1}{\varepsilon^2}\beta e^{-\frac{\beta t}{\varepsilon^2}}(I - \sigma\beta) + L e^{Lt}[I + \varepsilon^2 \sigma L], \end{aligned}$$

where

$$I := -U_{0z} - \sigma f'(U_0) - \frac{\sigma^2}{2}(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}) f''(\theta).$$

Lemma 4.6 and the hypothesis $\sigma < \sigma_2$ (cf. (4.55)) yield

$$I \geq 3\sigma\beta - \frac{\sigma^2}{2}(\beta + 1)|f''(\theta)| \geq 2\sigma\beta,$$

so that

$$R_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\frac{\beta t}{\varepsilon^2}} + 2\sigma\beta L e^{Lt}. \tag{4.63}$$

<u>Estimate of the term $\mathbf{S_2}$</u>

Using (4.48), we have

$$\begin{aligned} S_2 = &\frac{U_{0z}}{\varepsilon}\left[\hat{d}_t - \frac{\Delta \hat{d}}{m^2} - \nabla \hat{d} \cdot \nabla \left(\frac{1}{m^2}\right)\right] \\ &- \left[\frac{2\nabla U_{0z} \cdot \nabla \hat{d}}{m^2 \varepsilon} + \frac{1}{2} U_{0z} \frac{\nabla \hat{d}}{\varepsilon} \cdot \nabla \left(\frac{1}{m^2}\right) + \frac{\hat{U}_{1zz}|\nabla \hat{d}|^2}{m^2 \varepsilon} + \frac{1}{\varepsilon}\hat{U}_1 f'(U_0)\right] \\ = &\frac{U_{0z}}{\varepsilon}\left[\hat{d}_t - \frac{\Delta \hat{d}}{m^2} - \nabla \hat{d} \cdot \nabla \left(\frac{1}{m^2}\right)\right] - \frac{1}{\varepsilon}\frac{\hat{U}_{1zz}}{m^2}\left[|\nabla \hat{d}|^2 - 1\right] := S_{2a} + S_{2b}. \end{aligned} \tag{4.64}$$

The second term of (4.64) (denoted by S_{2b}) can be estimated by using a similar argument for the term S_1. We deduce that there exist ε_0 small enough and $\tilde{C}_2$ independent of ε, L such that

$$|S_{2b}| \leq \tilde{C}_2 \quad \text{in} \quad \overline{\Omega} \times [0, T]. \tag{4.65}$$

Next we estimate the first term in the right-hand-side of (4.64) which we denote by S_{2a}. By Lemma 4.1, we have

$$|\hat{d}_t - \frac{\Delta \hat{d}}{m^2} - \nabla \hat{d} \cdot \nabla \left(\frac{1}{m^2}\right)| \leq C_d |\hat{d}|$$

It follows that

$$|S_{2a}| \leq C_d |\hat{d}| \frac{|U_{0z}|}{\varepsilon} \leq C_d \hat{C}_1 \frac{|\hat{d}|}{\varepsilon} e^{-\lambda_1 |\frac{\hat{d}}{\varepsilon} + p|} \leq C_d \hat{C}_1 \max_{\xi \in \mathbf{R}} |\xi| e^{-\lambda_1 |\xi + p|}.$$

An elementary observation shows that the function $g(\xi) := |\xi| e^{-\lambda_1 |\xi + p|}$ satisfies

$$\max_{\xi \in \mathbf{R}} g(\xi) \leq \max\{g(-p), g(\frac{1}{\lambda_1}), g(-\frac{1}{\lambda_1})\} \leq \max\{|p|, \frac{1}{\lambda_1}\} \leq |p| + \frac{1}{\lambda_1}.$$

Thus,

$$|S_{2a}| \leq C_d \hat{C}_1 (|p| + \frac{1}{\lambda_1}) \leq C_d \hat{C}_1 (1 + K + e^{Lt} + \frac{1}{\lambda_1}), \tag{4.66}$$

so that

$$|S_{2a}| \leq \tilde{C}_3 e^{Lt} + \tilde{C}_4, \tag{4.67}$$

where $\tilde{C}_3 := C_d \hat{C}_1 (1 + K + \frac{1}{\lambda_1}), \tilde{C}_4 := C_d \hat{C}_1$.

