INVISCID TRAVELING WAVES OF MONOSTABLE NONLINEARITY

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Abstract. Inviscid traveling waves are ghost-like phenomena that do not appear in reality because of their instability. However, they are the reason for the complexity of the traveling wave theory of reaction-diffusion equations and understanding them will help to resolve related puzzles. In this article, we obtain the existence, the uniqueness and the regularity of inviscid traveling waves under a general monostable nonlinearity that includes non-Lipschitz continuous reaction terms. Solution structures are obtained such as the thickness of the tail and the free boundaries.

keywords: vanishing viscosity method, Fisher-KPP equation, minimum wave speed

1. Phantom of traveling waves

Traveling wave solutions with a monostable nonlinearity have been intensively studied (see [3]). For example, consider a reaction diffusion equation,

$$u_t = d(u^m)_{xx} + u^\beta(1-u^\alpha), \quad t,\alpha,\beta,d,m>0,\ x\in\mathbf{R}, \tag{1.1}$$

where subindexes indicate partial derivatives. The usual traveling wave phenomenon is produced by a correlation between diffusion and reaction. However, there are phantom-like traveling waves for any speed $c\in\mathbf{R}$ which are produced entirely by reaction ($d=0$). The reason why the reaction-diffusion equation admits a traveling wave of any speed greater than a minimum one, $|c|\ge c^*>0$, is related to such traveling waves.

These phantom-like traveling wave solutions satisfy an inviscid equation,

$$v_t = v^\beta(1-v^\alpha), \quad t>0,\ x\in\mathbf{R}, \tag{1.2}$$

where $v(x,\cdot)$ solves the the ODE independently for each $x\in\mathbf{R}$. Consider a traveling wave solution of speed $c>0$, $v(x,t)=v(x-ct)$ (here, we are abusing notation by using the same "v" for the traveling wave profile). Then, $v=v(z)$ satisfies

$$cv' + v^\beta(1-v^\alpha)=0, \qquad \alpha,\beta>0, z\in\mathbf{R}. \tag{1.3}$$

We restrict our study to a traveling wave with monotonicity. The solution is global and unique at least for $\beta\ge 1$ by the Cauchy Lipschitz theorem and satisfies boundary conditions

$$\lim_{z\to-\infty} v(z)=1, \quad \lim_{z\to\infty} v(z)=0, \quad v(0)=0.5. \tag{1.4}$$

Here, we have chosen a decreasing traveling wave. Since a traveling wave is invariant in translation, the extra condition $v(0)=0.5$ is taken for the uniqueness. An inviscid traveling wave, denoted by $v=v_{c,\alpha,\beta}$, depends on three parameters, c,α,β.

For the Fisher equation case ($\alpha=\beta=1$) the inviscid traveling wave is simply the logistic function given in (2.3). This information of inviscid traveling waves was the key to obtain the connection between viscous and inviscid traveling waves (see [5]). The purpose of this paper is to obtain the properties of inviscid traveling waves required to show similar connections in the general setting of the above.

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2. Three examples of inviscid traveling waves

We consider three cases of inviscid traveling waves solutions. They may have an algebraic tail, an exponential tail, or a free boundary, respectively.

Case 1. $\alpha = 1, \beta = 2$ (algebraic tail). Separate variables in (1.3) and obtain $-\frac{c}{v^2(1-v)}v' = 1$. Integrate both sides on $(0, z)$ and obtain

$$-\int_0^z \frac{cv'(s)}{v^2(s)(1-v(s))}ds = z.$$

A change of variable and the condition $v(0) = 1/2$ yield that

$$-\int_0^z \frac{cv'(s)}{v^2(s)(1-v(s))}ds = -\int_{v(0)}^{v(z)} \frac{c}{v^2(1-v)}dv = c\Big(\frac{1}{v(z)} + \log\frac{1-v(z)}{v(z)} - 2\Big).$$

Therefore, we have

$$c\Big(\frac{1}{v(z)} + \log\frac{1-v(z)}{v(z)} - 2\Big) = z. \tag{2.1}$$

From this implicit formula, one may easily check the boundary conditions (1.4) and, furthermore,

$$\lim_{z\to\infty} zv_{c,\alpha=1,\beta=2}(z) = c. \tag{2.2}$$

Thus, the traveling wave has an algebraic tail $v_{c,\alpha=1,\beta=2}(z) \cong cz^{-1}$ for z large.

