DISCONTINUOUS NONLINEARITY AND FINITE TIME EXTINCTION

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Abstract. The super- and sub-soltuion theory is developed when the nonlinear reaction function is discontinuous at stable steady states. The solution is defined in a weak sense using a notion of set-valued integral. The existence and the uniqueness of the weak solution are obtained together with a comparison principle. The lack of Lipschitz continuity of the problem forces the solution to reach such stable steady states in a finite time. This discontinuity driven dynamics produces physically interesting phenomena such as finite time extinction, free boundaries, and compactly supported solutions. The developed theory is applied to the Allee effect and a few criterions for the initial population distribution are found, which decide the extinction, survival, expansion, and blowup of the population.

1. Introduction

The purpose of this paper is to develop super- and sub-solution theories for an elliptic equation

$$0 = \Delta u + f(u) \tag{1.1}$$

and for a parabolic equation

$$v_t = \Delta v + f(v) \tag{1.2}$$

when the nonlinear reaction function f has discontinuity. The essential difference made by a discontinuous reaction term comes up when f is discontinuous at a steady state. We restrict our study to a case when the zero value, $s = 0$, is a *stable* steady state and f is discontinuous at it. One can easily extend the theory for a general case with multiple discontinuities. However, if f has a discontinuity at an *unstable* steady state, the theory in this paper fails.

The initial motivation of introducing such a discontinuous reaction term is to obtain a finite time extinction phenomenon (see Section 2). This finite time extinction phenomenon gives a free boundary to parabolic problems and a compactly supported solution to elliptic problems. As an application of the super- and sub-solution theory in the paper, we consider the Allee effect and find a criterion for the initial population distribution that dicides extinction or survival of the population.

More specifically, we assume that there exist two points a^* and h such that $0 < a^* < h < \infty$ and f satisfies the following hypotheses:

$$
\begin{aligned}
&f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^-) \cap \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+),\ \lim_{s \to 0^-} f(s) = f^* \geq 0,\ \lim_{s \to 0^+} f(s) = -f_* < 0, \\
&f(s) < 0 \text{ on } (0, a^*),\ f(s) > 0 \text{ on } (a^*, h],\ \text{and } \int_0^h f(s)ds = 0.
\end{aligned} \tag{1.3}
$$

Here, $\mathbb{R}^- = (-\infty, 0)$, $\mathbb{R}^+ = (0, +\infty)$, and $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^-)$ and $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+)$ are collection of Lipschitz continuous functions on $(-S, 0)$ and $(0, S)$, respectively, for any $S > 0$. Since $0 \notin \mathbb{R}^- \cup \mathbb{R}^+$, the origin is the only possible discontinuity point of f.

The first line in (1.3) shows that f has a jumping-downward discontinuity at $s = 0$ from a nonnegative left side limit, $f^* \geq 0$, to a negative right side limit, $-f_* < 0$. Since f decreases at the discontinuity, the reaction term reduces the variation of a solution even if f is discontinuous. Furthermore, the signs of the left and right side limits make the trivial state $s = 0$ stable. In particular, the strict negativity $-f_* < 0$ makes a finite time extinction possible and produces a free boundary. The second line in (1.3) is about the sign of the reaction function in the interval $(0, h]$. It implies that the second steady state is $s = a^*$ and is unstable. It also implies that the third steady state is larger than h if exists. Notice that the actual value of f at $s = 0$ is not involved in the hypotheses. Under the conditions in (1.3), the potential,

$$
F(s) := \int_0^s f(\tau)d\tau, \tag{1.4}
$$

satisfies the following properties:

$$
\begin{aligned}
&F(0) = F(h) = 0, \quad F(s) < 0 \quad \text{for } 0 < s < h, \\
&F(s) \text{ is strictly increasing at least in a small interval } (h, h + \epsilon).
\end{aligned}
$$

The detailed definition of the solution to (1.1) or (1.2) with a discontinuous reaction term are respectively given in Sections 3 and 5. Here, we briefly introduce basic ideas in the definition. The solution definition is given for more general reaction functions. Assume that f has left and right side limits at each point and denote them by

$$
f(s-) := \lim_{y \to s,\ y < s} f(y), \quad f(s+) := \lim_{y \to s,\ y > s} f(y). \tag{1.5}
$$

Define

$$
\begin{aligned}
\overline{f}(s) &:= \max(f(s-), f(s+)\,), && (1.6) \\
\underline{f}(s) &:= \min(f(s-), f(s+)\,), && (1.7) \\
\{f\}(s) &:= \{\ell \in \mathbb{R} : \underline{f}(s) \leq \ell \leq \overline{f}(s)\}. && (1.8)
\end{aligned}
$$

The three are identical if f is continuous at $s \in \mathbb{R}$ and $\{f\}(s)$ is a closed interval if f is discontinuous at s. To illustrate the idea behind the definition, let us state the classical definition of a weak solution in a *formal way*:

If

$$\iint \big(v_t - \Delta v - f(v)\big)\phi\, dxdt \le 0 \le \iint \big(v_t - \Delta v - f(v)\big)\phi\, dxdt$$

for all test function ϕ, then v is called a (weak) solution of (1.2). However, if f is not continuous, this definition does not guarantee the existence of a solution. In this paper we call v a solution of (1.2) if

$$\iint \big(v_t - \Delta v - \overline{f}(v)\big)\phi\, dxdt \le 0 \le \iint \big(v_t - \Delta v - \underline{f}(v)\big)\phi\, dxdt \tag{1.9}$$

for all *nonnegative* test functions. If the first inequality is satisfied for all nonnegative test functions, then v is called a sub-solution. If the second inequality is satisfied, then v is called a super-solution. The relation (1.9) can be written equivalently as

$$0 \in \iint \big(v_t - \Delta v - \{f\}(v)\big)\phi\, dxdt,$$

where the right side integral is set-valued, i.e.,

$$\left\{ \iint \big(v_t - \Delta v - g(v)\big)\phi\, dxdt \;\text{ for all } g \text{ such that } \; g(v) \in \{f\}(v) \right\}.$$

These formal expressions are valied if $\partial_t v \in L^2(Q)$ on every compact set $Q \Subset \mathbb{R} \times [0,\infty)$ (see Definition 5.1 for a general case). Solutions of the elliptic problem (1.1) can be similarly defined (see Definition 3.1). The uniqueness and the existence are not guaranteed in general and we need hypotheses corresponding to (1.3).

The paper consists as follows. In Section 2, we motivate the use of discontinuous reaction function in terms of population extinction phenomena. We construct a logistic equation type population dynamics that allows finite time extinction. It is required that the extinction state, $s = 0$, should be a stable steady state and the Lipschitz continuity of the reaction term f should be broken at it. The key example in this paper is

$$\frac{du}{dt} = f(u), \quad f(s) := (s - a^*)(1 - s)\chi_{\{s>0\}}.$$

If $a^* = 0$, it is the classical logistic equation and the extinction state, $s = 0$, is an unstable steady state. If $a^* > 0$, then $f(s)$ has a discontinuity at $s = 0$, which becomes a stable steady state. If $0 < a^* < 1/3$, then $f(s) := (s - a^*)(1 - s)\chi_{\{s>0\}}$ satisfies the conditions in (1.3) for some $h < 1$.

In Section 3, we consider the elliptic problem (1.1) with a reaction term f that satisfies (1.3). We first show in Theorem 3.1 that the nonnegative solution in a bounded domain with homogeneous (or zero) Dirichlet boundary condition is a sub-solution in the whole space. Then, we investigate the solution behavior in one space dimension,

$$0 = u_{xx} + f(u), \quad u > 0, \quad 0 < x < L, \tag{1.10}$$

with initial-boundary conditions

$$u(0) = u(L) = 0, \; u'(0) = \gamma \geq 0, \tag{1.11}$$

where the domain size L is unknown and decided by the slope (or shooting angle) γ and the positivity of the solution, i.e., we may set $L = L(\gamma)$. The solution of (1.10), denoted by Ψ_γ, exists uniquely (see Theorems 3.2 and 3.3) and is called a Dirichlet solution. The Dirichlet solutions are sub-solutions in $\mathbb{R}$ as shown in Theorem 3.1. In particular, when $\gamma = 0$, Ψ_0 satisfies the homogeneous Neumann boundary condition and hence is a (weak) solution in $\mathbb{R}$. This solution is $C^{1,\alpha}(\mathbb{R})$ for all $0 < \alpha < 1$ and is called the Neumann-Dirichlet solution in the paper.