Estimate of the term $\mathbf{R_2}$

Substituting $p_t = -\dfrac{q}{\sigma \varepsilon^2}$ and then replacing q by its explicit form (4.53), we obtain

$$\begin{aligned} R_2 &= \frac{q}{\varepsilon}\Big[- \hat{U}_{1z} - \sigma \hat{U}_1 f''(\theta) \Big] \\ &= \frac{1}{\varepsilon} \sigma (\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}) \Big[- \hat{U}_{1z} - \sigma \hat{U}_1 f''(\theta) \Big]. \end{aligned}$$

Since the last factor in the above expression is uniformly bounded in $\overline{\Omega} \times [0, T]$, it follows that there exists a constant $\tilde{C}_5$ such that

$$|R_2| \leq \tilde{C}_5 \Big[\frac{\beta}{\varepsilon} e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon L e^{Lt} \Big]. \tag{4.68}$$

Estimate of the terms $\mathbf{R_3}$ and $\mathbf{R_4}$

It is easy to see that all the terms in the expressions of R_3, R_4 are bounded, so that there exists a constant $\tilde{C}_6$ such that

$$|R_3| + |R_4| \leq \tilde{C}_6. \tag{4.69}$$

Combination of the above estimates

Collecting the estimates (4.62), (4.63), (4.65), (4.67), (4.68), (4.69), we obtain

$$\begin{aligned} \mathcal{L}(u^{\varepsilon,+}) &\geq \left[\frac{\sigma \beta^2}{\varepsilon^2} - \tilde{C}_5 \frac{\beta}{\varepsilon} \right] e^{-\frac{\beta t}{\varepsilon^2}} + \Big[2\sigma\beta L - \varepsilon \tilde{C}_5 L - \tilde{C}_3 \Big] e^{Lt} - \tilde{C}_1 - \tilde{C}_2 - \tilde{C}_4 - \tilde{C}_6 \\ &= \left[\frac{\sigma \beta^2}{\varepsilon^2} - \tilde{C}_5 \frac{\beta}{\varepsilon} \right] e^{-\frac{\beta t}{\varepsilon^2}} + \left[\frac{2\sigma\beta}{3} L - \varepsilon \tilde{C}_5 L \right] e^{Lt} \qquad (4.70) \\ &\quad + \left[\frac{2\sigma\beta}{3} L - \tilde{C}_3 \right] e^{Lt} + \left[\frac{2\sigma\beta}{3} L e^{Lt} - \tilde{C}_7 \right], \qquad (4.71) \end{aligned}$$

where $\tilde{C}_7 := \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_4 + \tilde{C}_6$. We first choose L large enough such that the terms in (4.71) are positive, then we choose ε_0 small enough (which satisfies (4.58), (4.61)) such that the two terms in (4.70) are positive, thus we obtain $\mathcal{L}(u^{\varepsilon,+}) \geq 0$. We complete the proof of Lemma 4.7.

4.5. **Proof of Theorem 1.2.** Let us briefly explain the idea of the proof. As we see in the proof of Theorem 1.1, the solution u^ε is sandwiched between the pair of sub- and super-solutions $\hat{w}^{\varepsilon,\pm}$ and develops steep transition layers at the time t^ε. In this section, by choosing an appropriate constant K in the expression (4.52) of $u^{\varepsilon,\pm}$, we will show that $\hat{w}^{\varepsilon,\pm}$ at the time t^ε is sandwiched between the sub-solution and super-solution $u^{\varepsilon,\pm}$ at the time $t = 0$, namely

$$u^{\varepsilon,-}(x,0) \leq \hat{w}^{\varepsilon,-}(x,t^\varepsilon) \leq u^\varepsilon(x,t^\varepsilon) \leq \hat{w}^{\varepsilon,+}(x,t^\varepsilon) \leq u^{\varepsilon,+}(x,0). \tag{4.72}$$

Therefore, Lemma 4.7 and the comparison principle implies the results in Theorem 1.2.

We need the following auxiliary results.

Lemma 4.8. *Given $M > 0$, then there exist M', ε_0 such that*

$$\bar{d}(x,0) \geq M'\varepsilon \quad \textit{implies} \quad u_0(x) \leq a - M\varepsilon \tag{4.73}$$

$$\bar{d}(x,0) \leq -M'\varepsilon \quad \textit{implies} \quad u_0(x) \geq a + M\varepsilon, \tag{4.74}$$

for all $x \in \overline{\Omega}$ and every $\varepsilon \in (0, \varepsilon_0)$.

Proof. We only prove (4.73). We deduce from (1.6) and (1.7) that $\dfrac{\partial u_0}{\partial n} < 0$ on Γ_0. We can choose d^* small enough such that in the neighborhood $V := \{x \in \Omega : |\bar{d}(x,0)| < d^*\}$ of Γ_0 there holds

$$-k := \inf_{x \in V} \frac{\partial u_0}{\partial n} < 0. \tag{4.75}$$

We consider two regions: $\{x \in \Omega : M'\varepsilon \leq \bar{d}(x,0) \leq d^*\}$ and $\{x \in \Omega : \bar{d}(x,0) \geq d^*\}$.