Case 2. $\alpha = 1, \beta = 1$ (exponential tail). In this case the equation (1.3) is written as

$$v' = -\frac{1}{c}v(1-v), \quad z \in \mathbf{R}.$$

The traveling wave is the logistic function and is given by

$$v_{c,\alpha=1,\beta=1}(z) = \big(1 + \exp\big(z/c\big)\big)^{-1}. \tag{2.3}$$

This solution satisfies the conditions in (1.4) and $v_{c,\alpha=1,\beta=1}(z) \cong e^{-\frac{1}{c}z}$ for z large.

Case 3. $\alpha = 1, \beta = 0.5$ (free boundary). We separate variables and obtain

$$-\frac{c}{v^{1/2}(1-v)}v' = 1.$$

Integrate it over $(0, z)$ and obtain

$$-\int_{v(0)}^{v(z)} \frac{c}{v^{1/2}(1-v)}dv = c\Big(\log\frac{1-\sqrt{v(z)}}{1+\sqrt{v(z)}} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big) = z. \tag{2.4}$$

The solution is explicitly given by

$$v(z) = \Big(\frac{2}{1+e^{\frac{z}{c}+\log\frac{\sqrt{2}-1}{\sqrt{2}+1}}} - 1\Big)_+^2. \tag{2.5}$$

This solution has a free boundary at $z_0 = c\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ and is positive for $z \in (-\infty, z_0)$.

Remark 2.1 (Regularity of traveling wave near free boundary)**.** *Formally speaking, the right hand side of* (1.3) *is bounded near the free boundary for all* $\alpha, \beta > 0$. *Hence, we expect* C^1 *regularity of the traveling wave. Consider the derivative of* (1.3),

$$\begin{aligned} c^2v'' &= -c\big(v^\beta(1-v^\alpha)\big)' = -c\beta v^{\beta-1}(1-v^\alpha)v' + c\alpha v^\beta v^{\alpha-1}v' \\ &= \beta v^{2\beta-1}(1-v^\alpha)^2 - \alpha v^{\alpha+2\beta-1}(1-v^\alpha). \end{aligned}$$

The most singular term near the free boundary is $v^{2\beta-1}$ *which is unbounded for* $\beta < \frac{1}{2}$. *In fact, the traveling wave given in* (2.5) *is the border case with* C^2 *regularity.*

3. Existence and uniqueness of traveling waves

If $\beta < 1$ and the solution of (1.2) is not unique in general. However, the traveling wave is unique for all $\beta > 0$. We show the uniqueness and the existence of a traveling wave solution for all $\beta > 0$, where the Cauchy Lipschitz theorem covers the case with $\beta \geq 1$.

Theorem 3.1 (Existence, Uniqueness, and C^1 regularity)**.** *For any $\alpha, \beta, c > 0$, there exists a unique solution $v = v_{c,\alpha,\beta} \in C^1(\mathbf{R})$ of (1.3)–(1.4). Furthermore,*

(1) $v_{c,\alpha,\beta}(z) < 1$ *for all* $z \in \mathbf{R}$.
(2) *If* $0 < \beta < 1$*, there exists* $z_0 < \infty$ *such that* $v_{c,\alpha,\beta}(z) > 0$ *iff* $-\infty < z < z_0$.
(3) *If* $\beta \geq 1$*,* $v_{c,\alpha,\beta}(z) > 0$ *for* $-\infty < z < z_0 := \infty$.
(4) $v_{c,\alpha,\beta}(z)$ *is strictly decreasing for* $-\infty < z < z_0$.
(5) $v_{c,\alpha,\beta}(z) \to 1$ *for* $z < 0$ *and* $v_{c,\alpha,\beta}(z) \to 0$ *for* $z > 0$ *as* $c \to 0$.

Proof. Rewrite the equation (1.3) as $-\frac{cv'}{v^\beta(1-v^\alpha)} = 1$ and integrate it on $(0, z)$ to obtain

$$-\int_0^z \frac{cv'(s)}{v^\beta(s)(1-v^\alpha(s))} ds = -\int_{v(0)}^{v(z)} \frac{c}{y^\beta(1-y^\alpha)} dy = z. \tag{3.1}$$

Let $G(v) := -\int_{v(0)}^{v} \frac{c}{y^\beta(1-y^\alpha)} dy$. Then we have $G(v(z)) = z$. Since $G'(v) = -\frac{c}{v^\beta(1-v^\alpha)} < 0$ for $0 < v < 1$, $G(v)$ is continuous and strictly decreasing on the unit interval $(0, 1)$. Therefore, G is invertible on $(0, 1)$ and v is given uniquely as its inverse function, i.e., $v(z) = G^{-1}(z)$. This also implies that v is $C^1(\mathbf{R})$ and strictly decreasing as long as $0 < v < 1$. Furthermore, $G(v) \to 0$ as $c \to 0$ and hence its inverse $v(z)$ satisfies (5).