In Section 4, we investigate two example cases, $f(s) = (s - a^*)\chi_{\{s>0\}}$ and $(s - a^*)(1 - s)\chi_{\{s>0\}}$. The solution structure of the two cases are investigated in more details in Theorems 4.1 and 4.2. Numerically computed Dirichlet and Neumann-Dirichlet solutions are given Figures 2, 4, and 5 for a comparison.

In Section 5, we study the parabolic problem (1.2) under the hypotheses (1.3). The existence and the uniqueness of the solution are proved in Theorems 5.1 and 5.2. In particular, the comparison property between super- and sub-solutions in Theorem 5.2 is used as a key theorem. The evolution of the solution is studied in one space dimension.

Finally, we show that the Neumann-Dirichlet solution Ψ_0 provides the extinction criterion. It is shown in Theorem 6.2 that, if $v_0 \leq a\Psi_0$ for some $0 < a < 1$, the population goes extinct in a finite time. On the other hand, if $v_0 \geq b\Psi_0$ for some $b > 1$, the support of the solution expands (see Theorem 6.4), and the population grows (see Theorem 6.6). The proof of these theorems are based on constructions of the related super- or sub-solutions which are given in Lemmas 6.1, 6.3, and 6.5. Numerical simulations in Figure 6 show some of these evolutionary dynamics.

The solution of the elliptic problem,

$$0 = \Delta u + f(u), \quad x \in \mathbb{R}^n,$$

has been studied intensively from many different contexts when f is continuous. In particular, the case with $f(s) = -s + s^p$, $1 < p < \frac{n+2}{n-2}$ (see for instance [1, 15, 17] and the references therein) and its perturbed problems (see [6, 18] and the references therein) have been extensively studied. This nonlinearity satisfies the hypotheses of (1.3) except the discontinuity. The Neumann-Dirichlet solution considered in this paper for an equation with discontinuous nonlinearity plays the role of the ground state solution of the equation with continuous nonlinearity. Non-Lipschitzian nonlinearity such as $f(s) = -s^{-\beta} + \lambda s^p$ with $0 < \beta < 1, 1 \leq p < \frac{n+2}{n-2}$ and $\lambda > 0$ for $n \geq 3$ has been studied in a bounded domain with smooth boundary (see [10]). Existence of solutions to an elliptic problem with discontinuous nonlinearities has been studied by using fixed point index in [22].

2. Modeling Allee effect with finite time extinction

The logistic equation is widely accepted as a population model, which is written by

$$\frac{du}{dt} = u - u^2 \ \text{ for } \ t > 0, \quad u = u_0 \ \text{ for } \ t = 0.$$

The linear term models the positive effect of population growth and the quadratic one the negative effect of self-competition. The solution converges as $t \to \infty$ to the unique stable steady state $u = 1$ for any positive initial value $u_0 > 0$. In ecology, the population of a species often goes extinct if the initial population size is less than a critical value, say $u_0 < a^*$ (see [20]). This phenomenon is called the *Allee effect* and became one of key ecological issues due to the crisis of animal conservation (see [3, 4, 14]). Many mathematical models have been considered to explain the Allee effect and the majority of them are based on the Allen-Cahn type bistable nonlinearity such as

$$g(u) = u(u - a^*)(1 - u) = -a^*u + (1 + a^*)u^2 - u^3, \quad 0 < a^* < 1$$

(see [5, 13]). However, in this model, the quadratic term is positive and the linear one is negative, which are opposite signs of the logistic equation. Furthermore, the population does not go extinct in a finite time even if $u_0 < a^*$. The population approaches to zero asymptotically. In fact, most population models, if not all of them, do not have the finite time extinction phenomenon for the following reason.

Let $f(s)$ be globally Lipschitzian and $s = 0$ be a stable steady state. Then, the Cauchy-Lipschitz theorem implies that the solution of an ordinary differential equation,

$$\frac{du}{dt} = f(u), \quad u(0) = u_0,$$

uniquely exists for $-\infty < t < \infty$. Suppose that the solution becomes zero in a finite time with a non-zero initial value $u_0 \neq 0$. Then, it implies that the solution is not unique in $\mathbb{R}$ since the trivial one is also a sonlution. In other words, to include a finite time extinction phenomenon, one should consider a population model without the Lipschitz continuity.

Consider a second order polynomial $f(s) = r_2 s^2 + r_1 s + r_0$ as an approximation of the population dynamics and find a one with the following properties:

(1) the solution converges to zero as $t \to +\infty$ if $u_0 < a^*$;
(2) the solution converges to 1 as $t \to +\infty$ if $u_0 > a^*$; and

(3) the solution stays zero for all t if $u_0 = 0$.

One may easily check that r_2 should be negative and, after a time scaling, f is given by

$$f(s) = \begin{cases} (s-a^*)(1-s), & s > 0, \\ 0, & s < 0. \end{cases}$$

Therefore, by using the characteristic function χ, the population dynamics is written by

$$\frac{du}{dt} = (u-a^*)(1-u)\chi_{\{u>0\}} = \left(-u^2 + (1+a^*)u - a^*\right)\chi_{\{u>0\}}, \tag{2.1}$$

where the quadratic term is negative and the linear one is positive. The key feature of this nonlinearity is the negative constant term $-a^*$ which makes the reaction function discontinuous at the stable steady state $u = 0$ and gives the finite time extinction dynamics. If $0 < a^* < 1/3$, the hypotheses in (1.3) are satisfied. This population dynamics is the key example throughout the paper.

Similar discontinuous population dynamics can be found from equations with harvesting terms. The harvesting term is often independent of population density and given by a constant term as in (2.1) (see [2, 9, 12]). To add finite time extinction to a logistic type population dynamics, one may consider a Hölder continuous reaction function such as $\frac{du}{dt} = -u^2 + u - c_u u^p$ with $c_u > 0$. If $0 < p < 1$, this population dynamics is not Lipschitzian and the solution may go extinct in a finite time.

For a widly spreaded population, the population size does not guarantee the survival of the population. The pattern of the spatial distribution is more important. Aggregation behavior of biological organisms such as bacteria patterns are understood as a survival strategy against such effects. Reaction-diffusion equations are often considered to model the effect of the spatial heterogeneity.We compare the extinction process of two cases: for $a^* = 0.3$,

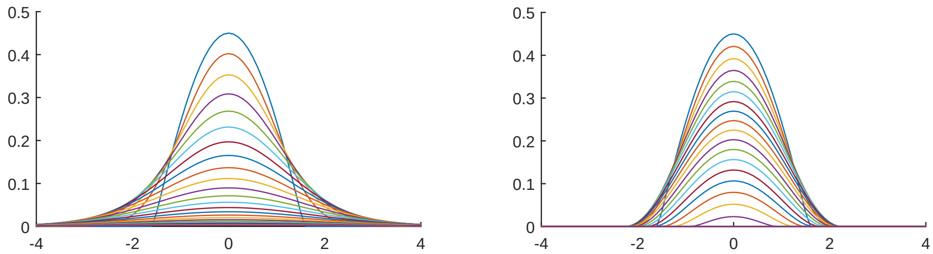
$$v_t = \Delta v + v(v-a^*)(1-v), \tag{2.2}$$

$$u_t = \Delta u + \ (u-a^*)(1-u)\chi_{\{u>0\}}, \tag{2.3}$$

$$u(x,0) = v(x,0) = 0.45\cos(x)\chi_{\{-\pi/2<x<\pi/2\}}.$$

The reaction function of the first equation (2.2) is a smooth bistable nonlinearity. Therefore, $v(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$. The total population decays to zero asymptotically, i.e., $\int v(x,t)dx \to 0$ as $t \to \infty$. In Figure 1(a), the intermediate profiles to extinction are given. The solution support expands as the total population decreases asymptotically (see [13, Figure 4(a)]).

The reaction function of the second equation (2.3) is the discontinuous nonlinearity of this paper and we obtain extinction in a finite time. In Figure 1(b), the intermediate profiles to extinction are given. The solution



(a) Intermediate profiles to extinction of (2.2).