Case 1. In the region $\{x \in \Omega : \bar{d}(x,0) \geq d^*\}$, we have

$$\tilde{a} := \sup\{u_0(x) : x \in \Omega, \ \ |\bar{d}(x,0)| \geq d^*\} < a.$$

Choosing $\varepsilon_0 \leq (a - \tilde{a})/M$, we deduce that $u_0(x) \leq \tilde{a} \leq a - M\varepsilon$.

Case 2. Consider $\{x \in \Omega : M'\varepsilon \leq \bar{d}(x,0) \leq d^*\}$. By the mean value theorem, we have

$$u_0(x) - a = \bar{d}(x,0) \frac{\partial u_0}{\partial n}(\theta) \ \text{ for some } \ \theta \in V.$$

Therefore, using (4.75), we obtain $u_0(x) \leq a - kM'\varepsilon \leq a - M\varepsilon$, provide that $M' \geq \dfrac{M}{k}$. □

Remark 4.1. The above lemma implies that $\{x \in \Omega : |u_0(x) - a| \leq M\varepsilon\} \subset \{x \in \Omega : |\bar{d}(x,0)| \leq M'\varepsilon\}$, so that its thickness is of order ε.

The proof of Theorem 1.2 is divided in two steps.

Step 1. We will prove that

$$u^{\varepsilon,-}(x,t) \leq u^\varepsilon(x, t^\varepsilon + t) \leq u^{\varepsilon,+}(x,t) \ \text{ for all } \ x \in \overline{\Omega}, \quad t \in [0, T - t^\varepsilon]. \tag{4.76}$$

We apply Theorem 1.1 by replacing η by $\dfrac{\sigma\beta}{3}$ and let M be the corresponding constant. Lemmas 4.8 and 4.2 imply that there exists $M' > 0$, ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\hat{d}(x,0) \geq M'\varepsilon \quad \text{implies} \quad u_0(x) \leq a - M\varepsilon,$$

$$\hat{d}(x,0) \leq -M'\varepsilon \quad \text{implies} \quad u_0(x) \geq a + M\varepsilon,$$

where we recall that $\hat{d}$ is the modified signed distance to $\Gamma_{[0,T]}$. It follows from (1.13) that

$$-\frac{\sigma\beta}{3} \le u^\varepsilon(x,t^\varepsilon) \le 1+\frac{\sigma\beta}{3} \quad \text{for all} \quad x\in\Omega.$$

We deduce from (1.15) that

$$u^\varepsilon(x,t^\varepsilon) \le v^+(x) := \begin{cases} 1+\dfrac{\sigma\beta}{3} & \text{if} \quad \hat{d}(x,0)\le M'\varepsilon \\ \dfrac{\sigma\beta}{3} & \text{if} \quad \hat{d}(x,0) > M'\varepsilon. \end{cases} \tag{4.77}$$

Next we apply (1.14) to deduce that

$$u^\varepsilon(x,t^\varepsilon) \ge v^-(x) := \begin{cases} 1-\dfrac{\sigma\beta}{3} & \text{if} \quad \hat{d}(x,0) < -M'\varepsilon \\ -\dfrac{\sigma\beta}{3} & \text{if} \quad \hat{d}(x,0) \ge -M'\varepsilon. \end{cases} \tag{4.78}$$

Next we show that

$$u^{\varepsilon,-}(x,0) \le v^-(x) \quad \text{and} \quad v^+(x)\le u^{\varepsilon,+}(x,0). \tag{4.79}$$

Let $\overline{m}, \underline{m}$ are defined in (4.50). We have

$$\begin{aligned} u^{\varepsilon,+}(x,0) &= U_0(x,0,\frac{\hat{d}(x,0)}{\varepsilon}-K) + \varepsilon\hat{U}_1(x,0,\frac{\hat{d}(x,0)}{\varepsilon}-K) + \sigma(\beta+\varepsilon^2 L) \\ &= \Phi_0(m(x,0)\frac{\hat{d}(x,0)}{\varepsilon} - m(x,0)K) + \varepsilon\hat{U}_1(x,0,\frac{\hat{d}(x,0)}{\varepsilon}-K) + \sigma(\beta+\varepsilon^2 L) \\ &\ge \Phi_0(\overline{m}\frac{\hat{d}(x,0)}{\varepsilon} - \underline{m}K) + \varepsilon\hat{U}_1(x,0,\frac{\hat{d}(x,0)}{\varepsilon}-K) + \sigma\beta, \end{aligned} \tag{4.80}$$

where we have used the monotonicity of Φ_0 in the last inequality.