Now we check the boundary conditions in (1.4). Let $s = y^\alpha$. Then, the integral is written as

$$G(v) = -\int_{v(0)}^{v} \frac{c}{y^\beta(1-y^\alpha)} dy = -\int_{v^\alpha(0)}^{v^\alpha} \frac{c}{\alpha s^{(\alpha+\beta-1)/\alpha}(1-s)} ds.$$

Therefore,

$$\lim_{v\to 1^-} G(v) = -\infty, \quad \lim_{z\to-\infty} v(z) = 1, \quad \text{and} \quad v(z) < 1.$$

Similarly we can compute

$$\lim_{v\to 0^+} G(v) = -\lim_{v\to 0^+} \int_{v(0)}^{v} \frac{c}{y^\beta(1-y^\alpha)} dy = \begin{cases} \infty, & \beta \geq 1, \\ z_0, & \beta < 1. \end{cases}$$

Therefore, $v(z) > 0$ if $\beta \geq 1$, and $\lim_{z\to z_0} v(z) = 0$ if $0 < \beta < 1$. Furthermore, the nonnegativity and monotonicity of v implies $v(z) = 0$ for all $z \geq z_0$ if $0 < \beta < 1$. □

4. Free boundary and tail thickness

The thickness of the tail or the free boundary decides the asymptotic behavior of a traveling wave. The thickness is proportional to the wave speed in a certain sense. In this section we define the thickness in a way to be identical to the wave speed.

Definition 4.1. *The limit $\lambda := \lim_{z\to\infty}(\beta - 1)v^{\beta-1}(z)z$ is called the algebraic tail thickness of v of degree $\frac{1}{\beta-1}$ if the limit exists as $z \to \infty$.*

If $\lambda \neq 0$, such a traveling wave asymptotically satisfies

$$v(z) \cong \Big(\frac{\beta-1}{\lambda} z\Big)^{-\frac{1}{\beta-1}} \quad \text{for} \;\; z \cong \infty. \tag{4.1}$$

The traveling wave implicitly given by (2.1) is an example with $\alpha = 1$ and $\beta = 2$. In the case, the tail thickness is $\lim_{z\to\infty}(\beta - 1)v(z)^{\beta-1}z = \lim_{z\to\infty} v_{c,\alpha=1,\beta=2}(z)z = c$. The algebraic tail thickness of the traveling wave $v_{c,\alpha=1,\beta=2}$ of degree $\frac{1}{\beta-1}(= 1)$ is $\lambda(c, \alpha = 1, \beta = 2) = c$, i.e., the wave speed. We will see it is true for all $\alpha > 0$ and $\beta > 1$.

Theorem 4.1. *Let $\alpha > 0$ and $\beta > 1$. Then, the traveling wave $v_{c,\alpha,\beta}$ has the algebraic tail thickness of degree $\frac{1}{\beta-1}$ which is same as the wave speed c, i.e.,*

$$\lambda(c, \alpha > 0, \beta > 1) \Big(:= \lim_{z\to\infty} (\beta - 1) v^{\beta-1}_{c,\alpha,\beta}(z) z \Big) = c. \tag{4.2}$$

Proof. The relation (3.1) is written as

$$\begin{aligned} z &= -\int_0^z \frac{cv'(s)}{v^\beta(s)(1 - v^\alpha(s))} ds = -\frac{c}{1-\beta} \int_0^z \frac{d}{ds}[v^{1-\beta}(s)] \frac{1}{(1 - v^\alpha(s))} ds \\ &= -\frac{cv^{1-\beta}}{(1-\beta)(1 - v^\alpha)}\Big|_0^z - \frac{c}{\beta - 1} \int_0^z \frac{\alpha v^{\alpha-\beta}(s)}{(1 - v^\alpha(s))^2} v'(s)\, ds. \end{aligned} \tag{4.3}$$