(b) Intermediate profiles to extinction of (2.3).

FIGURE 1. Extinction dynamics. (a) The support size expands when smooth bistable nonlinearity is used. (b) It shrinks when a discontinuous one is used.

support expands in the first stage of evolution due to the diffusion and the singularity of the initial value at the boundary of the support. However, as soon as a certain profile is obtained, the support shrinks back and the total population decreases, which is a more realistic extinction pattern. Finally, the total population becomes zero in a finite time.

3. Elliptic problem for a steady state solution

3.1. **Super- and sub-solutions in $\mathbb{R}^n$.** In this section we define the solution of an elliptic problem,

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^n, \tag{3.1}$$

in a weak sense when the nonlinear function f is discontinuous. Let Ω be an open set. We denote by $C^k(\Omega)$ the set of all functions that has continuous partial derivatives up to the order k in Ω. If the partial derivatives are uniformly bounded on any bounded subset of Ω, it is denoted by $C^k(\overline{\Omega})$. Let $C^k_0(\Omega) = \{\phi \in C^k(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}$, and $C^k_c(\Omega) = \{\phi \in C^k(\Omega) : \operatorname{supp}(\phi) \Subset \Omega\}$.

Definition 3.1. *Suppose that one sided limits of f exist and let $\overline{f}, \underline{f}$ and $\{f\}$ be given by* (1.6),(1.7) *and* (1.8)*, respectively.*

(*i*) *A function $u \in H^1(\mathbb{R}^n)$ is called a sub-solution of* (3.1) *if*

$$\int (\nabla u \cdot \nabla \phi - \overline{f}(u)\phi)dx \le 0 \tag{3.2}$$

for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n)$.

(*ii*) *A function $u \in H^1(\mathbb{R}^n)$ is called a super-solution of* (3.1) *if*

$$0 \le \int (\nabla u \cdot \nabla \phi - \underline{f}(u)\phi)dx \tag{3.3}$$

for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n)$.

(*iii*) *A function* $u \in H^1(\mathbb{R}^n)$ *is called a (weak) solution of* (3.1) *if it is a super- and sub-solution at the same time.*

The above definition holds for a general discontinuous reaction function $f(u)$ which has one sided limits. However, further assumptions are required for the existence and the uniqueness and we study the solution structure under the assumption (1.3). Consider nonnegative solution of the elliptic problem in a smooth domain Ω,

$$\begin{cases} -\Delta u &= f(u),\ u > 0, \qquad x \in \Omega \subset \mathbb{R}^n, \\ \quad u &= 0, \qquad\qquad\qquad x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

The hypotheses on the nonlinearity in (1.3) do not guarantee neither the existence nor the uniqueness of the elliptic problem, which also depends on the domain. For example, if $f(s) = (s - a^*)\chi_{\{s>0\}}$ and the space dimension is $n = 1$, the domain size should be $\pi < |\Omega| \le 2\pi$ to have a solution (see Theorem 4.1(i)). We study this issue in one space dimension in Section 3.2. We first show that, if exists, such a solution is a sub-solution in $\mathbb{R}^n$.

Theorem 3.1. *Let* f *satisfy* (1.3)*,* u *be nonnegative and continuous in* $\mathbb{R}^n$*, and* $\Omega := \{x \in \mathbb{R}^n : u(x) > 0\}$*. Suppose that* $\partial\Omega$ *is bounded and of* C^1*, and the Hausdorff measure* $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ *(or simply denoted by* $|\partial\Omega| < \infty$*). If* $u \in C^2(\Omega)$ *and* $-\Delta u = f(u)$ *in* Ω*, then* u *is a sub-solution of* (3.1)*. Furthermore,* u *is a solution of* (3.1) *if and only if* $\frac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$*.*

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative test function. Divide the integral in (3.2) into two parts, $A = \operatorname{supp}(\phi) \setminus \Omega$ and $B = \operatorname{supp}(\phi) \cap \Omega$, which are called outer and inner domains, respectively. Then, the integral on the outer domain becomes

$$\int_A (\nabla u \cdot \nabla\phi - \overline{f}(u)\phi)dx = -f^* \int_A \phi\, dx \le 0.$$

The integral on the inner domain becomes

$$\int_B (\nabla u \cdot \nabla\phi - \overline{f}(u)\phi)dx = \int_{\partial B} \frac{\partial u}{\partial \nu}\phi\, ds + \int_B (-\Delta u - \overline{f}(u))\phi dx.$$

Since $u > 0$ on Ω and $u = 0$ on $\mathbb{R}^n \setminus \Omega$, we see that $\frac{\partial u}{\partial \nu} \le 0$ on $\partial\Omega$. The boundary ∂B is divided into two parts, $\partial B \cap \partial\Omega$ and $\partial B \cap \partial\big(\operatorname{supp}(\phi)\big)$. Furthermore, since $-\Delta u = f(u)$ and $u > 0$ in B,

$$\int_B (\nabla u \cdot \nabla\phi - \overline{f}(u)\phi)dx = \int_{\partial B \cap \partial\Omega} \frac{\partial u}{\partial \nu}\phi\, ds \le 0.$$

Therefore, (3.2) is satisfied for all nonnegative test functions and hence u is a sub-solution.

Next, consider the integral in (3.3). The integral in the outer domain becomes

$$\int_A (\nabla u \cdot \nabla\phi - \underline{f}(u)\phi)dx = f_* \int_A \phi\, dx \ge 0.$$

(i) Suppose that $\frac{\partial u}{\partial \nu}=0$ on $\partial\Omega$. Then, the inner integral becomes

$$\int_B (\nabla u\cdot\nabla\phi - \underline{f}(u)\phi) = \int_{\partial B}\frac{\partial u}{\partial \nu}\phi + \int_B(-\Delta u - \underline{f}(u))\phi = \int_{\partial B\cap\partial\Omega}\frac{\partial u}{\partial\nu}\phi\ = 0.$$

Therefore, u is a super-solution and hence is a solution.

(ii) Now suppose that Ω is bounded and $\frac{\partial u}{\partial \nu}(x_0)\neq 0$ at a point $x_0\in\partial\Omega$. Let ϕ^ϵ be a test function such that $\phi^\epsilon\geq 0$, $|\phi^\epsilon|\leq 1$, $\phi^\epsilon(x)=1$ for $x\in\Omega$, and $\phi^\epsilon(x)=0$ for all x such that $\inf_{y\in\Omega}|x-y|>\epsilon$. Then, the outer integral becomes

$$\int_A(\nabla u\cdot\nabla\phi^\epsilon - \underline{f}(u)\phi^\epsilon)dx = f_*\int_A\phi^\epsilon dx \leq \epsilon f_*|\partial\Omega| + O(\epsilon^2),$$

which converges to 0 as $\epsilon\to 0$. On the other hand, since u is smooth and $\frac{\partial u}{\partial \nu}(x_0)<0$, there exist $c>0$ and $\delta>0$ such that $\frac{\partial u}{\partial\nu}(x)<-c$ for $x\in B(x_0,\delta)$. Then, the inner integral becomes

$$\begin{aligned}\int_B(\nabla u\cdot\nabla\phi^\epsilon - \underline{f}(u)\phi^\epsilon)dx &= \int_{\partial B\cap\partial\Omega}\frac{\partial u}{\partial\nu}\phi^\epsilon ds \leq \int_{\partial B\cap\partial\Omega\cap B(x_0,\delta)}\frac{\partial u}{\partial\nu}\phi^\epsilon ds\\ &\leq -c|\partial B\cap\partial\Omega\cap B(x_0,\delta)|<0,\end{aligned}$$

which is independent of the parameter ϵ. Hence, by taking $\epsilon>0$ small enough, we can find a test function ϕ^ϵ such that $\int(\nabla u\cdot\nabla\phi^\epsilon - \underline{f}(u)\phi^\epsilon)dx<0$. Therefore, u is not a super-solution.