Let $\hat{C}_3$ be defined in Lemma 4.5 and choose ε_0 such that $\varepsilon_0\hat{C}_3 \le \frac{\sigma\beta}{3}$. Then

$$\varepsilon\hat{U}_1(x,0,\frac{\hat{d}(x,0)}{\varepsilon}-K) \le \frac{\sigma\beta}{3}.$$

The inequality (4.80) and the positivity of Φ_0 imply that

$$u^{\varepsilon,+}(x,0) \ge \frac{2\sigma\beta}{3} \quad \text{for all} \quad x\in\overline{\Omega}. \tag{4.81}$$

On the other hand, since $\Phi_0(-\infty)=1$, we may choose K large enough such that

$$\Phi_0(\overline{m}M' - \underline{m}K) \ge 1-\frac{\sigma\beta}{3}.$$

As a consequence, using (4.80) and (4.77), we obtain

$$u^{\varepsilon,+}(x,0) \ge \Phi_0(\overline{m}M' - \underline{m}K) + \frac{2\sigma\beta}{3} \ge 1+\frac{\sigma\beta}{3} \ge v^+(x)$$

for all $x\in\overline{\Omega}$ such that $\hat{d}(x,0)\le M'\varepsilon$ and all $\varepsilon\in(0,\varepsilon_0)$. This together with (4.81) implies the second inequality in (4.79). The first inequality in (4.79) can be proven in a similar way. Therefore,

$$u^{\varepsilon,-}(x,0) \le u^\varepsilon(x,t^\varepsilon) \le u^{\varepsilon,+}(x,0) \quad \text{for all} \quad x\in\overline{\Omega},$$

which together with Lemma 4.7 and the comparison principle imply that

$$u^{\varepsilon,-}(x,t) \le u^\varepsilon(x,t^\varepsilon+t) \le u^{\varepsilon,+}(x,t) \quad \text{for all} \quad x\in\overline{\Omega}, \quad t\in[0,T-t^\varepsilon].$$

Step 2. Choose ε_0, σ small enough such that

$$\varepsilon_0 \hat{C}_3 + \sigma(\beta + \varepsilon_0^2 L e^{LT}) \leq \frac{\eta}{2}. \tag{4.82}$$

Then it is easy to check that

$$-\frac{\eta}{2} \leq u^{\varepsilon,-}(x,t) \leq u^\varepsilon(x, t^\varepsilon + t) \leq u^{\varepsilon,+}(x,t) \leq 1 + \frac{\eta}{2} \text{ for all } (x,t) \in \Omega \times [0, T - t^\varepsilon],$$

which implies (1.13). It remains to claim that there exists a constant $C > 0$ such that

$$\bar{d}(x, t + t^\varepsilon) \geq C\varepsilon \quad \text{implies} \quad u^\varepsilon(x, t + t^\varepsilon) \leq \eta,$$
$$\bar{d}(x, t + t^\varepsilon) \leq -C\varepsilon \quad \text{implies} \quad u^\varepsilon(x, t + t^\varepsilon) \geq 1 - \eta.$$

Note that

$$\bar{d}(x, t + t^\varepsilon) \geq \varepsilon C \quad \Leftrightarrow \quad \hat{d}(x, t + t^\varepsilon) \geq \varepsilon C \quad \Rightarrow \quad \hat{d}(x,t) \geq \frac{\varepsilon C}{2}$$

for all $(x,t) \in \Omega \times [0, T - t^\varepsilon]$ and every $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough. Thus the inequality $\bar{d}(x, t + t^\varepsilon) \geq \varepsilon C$ implies in view of (4.82)

$$\begin{aligned} u^{\varepsilon,+}(x,t) &= \Phi_0(m(x,t)\frac{\hat{d}(x,t)}{\varepsilon} + m(x,t)p(t)) + \varepsilon \hat{U}_1(x, t, \frac{\hat{d}(x,t)}{\varepsilon} + p(t)) + \sigma(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}) \\ &\leq \Phi_0(\underline{m}\frac{C}{2} - \overline{m} \max_{t \in [0,T]} |p(t)|) + \varepsilon \hat{C}_3 + \alpha(\beta + \varepsilon^2 L e^{LT}) \\ &\leq \Phi_0(\underline{m}\frac{C}{2} - \overline{m}(1 + e^{LT} + K)) + \frac{\eta}{2}. \end{aligned}$$

Choosing C large enough such that $\Phi_0(\underline{m}\frac{C}{2} - \overline{m}(1 + e^{LT} + K)) \leq \frac{\eta}{2}$, we deduce that for all $(x,t) \in \Omega \times [0, T - t^\varepsilon]$ such that $\bar{d}(x, t + t^\varepsilon) \geq \varepsilon C$, we have $u^{\varepsilon,+}(x,t) \leq \eta$ and hence $u^\varepsilon(x, t + t^\varepsilon) \leq \eta$. Similarly, the inequality $\bar{d}(x, t + t^\varepsilon) \geq -\varepsilon C$ implies that $u^\varepsilon(x, t + t^\varepsilon) \geq 1 - \eta$. This completes the proof of Theorem 1.2.

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