As $z \to \infty$, $v(z) \to 0$ and the last term in (4.3) is estimated by, if $\alpha - \beta \neq -1$,

$$\Big| \int_{v(0)}^{v(z)} \frac{\alpha v^{\alpha-\beta}}{(1 - v^\alpha)^2}\, dv \Big| \lesssim \int_{v(z)}^{v(0)} v^{\alpha-\beta}\, dv \lesssim v(z)^{\alpha-\beta+1} + v(0)^{\alpha-\beta+1},$$

where the inequality $\lesssim$ denotes that a positive constant is omitted in the inequality. If $\alpha - \beta = -1$, the upper bound is replaced by $|\ln v(z)| + |\ln v(0)|$. Multiply the both sides of (4.3) by $(\beta - 1)v(z)^{\beta-1}$ and obtain

$$(\beta - 1)v^{\beta-1}(z)z = c\frac{1}{1 - v^\alpha(z)} - cv^{\beta-1}(z)\Big(\frac{v^{1-\beta}(0)}{1 - v^\alpha(0)} - \int_0^z \frac{\alpha v^{\alpha-\beta}(s)}{(1 - v^\alpha(s))^2} v'(s)\, ds\Big).$$

Take limit as $z \to \infty$ and obtain $\lim_{z\to\infty}(\beta - 1)v^{\beta-1}(z)z = c$. □

Definition 4.2. *The limit $\lambda := \lim_{z\to\infty} \frac{z}{-\ln v(z)}$ is called the exponential tail thickness of v if the limit exists as $z \to \infty$.*

If $\lambda \neq 0$, such a traveling wave asymptotically satisfies

$$v(z) \cong e^{-\frac{1}{\lambda} z} \quad \text{for} \quad z \cong \infty. \tag{4.4}$$

The traveling wave explicitly given by (2.3) is an example with $\alpha = \beta = 1$. In this case the traveling wave has an exponential tail thickness of

$$\lambda(c, \alpha = 1, \beta = 1) = c.$$

In the following theorem we show that the tail thickness is independent of $\alpha > 0$.

Theorem 4.2. *Let $\alpha > 0$ and $\beta = 1$. Then, the traveling wave $v_{c,\alpha,\beta=1}$ has the exponential tail thickness which is same as the wave speed c, i.e.,*

$$\lambda(c, \alpha > 0, \beta = 1) \left(:= \lim_{z\to\infty} \frac{z}{-\ln v_{c,\alpha,1}(z)} \right) = c. \tag{4.5}$$

Proof. From (3.1),

$$z = -c \int_{v(0)}^{v(z)} \Big(\frac{d}{dv} \ln v\Big)(1 - v^\alpha)^{-1}\, dv.$$

Integrate it by parts and obtain

$$z = -c\frac{\ln v(z)}{1 - v^\alpha(z)} + c\frac{\ln v(0)}{1 - v^\alpha(0)} + c\int_{v(0)}^{v(z)} \frac{\alpha v^{\alpha-1} \ln v}{(1 - v^\alpha)^2}\, dv.$$

Note that the third term is bounded as $v(z) \to 0$ since $\alpha > 0$. Divide both sides by $-\ln v(z)$, take the limit as $z \to \infty$, and obtain $\lim_{z\to\infty} \frac{z}{-\ln v(z)} = c$. □

Lastly, consider a traveling wave that has a free boundary of its support at $z = z_0$.

Definition 4.3. *The limit $\lambda := \lim_{z\to z_0^-}(\beta - 1)v^{\beta-1}(z)(z - z_0)$ is called the thickness of a free boundary of v of degree $\frac{1}{1-\beta}$ as $z \to z_0^-$ if the limit exists.*

If $\lambda \neq 0$, such a traveling wave satisfies

$$v(z) \cong \Big(\frac{1-\beta}{\lambda}(z_0 - z)\Big)^{\frac{1}{1-\beta}} \quad \text{for} \quad z_0 - \epsilon < z < z_0. \tag{4.6}$$

The traveling wave explicitly given by (2.5) is an example with $\alpha = 1$ and $\beta = 0.5$. In this case the traveling wave has a free boundary at $z_0 = c \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ and

$$\begin{aligned} \lambda(c, \alpha = 1, \beta = 1/2) &= -\lim_{z\to z_0} \frac{1}{2} v^{-\frac{1}{2}}(z - z_0) = -\lim_{z\to z_0} \frac{1}{2}\left(\frac{2}{1 + e^{\frac{1}{c}(z-z_0)}} - 1\right)^{-1}(z - z_0) \\ &= -\lim_{z\to z_0} \frac{1}{2}\left(\frac{1 + e^{\frac{1}{c}(z-z_0)}}{1 - e^{\frac{1}{c}(z-z_0)}}\right)(z - z_0) = c. \end{aligned}$$

Therefore, the front thickness of this inviscid traveling wave is same as its traveling wave speed. This holds for all $\beta < 1$.