$(ii)'$ Suppose that Ω is unbounded. Then, the function ϕ^ϵ constructed above is not compactly supported and hence is not an admissible test function anymore. However, the argument holds by replacing Ω in the definition of ϕ^ϵ by $B(0,R)\cap\Omega$ for $R>0$ large enough so that $\partial\Omega\subset B(0,R)$. Since $\partial\Omega$ is bounded, there exists such a radius $R>0$. □

One of special features provided by a discontinuous nonlinearity is that the solution satisfies both of homogeenous Dirichlet and homogeneous Neumann boundary conditions. The finite time extinction behavior of the solution gives a free boundary to the population front and the homogeneous Neumann boundary condition is satisfied along this free boundary of the solution support.

3.2. **Nonnegative Dirichlet solutions in $\mathbb{R}$.** In this section we consider a nonnegative solution of a Dirichlet boundary value problem in one space dimension,

$$\begin{cases} -u''=f(u),\ u>0, & x\in(a,b),\\ \ \ u=0. & x\notin(a,b).\end{cases}\tag{3.4}$$

Notice that u is a classical solution in the bounded open interval (a,b) with the boundary condition $u(a)=u(b)=0$ and is extended to $\mathbb{R}$ by simply letting $u=0$ outside of the interval (a,b). After the extension, u is called a *Dirichlet solution*. More precisely, we take the following definition.

Definition 3.2. *Let $I = (a, b)$.*

(*i*) *A function u is called a Dirichlet solution if $u \in C(\mathbb{R}) \cap C^2(I)$ and satisfies* (3.4) *in the classical sense for $x \in I$.*

(*ii*) *If a Dirichlet solution u satisfies homogeneous Neumann boundary conditions,*

$$u'(a) = u'(b) = 0, \tag{3.5}$$

or equivalently, $u \in C^1(\mathbb{R}) \cap C^2(I)$, then it is called a Neumann-Dirichlet solution.

Remark 3.3. *Theorem 3.1 implies that a Dirichlet solution is a subsolution and a Neumann-Dirichlet solution is a (weak) solution (see Definition 3.1).*

After making a translation, we may take $a = 0$ and $b - a = L$. Then we write (3.4) as

$$\begin{cases} -u'' = f(u),\ u > 0, & x \in (0, L) \\ \quad u = 0, & x \notin (0, L). \end{cases} \tag{3.6}$$

Note that the Dirichlet solution of (3.6) is not unique in general for a given domain size $L > 0$ (see Section 4.2). However, we will see in the following theorem that a Neumann-Dirichlet solution exists uniquely only for a unique domain size.

Theorem 3.2. *Let f satisfy* (1.3)*, F be its antiderivative given in* (1.4)*, and*

$$L_0 = 2 \int_0^h \frac{du}{\sqrt{-2F(u)}}.$$

(*i*) *If $L \neq L_0$, a Dirichlet solution of* (3.6) *is not a Neumann-Dirichlet solution.*

(*ii*) *If $L = L_0$, there exists a unique Neumann-Dirichlet solution of* (3.6)*.*

Proof. Step 1. First we show the uniqueness and investigate the structure of the Neumann-Dirichlet solution. Suppose that u is a Neumann-Dirichlet solution, i.e., $u \in C^1(\mathbb{R})$ and satisfies (3.6) and is in $C^1(\mathbb{R})$. Since u is continuous in $\mathbb{R}$ and has a compact support, $f(u)$ is integrable. Since u' is continuous on $(-\infty, \infty)$ and $u'(x) = 0$ for $x \notin [0, L]$, we see that u satisfies the Neumann condition (3.5). We also see that u'' is continuous for $0 < x < L$. Multiplying the differential equation in (3.6) by $2u'(x)$ we obtain

$$\frac{d}{dx}|u'|^2 = -2f(u)u', \quad 0 < x < L.$$

Integrate it on the interval $(0, x)$ with the initial conditions $u(0) = u'(0) = 0$ and obtain

$$\begin{aligned}|u'(x)|^2 &= -2\int_0^x f(u(s))u'(s)ds \\ &= -2\int_0^x f(u(s))du(s) = -2\int_0^{u(x)} f(u)du = -2F(u(x)).\end{aligned}$$

Then $F(u(x)) \le 0$ for $0 < x < L$. Thus $0 < u(x) \le h$ for all $0 < x < L$.

Since u is a Dirichlet solution, there exists $x_0 \in (0, L)$ which is the first maximum point of $u(x)$ in $(0, L)$. Hence, $u'(x_0) = 0$. Since $u(x) > 0$ for $0 < x < x_0$, we have $F(u(x)) < 0$, hence $|u'(x)|^2 = -2F(u(x)) > 0$. Thus x_0 is the first zero point of $u'(x)$ in $(0, L)$. So

$$F(u(x)) < 0 \quad \text{for } 0 < x < x_0, \quad F(u(x_0)) = 0.$$

Hence $u(x_0) = h$ and

$$u''(x_0) = -f(h) < 0.$$

Therefore, there exists a small constant $\delta > 0$ such that $u'(x) < 0$ for $x_0 < x \le x_0 + \delta$. Let us denote by x_1 the first zero point of $u'(x)$ in the interval $(x_0, L]$. Then, $u'(x) < 0$ for $x \in (x_0, x_1)$ and $u'(x_1) = 0$. Hence $F(u(x_1)) = 0$. This is possible only if $u(x_1) = 0$ and hence $x_1 = L$. In summary,

$$u'(x) > 0 \quad \text{for } 0 < x < x_0, \quad u'(x) < 0 \quad \text{for } x_0 < x < L,$$

and hence

$$u'(x) = \begin{cases} \sqrt{-2F(u(x))} & \text{for } 0 < x < x_0, \\ -\sqrt{-2F(u(x))} & \text{for } x_0 < x < L. \end{cases} \tag{3.7}$$

For $0 < x < x_0$, we have

$$\frac{du(x)}{\sqrt{-2F(u(x))}} = dx.$$

This implies

$$x_0 = \int_0^{x_0} dx = \int_0^{x_0} \frac{du(x)}{\sqrt{-2F(u(x))}} = \int_0^{u(x_0)} \frac{du}{\sqrt{-2F(u)}} = \int_0^h \frac{du}{\sqrt{-2F(u)}}.$$

Similarly,

$$\frac{du(x)}{\sqrt{-2F(u(x))}} = -dx, \quad x_0 < x < L.$$

Therefore,

$$\begin{aligned} x_0 - L &= -\int_{x_0}^{L} dx = \int_{x_0}^{L} \frac{du(x)}{\sqrt{-2F(u(x))}} = \int_{u(x_0)}^{u(L)} \frac{du}{\sqrt{-2F(u)}} \\ &= \int_{h}^{0} \frac{du}{\sqrt{-2F(u)}} = -\int_{0}^{h} \frac{du}{\sqrt{-2F(u)}} = -x_0. \end{aligned}$$

In conclusion, the domain size is

$$L = 2x_0 = 2\int_0^h \frac{du}{\sqrt{-2F(u)}} = L_0$$

and the solution u is symmetric with respect to $x = L_0/2$. The solution is given by the implicit formula

$$\begin{cases} x = \displaystyle\int_0^u \frac{ds}{\sqrt{-2F(s)}} & \text{if} \quad 0 < x < L_0/2, \\ L_0 - x = \displaystyle\int_0^u \frac{ds}{\sqrt{-2F(s)}} & \text{if} \quad L_0/2 < x < L_0. \end{cases} \tag{3.8}$$

The above discussion shows that, if $L \neq L_0$, the Dirichlet solution does not satisfy a Neumann boundary condition. Furthermore, if $L = L_0$, the Neumann-Dirichlet solution must be given by the formula (3.8) if exists. Therefore, the uniqueness and the structure of the Neumann-Dirichlet solution follow this implicit formula.

Step 2. Now we show that $u(x)$ given implicitly by (3.8) is a Neumann-Dirichlet solution which completes the existence part. By a direct computation, we see that u satisfies (3.7) and hence (3.6). Since $F(0) = 0$, (3.7) implies that

$$\lim_{x \to 0^+} u'(x) = 0, \qquad \lim_{x \to L_0^-} u'(x) = 0.$$

After extending the solution for $x \in \mathbb{R}\backslash[0, L_0]$ with the zero value, we obtain $u \in C^1(\mathbb{R}) \cap C^2((0, L_0))$. Hence u is a Neumann-Dirichlet solution. □

Notation. We denote the unique Neumann-Dirichlet solution in Theorem 3.2 by Ψ_0, which is the weak solution in Definition 3.1.

Remark 3.4. *Since the Neumann-Dirichlet solution u and its derivative u' are continuous in $\mathbb{R}$ and satisfy the differential equation in* (3.6) *for all x except $x = 0, L_0$, we see that there exists a constant C such that*

$$|f(u(x))| \le C.$$

This estimate implies that

$$|u''(x)| \le C$$

for $x \neq 0, L_0$. Thus $u \in C^{1,\alpha}(\mathbb{R})$ for all $0 < \alpha < 1$.

Now we consider a Dirichlet solution which is not differentiable at the boundary of its support. Since a nonnegative solution of the problem is symmetric with respect to the middle of its support, we have[1]

$$u'(0^+) = -u'(L^-) > 0.$$

Therefore, we may consider the problem (3.6) with an extra condition on the boundary

$$u'(0) = \gamma > 0. \tag{3.9}$$

Remember that the solution of (3.6) with (3.9) may exist only when γ and L are appropriately related. We now consider the problem with (3.9) and the boundary $L > 0$ is the hitting point after shooting the solution with the angle $\gamma > 0$, i.e.,

$$\begin{cases} -u'' = f(u), \ u > 0, & x \in (0, L), \\ \ u = 0, & x \notin (0, L), \\ \ u'(0) = \gamma. \end{cases} \tag{3.10}$$

The sign of $f(s)$ for $s > h$ is not mentioned in (1.3). Let b^* be the second smallest zero, i.e., $a^* < b^*$, $f(b^*) = 0$, and $f(s) > 0$ on (a^*, b^*). If there is none, we set $b^* = \infty$. The second zero b^* is a stable steady state if $b^* < \infty$. Define

$$r_\infty := \sqrt{2F(b^*)}, \quad F(b^*) = \int_0^{b^*} f(s)\, ds. \tag{3.11}$$

Now we consider Dirichlet solutions bounded by b^*.

Theorem 3.3. *Let f satisfy* (1.3) *and F be the antiderivative of f. Suppose that $f(s) > 0$ for $a^* < s < b^* \le \infty$. Then, for r_∞ given by* (3.11)*, we have the following conclusions:*

(*i*) *If $\gamma > \gamma_\infty$, there is no solution of* (3.10) *bounded by b^*, i.e.,* $\max u \ge b^*$ *if there is a solution.*
(*ii*) *If $0 < \gamma < \gamma_\infty$, there exists a unique domain size $L = L(\gamma) > 0$ such that* (3.10) *has a solution. This solution is also unique.*

Proof. First, we obtain that

$$|u'(x)|^2 = \gamma^2 - 2F(u(x)), \quad x \in (0, L).$$

(*i*) Let $\gamma > \gamma_\infty$ and $\max u < b^*$. Then, $\gamma_\infty < \infty$ and

$$u'(x) = \sqrt{\gamma^2 - 2F(u)}$$

[1]This can also be proved as follows: Let u be a Dirichlet solution. Then for $0 < x_1 < x_2 < L$ we have

$$\frac{1}{2}u'(x_2)^2 - \frac{1}{2}u'(x_1)^2 = F(u(x_2)) - F(u(x_1)).$$

Letting $x_1 \to 0^+$ and $x_2 \to L^-$, and using $F(u(L)) = F(u(0)) = F(0) = 0$, we have

$$\frac{1}{2}u'(L^-)^2 - \frac{1}{2}u'(0^+)^2 = 0.$$

is bounded away from zero. Hence, $u(x)$ increases strictly on $(0, L)$, and $u(L) \neq 0$ for any $L > 0$. This shows that there is no Dirichlet solution of (3.6) with its support $[0, L]$ for any $L > 0$. Therefore, $\max u \geq b^*$ if $\gamma > \gamma_\infty$.

(*ii*) Next, we assume $0 < \gamma < \gamma_\infty$. Then, the algebraic equation

$$\gamma^2 - 2F(y) = 0$$

has at least one positive root $y < b^*$. We denote the smallest positive root by $y = h_\gamma$. Then, as we did in the proof of Theorem 3.2, we can show that there exists a positive solution u of (3.6) which has its maximum at $x_{0,\gamma}$ and

$$u(x_{0,\gamma}) = h_\gamma \quad \text{with} \quad \operatorname{supp}(u) = [0, 2x_{0,\gamma}].$$

Thus u is a solution of (3.6) with $L = 2x_{0,\gamma}$ and it satisfies (3.9). The uniqueness is also obtained similarly. $\square$

We denote the unique solution of (3.10) in Theorem 3.3 by Ψ_γ. The Neumann-Dirichlet solution Ψ_0 can be considered as a special case with $\gamma = 0$. Recall that the maximum point $x_{0,\gamma}$ and value h_γ of the Dirichlet solution Ψ_γ is given by

$$x_{0,\gamma} = \int_0^{h_\gamma} \frac{dy}{\sqrt{\gamma^2 - 2F(y)}}.$$

This integral may either diverge or converge when $\gamma = \gamma_\infty$. If the integral diverges with $\gamma = \gamma_\infty$, then $x_{0,\gamma_\infty} = \infty$, and (3.10) has no solution with $\gamma = \gamma_\infty$. If the integral converges with $\gamma = \gamma_\infty$, then $x_{0,\gamma_\infty} < \infty$, and (3.10) has a solution with an initial condition $u'(0) = \gamma_\infty$, and the support of this Dirichlet solution is $[0, 2x_{0,\gamma_\infty}]$. Moreover, under the condition (1.3), then maximum of Ψ_{γ_∞} is the largest among all Dirichlet solutions. In fact, we have a monotonicity relation between the maximum and the shooting angle.

Corollary 3.4. *If* $\gamma_1 < \gamma_2$, *then* $\max \Psi_{\gamma_1} < \max \Psi_{\gamma_2}$.

Proof. Since the maximum value h_γ is the first positive root of the function $\gamma^2 - 2F(y)$, we see that $a^* < h < h_\gamma$, and $f(h_\gamma) > 0$. Hence, we can apply the implicit function theorem to conclude that h_γ is differentiable in γ for $\gamma \in [0, \gamma_\infty)$. Differentiation of the equality,

$$\gamma^2 - 2F(h_\gamma) = 0,$$

gives that

$$\frac{d}{d\gamma} h_\gamma = \frac{\gamma}{f(h_\gamma)} > 0 \quad \text{for all } 0 \leq \gamma < \gamma_\infty.$$

Therefore, for $0 \leq \gamma < \gamma_\infty$, the value h_γ is strictly increasing in γ and it achieves its maximum at $\gamma = \gamma_\infty$. $\square$

4. Two examples of population dynamics

It is clear that the support of a Dirichlet solution should be larger than a positve lower bound due to the Dirichlet boundary condition. In this section, we find the minimum domain sizes, denoted by $L_- > 0$, and a maximum one, denoted by $L_+ > 0$, such that there exists a nontrivial Dirichlet solution if and only if $L_- < L \le L_+$. The answer depends on the reaction term and we consider two examples in this section.

4.1. **Linear growth with discontinuity.** In this section we consider a linear reaction function,

$$f(s) = (s - a^*)\chi_{\{s>0\}}, \quad a^* > 0. \tag{4.1}$$

In this case $s = 0$ is a stable steady state and $s = a^*$ is an unstable one. This reaction function satisfies the hypotheses in (1.3) and its integral is

$$F(s) = \frac{1}{2}s^2 - a^*s, \quad s \ge 0,$$

where $F(h) = 0$ with $h = 2a^*$. Furthermore, the domain size of the Neumann-Dirichlet solution given in Theorem 3.2 is

$$L_0 = 2\int_0^{h=2a^*} \frac{ds}{\sqrt{-2F(s)}} = 2\pi.$$

We will see that this is the maximum domain size of all Dirichlet solutions.