Theorem 4.3. *Let $\alpha > 0$, $0 < \beta < 1$, and $z = z_0$ be the free boundary of $v_{c,\alpha,\beta}$. The thickness of the free boundary of degree $\frac{1}{1-\beta}$ is the wave speed c, i.e.,*

$$\lambda(c, \alpha > 0, \beta) \left(:= \lim_{z\to z_0^-} (\beta - 1) v_{c,\alpha,\beta}^{\beta-1}(z)(z - z_0) \right) = c. \tag{4.7}$$

Proof. Rewrite the equation (1.3) as $-\frac{cv'}{v^\beta(1-v^\alpha)} = 1$ and integrate it from z to z_0 to obtain

$$z_0 - z = \int_0^{v(z)} \frac{c}{v^\beta(1 - v^\alpha)}\, dv.$$

Integration by parts gives

$$z_0 - z = c\frac{v^{1-\beta}(z)}{(1-\beta)(1 - v^\alpha(z))} - \frac{c}{1-\beta}\int_0^{v(z)} \frac{\alpha v^{\alpha-\beta}}{(1 - v^\alpha)^2}\, dv. \tag{4.8}$$

The second term is estimated by

$$\left| \int_0^{v(z)} \frac{\alpha v^{\alpha-\beta}}{(1 - v^\alpha)^2}\, dv \right| \lesssim v^{\alpha+1-\beta}(z).$$

Multiply $\frac{1-\beta}{v^{1-\beta}}$ to (4.8) and take limit as $z \to z_0^-$, which gives $\lim_{z\to z_0^-}(1-\beta)\frac{z_0-z}{v^{1-\beta}(z)} = c$. ☐

5. Discussions

If $m + \beta \geq 2$ and the wave speed c is greater than or equal to a minimum wave speed, $c^* > 0$, there exists a traveling wave solution of (1.1) (see [3, Theorem 10.5]), and its profile, $u(x - ct) = u(x,t)$, satisfies

$$d(u^m)'' + cu' + u^\beta(1 - u^\alpha) = 0, \qquad \alpha, \beta > 0, z \in \mathbf{R}, \tag{5.1}$$

$$\lim_{z\to-\infty} u(z) = 1, \quad \lim_{z\to\infty} u(z) = 0, \quad u(0) = 0.5. \tag{5.2}$$

The traveling wave profile, $u(z) = u_{d,m,c,\alpha,\beta}(z)$, is a member of five parameters family of functions and inviscid traveling waves of the paper provide a useful tool to understand the dynamics in the diffusive traveling waves. Take a diffusive traveling wave as an initial value and consider a Cauchy problem,

$$\begin{aligned} &u_t = d\big(u^m\big)_{xx} + u^\beta(1 - u^\alpha), \quad t > 0,\ x \in \mathbf{R}, \\ &u(x,0) = u_{d,m,c,\alpha,\beta}(x). \end{aligned} \tag{5.3}$$

The dynamics of this Cauchy problem can be viewed from its inviscid correspondence,

$$\begin{aligned} &v_t = v^\beta(1 - v^\alpha), \qquad t > 0,\ x \in \mathbf{R}, \\ &v(x,0) = v_{c,\alpha,\beta}(x). \end{aligned} \tag{5.4}$$

If $\beta < 1$, the initial value $v_{c,\alpha,\beta}(x)$ has a free boundary at $x = z_0$. The interesting part is that the reaction term is not Lipschitz continuous and the solution of (5.4) is not unique. The solution behavior for $x < z_0$ is determined since $v(x,0) > 0$. However, for $x \geq z_0$, the solution behavior is decided by the firing moment which could be given arbitrarily.[1] The non-uniqueness of solution comes from this arbitrariness. The unique traveling wave solution obtained in Theorem 3.1 is the one that the firing moment at $x > z_0$ is $t = \frac{x-z_0}{c}$. There is no dynamics in the model that gives such a firing and hence we call it a phantom-like traveling wave. The solution of the reaction-diffusion equation (5.3) is not unique, neither. The minimum traveling wave speed, $c = c^*$, is when the firing moment is given by the diffusion process. If there is a ghost which controls the firing moment, the ghost can only speed up the traveling wave, i.e., $c \geq c^*$, since the solution is determined as soon as the wave is fired by the diffusion. We may conjecture that the initial propagation speed of the free boundary of a Cauchy problem,

$$u_t = d(u^m)_{xx}, \quad u(x,0) = u_{d,m,c,\alpha,\beta}, \quad x \in \mathbf{R},\ t > 0,$$

is c if and only if $c = c^*$. Otherwise, slower.