Theorem 4.1. *Let $f(s) = (s - a^*)\chi_{\{s>0\}}$ with $a^* > 0$.*

(*i*) *There exists a Dirichlet solution with support $[0, L]$ if and only if $\pi < L \le 2\pi$.*

(*ii*) *A Dirichlet solution is unique for each domain size $L \in (\pi, 2\pi]$. A Dirichlet solution is the Neumann-Dirichlet solution if and only if $L = 2\pi$.*

(*iii*) *The slope $\gamma = u'(0^+)$ decreases as L increases and the limits are*

$$\gamma \to \infty \quad as \quad L \to \pi^+, \qquad \gamma \to 0 \quad as \quad L \to 2\pi^-.$$

The total mass decreases as L increases and the limits are

$$\int_0^L u\,dx \to \infty \quad as \quad L \to \pi^+, \qquad \int_0^L u\,dx \to 2\pi a^* \quad as \quad L \to 2\pi^-.$$

Proof. Step 1. For $f(s) = (s - a^*)\chi_{\{s>0\}}$ we have

$$F(s) = \frac{1}{2}s^2 - a^*s.$$

Therefore, by Theorem 3.2, the unique Neumann-Dirichlet solution u of (3.6) satisfying the Neumann boundary condition has a compact support

$[0, L_0]$, where

$$L_0 = 2\int_0^{2a^*} \frac{1}{\sqrt{2a^* s - s^2}} ds = 2\cos^{-1}\left(\frac{a^* - s}{a^*}\right)\Big|_0^{2a^*} = 2\pi.$$

Step 2. Now let u be a Dirichlet solution with its support $[0, L]$ and $u'(0) = \gamma > 0$. In a similar way as in the proof of Theorem 3.2, we obtain that

$$\int_0^x \frac{1}{2}\{(u')^2\}' dx = \int_0^x \{-\frac{1}{2}(u^2)' + a^* u'\} dx.$$

Since $u'(0) = \gamma$ and $u(0) = 0$, it follows that

$$\frac{1}{2}(u')^2 - \frac{1}{2}\gamma^2 = -\frac{1}{2}u^2 + a^* u,$$

or

$$(u')^2 = \gamma^2 + 2a^* u - u^2$$

for $x \in (0, L)$. It implies that

$$\gamma^2 + 2a^* u(x) - u^2(x) \geq 0 \quad \text{for all } x \in (0, L).$$

Therefore,

$$\xi_- \leq u(x) \leq \xi_+ \quad \text{for all } x \in (0, L),$$

where $\xi_- < \xi_+$ are the roots of the polynomial $y^2 - 2a^* y - \gamma^2 = 0$, i.e.,

$$\xi_\pm = a^* \pm \sqrt{a^{*2} + \gamma^2}.$$

Since $u(x) \geq 0$ for all $x \in (0, L)$, we see that

$$0 \leq u(x) \leq \xi_+ \quad \text{for all } x \in (0, L). \tag{4.2}$$

If $0 \leq u(x) < \xi_+$, then $(u')^2 = \gamma^2 + 2a^* u - u^2 > 0$.

Since $u(0) = u(L)$, u' has a zero point in the interior of the interval $(0, L)$ by the mean value theorem. Let $x_{0,\gamma}$ be the smallest positive zero point of u'. Then

$$u'(x_{0,\gamma}) = 0, \quad u'(x) > 0 \quad \text{for all } 0 < x < x_{0,\gamma}.$$

It implies that $u(x_{0,\gamma}) \geq \xi_+ > 2a^*$. Combining this with (4.2) we see that

$$u(x_{0,\gamma}) = \xi_+.$$

Therefore, $x_{0,\gamma}$ is the maximum point of u and $u'(x) < 0$ for $x > x_{0,\gamma}$ but less than the next critical point. However, the relation $u(x_{0,\gamma}) \geq \xi_+ > 2a^*$ and (4.2) implies that $u'(x) < 0$ as long as $x > x_{0,\gamma}$ and $u(x) > 0$. Hence we have

$$u'(x) < 0, \quad 0 \leq u(x) < \xi_+, \quad \forall x_{0,\gamma} < x < L.$$

By the symmetry of the positive solutions, we conclude that

$$L = 2x_{0,\gamma},$$

and

$$u'(x) = \begin{cases} \sqrt{\gamma^2 + 2a^*u - u^2} & \text{if } 0 \le x \le x_{0,\gamma}, \\ -\sqrt{\gamma^2 + 2a^*u - u^2} & \text{if } x_{0,\gamma} \le x \le L. \end{cases} \tag{4.3}$$

Now we may compute $x_{0,\gamma}$,

$$\begin{aligned} x_{0,\gamma} = u^{-1}(\xi_+) &= \int_0^{a^*+\sqrt{a^{*2}+\gamma^2}} \frac{1}{\sqrt{\gamma^2 + 2a^*y - y^2}} dy \\ &= \int_{-\frac{a^*}{\sqrt{a^{*2}+\gamma^2}}}^{1} \frac{1}{\sqrt{1-z^2}} dz = \frac{\pi}{2} + \sin^{-1}(\frac{a^*}{\sqrt{a^{*2}+\gamma^2}}). \end{aligned}$$

Therefore

$$L = L(\gamma) = 2x_{0,\gamma} = \pi + 2\sin^{-1}(\frac{a^*}{\sqrt{a^{*2}+\gamma^2}}). \tag{4.4}$$

In particular

$$\pi < L(\gamma) \le 2\pi \quad \text{for all } 0 \le \gamma < \infty,$$

and $L(\gamma) = 2\pi$ if and only if $\gamma = 0$.

Step 3. Define u using (4.3) and $u'(0) = \gamma$. We will show that u is a solution of (3.6) for any $\gamma > 0$ on $(0, L(\gamma))$ with $L(\gamma)$ determined by (4.4). As a consequence of (4.4) we see that $L(\gamma)$ is continuous and decreasing in γ, $L(\gamma) \to \pi$ as $\gamma \to \infty$, and $L(\gamma) \to 2\pi$ as $\gamma \to 0$. Hence, for each $L \in (\pi, 2\pi)$, there exists exactly one $\gamma > 0$ such that $L(\gamma) = L$ and (3.6) has a Dirichlet solution u with the support $[0, L(\gamma)]$ and $u'(0) = \gamma$.

The discussion above also shows that $0 < u(x) < \xi_+$ for all $x \in (0, L(\gamma)) \setminus \{x_{0,\gamma}\}$, hence

$$-u'' = u - a^* \quad \text{for all } x \in (0, L(\gamma)),$$

and

$$\int_0^{L(\gamma)} u dx = \int_0^{L(\gamma)} \{-u'' + a^*\} dx = 2\gamma + a^*\pi + 2a^* \sin^{-1}(\frac{a^*}{\sqrt{a^{*2}+\gamma^2}}).$$

Therefore,

$$\frac{d}{d\gamma}\Big(\int_0^{L(\gamma)} u dx\Big) = \frac{2\gamma^2}{a^{*2}+\gamma^2} > 0,$$

and

$$\lim_{\gamma \to 0^+} \int_0^{L(\gamma)} u dx = 2a^*\pi,$$

$$\lim_{\gamma \to \infty} \int_0^{L(\gamma)} u dx = +\infty.$$

If $L = 2\pi$ then $\gamma = 0$, and the unique Neumann-Dirichlet solution u has support $[0, 2\pi]$ with $u'(0) = u'(2\pi) = 0$.

In conclusion, we have actually showed that, for any $L \in (\pi, 2\pi]$ there exists exactly one $\gamma \geq 0$ such that $L = L(\gamma)$ and (3.6) has exactly one Dirichlet solution u satisfying $u'(0) = \gamma$. This Dirichlet solution is a Neumann-Dirichlet solution if and only if $\gamma = 0$ hence if and only if $L = 2\pi$. In particular it implies that when $L = 2\pi$ the Dirichlet solution is unique and is actually the Neumann-Dirichlet solution. In a similar way as we did in Theorem 3.2, we obtain that $u \in C^2((0, L(\gamma)))$. □

Remark 4.1. *Graphs of five Dirichlet solutions are given in Figure 2. If the reaction function is $f(u) = (u - a^*)\chi_{\{u>0\}}$, the domain size $L(\gamma)$ for a given shooting angle $\gamma \geq 0$ decreases as $\gamma \to \infty$. The maximum domain size is $L_+ = L(0) = 2\pi$, which is the size of the Neumann-Dirichlet solution. The domain size $L(\gamma) \to \pi$ as $\gamma \to \infty$. However, there is no Dirichlet solution with the minimum size.*

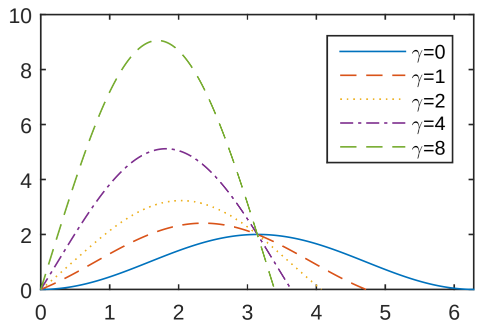


FIGURE 2. Dirichlet Solutions Ψ_γ when $f(u) = (u-1)\chi_{\{u>0\}}$. In the case $\gamma_\infty = \infty$.