If $\beta \geq 1$, the reaction term is Lipschitz and the solution of (5.4) is unique. The initial value $v_{c,\alpha,\beta}(x)$ is strictly positive and the growth (or firing) moment is already encoded in the tail. For example, for $x > 0$, the tail thickness is the initial value, $v(x,0) = v_{c,\alpha,\beta}(x) > 0$, where the solution of the ODE has the value $v(x,t) = 0.5$ when $t = \frac{x-0}{c}$. Therefore, the value $v = 0.5$ is traveling with speed c. Remember that, if $m > 1$ and $\beta \geq 1$, the traveling wave of (1.1) with the minimum speed is the only one without a tail (see [4, Theorem 3.1]). In other words, the propagation of the wave is entirely given by the diffusion, but not the tail of initial value.

The properties of inviscid traveling waves $v_{c,\alpha,\beta}$ naturally indicate some properties of traveling waves of diffusive traveling waves $u_{d,m,c,\alpha,\beta}$. For example, since $v_{c,\alpha,\beta}$ exists for any $c > 0$, it is expected that the minimum traveling wave speed, $c^* = c^*(d,m,\beta)$, converges to zero as $d \to 0$. In fact, this convergence has been shown for many cases such as $m + \beta = 2$. One may ask if the diagram of convergence corresponding to [5, Figure 1]) can be completed. For example, let $c > c^*(d,m,\beta)$. Then, there exists a diffusive traveling wave of the speed c. If such a traveling wave is related to the dynamics of the reaction term, it is expected that $u_{d,m,c,\alpha,\beta} \to v_{c,\alpha,\beta}$ as $d \to 0$. In fact such a limit has been shown for Fisher's equation case (see [5, Theorem 1]). As $d \to 0$, the diffusive traveling wave of the minimum speed is supposed to have zero wave speed and inviscid structure. It has been shown that $v_{c,\alpha,\beta} \to \chi_{(-\infty,0)}$ as $c \to 0$ (Theorem 3.1). Therefore, it is expected that $u_{d,m,c^*(d,m,\beta),\alpha,\beta} \to \chi_{(-\infty,0)}$ as $d \to 0$. This limit has been shown for Fisher's equation case (see [5, Theorem 3]). The vanishing viscosity limit has been mostly studied for under the effect of advection or hyperbolic problems (see [1, 2, 6]). In that case, even if the diffusion disappears, the advection gives a migration phenomenon. The inviscid traveling wave of this paper shows the possibility of generation of traveling wave phenomena without any migration mechanism which is quite a ghost-like one.

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References

[1] E. C. M. Crooks, *Front profiles in the vanishing-diffusion limit for monostable reaction-diffusion-convection equations*, Differential Integral Equations **23** (2010), no. 5-6, 495–512. MR 2654247

[2] Constantine M. Dafermos, *Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method*, Arch. Rational Mech. Anal. **52** (1973), 1–9. MR 0340837

[3] Brian H. Gilding and Robert Kersner, *Travelling waves in nonlinear diffusion-convection reaction*, Progress in Nonlinear Differential Equations and their Applications, vol. 60, Birkhäuser Verlag, Basel, 2004. MR 2081104

[4] D. Hilhorst, R. Kersner, E. Logak, and M. Mimura, *Interface dynamics of the Fisher equation with degenerate diffusion*, J. Differential Equations **244** (2008), no. 11, 2870–2889. MR 2418179

[5] Danielle Hilhorst and Yong-Jung Kim, *Diffusive and inviscid traveling waves of the Fisher equation and nonuniqueness of wave speed*, Appl. Math. Lett. **60** (2016), 28–35. MR 3505849

[6] Marshall Slemrod and Athanassios E. Tzavaras, *A limiting viscosity approach for the Riemann problem in isentropic gas dynamics*, Indiana Univ. Math. J. **38** (1989), no. 4, 1047–1074. MR 1029688

[1]The firing moment is the time that the solution becomes positive from the zero initial value.