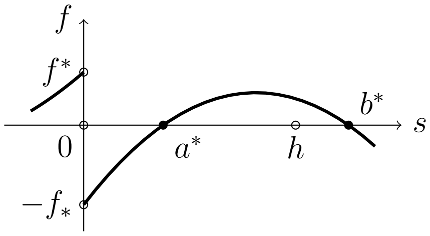
Remark 4.2. *Let u be a nonnegative solution of (3.1) defined on $\mathbb{R}$. Then, u is either an extension of the unique Neumann-Dirichlet solution by the zero, or is obtained by gluing the Neumann-Dirichlet solution and its translations in space.*

4.2. **Logistic growth with discontinuity.** In this section we consider a reaction function

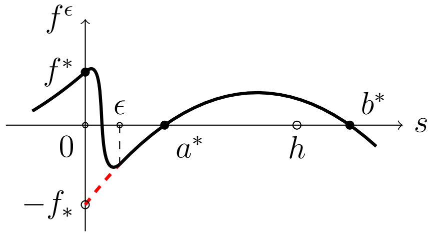
$$f(s) = \begin{cases} (s - a^*)(1 - s), & s > 0, \ 0 < a^* < 1/3, \\ > 0, & s < 0, \end{cases} \tag{4.5}$$

(see Figure 3(a)). In this case there are two stable steady states, $s = 0$ and 1, and one unstable one, $s = a^*$. Since $a^* < \frac{1}{3}$, we have $\int_0^1 f(s)ds > 0$. Therefore, there exists $h \in (a^*, 1)$ such that $\int_0^h f(s)ds = 0$.

Theorem 4.2. *Let f be given by (4.5). There exists $\gamma_\infty > 0$ and $L_- > 0$ such that there exists a solution of (3.10) if and only if $L \geq L_-$ and $0 \leq \gamma < \gamma_\infty$. There is only one solution for each $0 \leq \gamma < \gamma_\infty$ and, in fact, $\gamma_\infty = \sqrt{1/3 - a^*}$.*



(a) discontinuous nonlinearity f



(b) continuous approximation f^ϵ

FIGURE 3. Discontinuous f and its approximation f^ϵ

Proof. Step 1. We have

$$2F(y) = 2\int_0^y f(s)ds = -\frac{2}{3}y^3 + (1+a^*)y^2 - 2a^*y.$$

Denote

$$G(y,\gamma) = \gamma^2 - 2F(y) = \gamma^2 + \frac{2}{3}y^3 - (1+a^*)y^2 + 2a^*y.$$

For any $\gamma \geq 0$, $G(y,\gamma)$ has its minimum at $y = 1$ which is

$$G(1,\gamma) \begin{cases} < 0 & \text{if } 0 \leq \gamma < \gamma_\infty, \\ = 0 & \text{if } \gamma = \gamma_\infty, \\ > 0 & \text{if } \gamma > \gamma_\infty. \end{cases} \tag{4.6}$$

Hence

$$G(y,\gamma) \geq G(1,\gamma) \geq 0 \quad \text{for all } y > 0 \quad \text{if } \gamma \geq \gamma_\infty.$$

Therefore,

$$G(y,\gamma) \text{ has } \begin{cases} \text{one real zero if } \gamma > \gamma_\infty, \\ \text{two real zeros if } \gamma = \gamma_\infty, \\ \text{three real zeros if } 0 \leq \gamma < \gamma_\infty. \end{cases}$$

When $0 \leq \gamma < \gamma_\infty$, we denote the three real zeros of $G(y,\gamma)$ by $h_{j,\gamma}$, $j = 1,2,3$ in the increasing order, i.e.,

$$h_{1,\gamma} < 0 < h_{2,\gamma} < 1 < h_{3,\gamma}. \tag{4.7}$$

For each j, $h_{j,\gamma}$ is continuous with respect to γ in the interval $[0,\gamma_\infty)$. In particular, when $\gamma = 0$, $h_{j,0}$ are the roots of the equation

$$G(y,0) \equiv -2F(y) \equiv \frac{2}{3}y^3 - (1+a^*)y^2 + 2a^*y = 0.$$

Hence,

$$-2F(y) \equiv \frac{2}{3}y^3 - (1+a^*)y^2 + 2a^*y = \frac{2}{3}y(y-h_{2,0})(y-h_{3,0}).$$

Therefore, $G(y,\gamma) \equiv \gamma^2 - 2F(y)$ can be written in three ways;

$$\begin{aligned} G(y,\gamma) &= \frac{2}{3}(y-h_{1,\gamma})(y-h_{2,\gamma})(y-h_{3,\gamma}) \\ &= \frac{2}{3}y^3 - (1+a^*)y^2 + 2a^*y + \gamma^2 \\ &= \gamma^2 + \frac{2}{3}y(y-h_{2,0})(y-h_{3,0}). \end{aligned} \tag{4.8}$$

Step 2. We show that if $\gamma > \gamma_\infty$, then (3.6) has no Dirichlet solution. Suppose that there is a Dirichlet solution with support $[0,L]$ and $u'(0) = \gamma > \gamma_\infty$. Then,

$$|u'(x)|^2 = G(u(x),\gamma), \quad x \in (0,L) > 0.$$

Therefore,

$$|u'(x)| = \sqrt{G(u(x),\gamma)} = \sqrt{\gamma^2 + \frac{2}{3}u^3(x) - (1+a^*)u^2(x) + 2a^*u(x)}\,.$$

However, if $\gamma > \gamma_\infty$, we see from (4.6) that

$$|u'(x)|^2 = G(u(x),\gamma) > 0 \quad \text{for all } x \in (0,L).$$

Since $u'(0) = \gamma > 0$, we see that $u'(x) > 0$ for all $x \in (0,L)$. So u is strictly increasing on the interval of existence. Thus u can never vanish at any $x > 0$. Thus u is not a Dirichlet solution. In other words, there is no Dirichlet solution for (3.6) if $\gamma > \gamma_\infty$.

Step 3. Assume $0 \le \gamma < \gamma_\infty$. Then, $G(y,\gamma)$ has three real zeros $h_{j,\gamma}$, $j = 1,2,3$, which satisfy (4.7). By using the argument in the proof of Theorem 3.2 we can show that, (3.6) has a Dirichlet solution u with support $[0, 2x_{2,\gamma}]$, where $x_{2,\gamma}$ is a maximum point of u, and

$$u(x_{2,\gamma}) = h_{2,\gamma} \quad \text{and} \quad \operatorname{supp}(u) = [0, 2x_{2,\gamma}].$$

Using (4.8) we can write

$$x_{2,\gamma} = \int_0^{h_{2,\gamma}} \frac{dy}{\sqrt{G(y,\gamma)}} = \int_0^{h_{2,\gamma}} \frac{\sqrt{3/2}\; dy}{\sqrt{(y-h_{1,\gamma})(y-h_{2,\gamma})(h_{3,\gamma}-y)}}\,. \tag{4.9}$$

When $0 \le \gamma < \gamma_\infty$ each of the three roots $h_{j,\gamma}$ is simple, hence the integral in the right side of the above equality converges, thus $x_{2,\gamma}$ is finite and positive.

Moreover, u is the only Dirichlet solution with the support $[0, 2x_{2,\gamma}]$ and with the maximum value $h_{2,\gamma}$. In fact, since $h_{2,\gamma}$ is the first positive root of $G(y,\gamma)$, we see that $a^* < h_{2,\gamma} < 1$ and hence $f(h_{2,\gamma}) > 0$. We apply the implicit function theorem to conclude that $h_{2,\gamma}$ is differentiable in γ for $\gamma \in [0,\gamma_\infty)$. Differentiate the equality,

$$\gamma^2 - 2F(h_{2,\gamma}) = 0,$$

with respect to γ and find that

$$\frac{d}{d\gamma}h_{2,\gamma} = \frac{\gamma}{f(h_{2,\gamma})} > 0 \quad \text{for all } 0 \le \gamma < \gamma_\infty.$$

Therefore, for $0 \le \gamma < \gamma_\infty$, the value $h_{2,\gamma}$ is strictly increasing in γ. If there are two Dirichlet solutions of (3.6) having the same support $[0, 2x_{2,\gamma}]$ and the maximum value $h_{2,\gamma}$, then they must be identical.

Step 4. Now consider the case that $\gamma = \gamma_\infty$ and $h_{2,\gamma_\infty} = h_{3,\gamma_\infty} = 1$. In this case the right side of the equality in (4.9) diverges to infinity (we may say that in this case $x_{2,\gamma_\infty} = \infty$). Hence the solution of the equation with $u'(0) = \gamma_\infty$ is positive for all $0 < x < \infty$. Therefore, there is no Dirichlet solution with $u'(0) = \gamma_\infty$.

Step 5. Let $x_{2,\gamma}$ be given in (4.9). Since $h_{j,\gamma}$ is continuous on $[0, \gamma_\infty)$ for each j, it is easy to check that, as a function of γ, $x_{2,\gamma}$ is continuous for $0 \le \gamma < \gamma_\infty$. Note that

$$\lim_{\gamma\to\gamma_\infty^-} h_{2,\gamma} = 1, \qquad \lim_{\gamma\to\gamma_\infty^-} h_{3,\gamma} = 1.$$

Hence, we conclude from (4.9) that

$$\lim_{\gamma\to\gamma_\infty^-} x_{2,\gamma} = \lim_{\gamma\to\gamma_\infty^-} \int_0^{h_{2,\gamma}} \frac{\sqrt{3/2}dy}{\sqrt{(y-h_{1,\gamma})(y-h_{2,\gamma})(h_{3,\gamma}-y)}} = +\infty.$$

We set

$$L_- = 2 \min_{\gamma\in[0,\gamma_\infty)} x_{2,\gamma}.$$

Since $x_{2,\gamma}$ is positive and continuous with respect to γ, the minimum domain size L_- is also strictly positive. In fact, since $0 \le \gamma < \gamma_\infty$ is the only regime that a Dirichlet solution may exist, there is no Dirichlet solution if $0 < L < L_-$. If $L \ge L_-$, since $x_{2,\gamma}$ is continuous with respect to γ, there exists $\gamma \in [0, \gamma_\infty)$ (maybe not unique) such that $2x_{2,\gamma} = L$. For this γ, (3.6) has a Dirichlet solution with the support $[0, 2x_{2,\gamma}]$ and $u'(0) = \gamma$. □

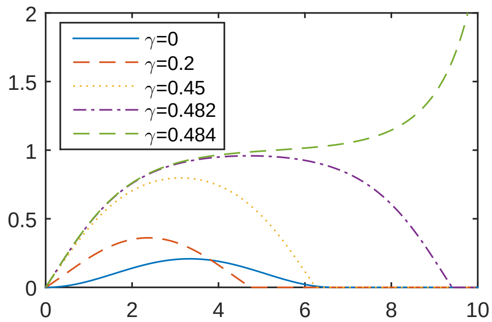


FIGURE 4. Dirichlet Solutions with $f(u) = (u-0.1)(1-u)\chi_{\{u>0\}}$. In this case, $\gamma_\infty \cong 0.483$.

There are two special domain sizes of interest related to the logistic growth with a discontinuity. The first one, denoted by L_0, is the domain size of the Neumann-Dirichlet solution which is obtained when $\gamma = 0$. The other is L_- given in Theorem 4.2, the minimum domain size of Dirichlet solutions. Therefore, obviously, $L_- \le L_0$. However, we do not know if $L_- < L_0$ or not. Suppose that $a^* > 0$ is small enough. Then, $h > 0$ is also small and the logistic type population dynamics in (4.5) becomes similar to the linear one in (4.1) at least for $u \in (0, h)$. Therefore, $L(\gamma) = 2x_{2,\gamma}$ decreases while $h_{2,\gamma}$ is small and hence $L_- < L_0$ (see Figure 4). This observation shows why positive Dirichlet solutions are not unique for a given domain size.

Unfortunately, it is not clear if $L_- < L_0$ when a^* is close enough to $1/3$. In Figure 5, Dirichlet solutions are given with $a^* = 0.3$. Numerical simulations still show $L_- \lesssim L_0$. However, the simulation is not convincing for either case.

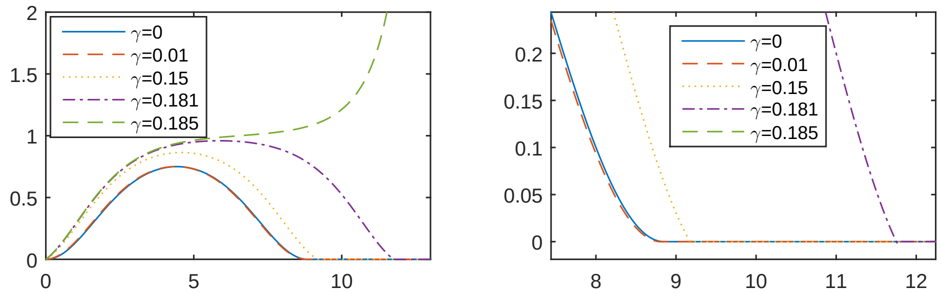


FIGURE 5. Dirichlet Solutions when $f(u) = (u - a^*)(1 - u)\chi_{\{u>0\}}$ and $a^* = 0.3$. Then, $\gamma_\infty = \sqrt{1/3 - a^*} \cong 0.183$.

5. Parabolic problem for population dynamics

In this section we consider the Cauchy problem of the parabolic equation (1.2) in $\mathbb{R}^n$,

$$\begin{cases} v_t = \Delta v + f(v), & x \in \mathbb{R}^n,\ t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.1}$$

and show the existence and the uniqueness of its solution. In particular, a comparison property between a super- and a sub-solution is obtained which will be used as a key tool in the following analysis. We have previously denoted the solution of the elliptic problem by u. Now we use v to denote the solution of this parabolic problem.

5.1. **Super- and sub-solutions in $\mathbb{R}^n$.** Notice that the discontinuity of f is not actually involved in obtaining nonnegative solutions of the elliptic problem (1.10)-(1.11) except the Neumann-Dirichlet solution. The discontinuity of f is truly activated for the parabolic problem and we are forced to consider a solution concept involving a set-valued function (like the Filippov solution for ordinary differential equations [11]).

We take a test function space denoted by

$$C_c^\infty(\mathbb{R}^n \times [0,T)) = \{\phi \in C^\infty(\mathbb{R}^n \times [0,T)) : \mathrm{supp}(\phi) \Subset \mathbb{R}^n \times [0,T)\}$$

and define solutions in a weak sense in the followings. We will mostly omit the word 'weak' and simply call super- and sub-solutions.

Definition 5.1. *Suppose that the left and right sided limits of f, given in* (1.5)*, exist and $v_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\overline{f}, \underline{f}$ and $\{f\}$ be given by* (1.6),(1.7) *and* (1.8)*, respectively, and a function v satisfy the following two conditions:*

(*i*) *For any compact domain $D \subset \mathbb{R}^n$ and $T > 0$,*

$$v \in C([0,T); L^2(D)) \cap L^2((0,T); H^1(D)).$$

(*ii*) $t \longmapsto \int e^{-|x|}v^2(x,t)dx$ *is continuous on* $[0,T)$ *and*

$$\sup_{0<t<T} \int e^{-|x|}\{v(x,t)^2 + |\nabla v|^2(x,t)\}dx < \infty.$$

Then,

$(iii)_1$ *The function v is called a weak sub-solution of the Cauchy problem* (5.1) *if, for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n \times [0,T))$,*

$$\iint (-v\phi_t + \nabla v \cdot \nabla \phi - \overline{f}(v)\phi)dxdt \le \int v_0(x)\phi(x,0)dx. \tag{5.2}$$

$(iii)_2$ *The function v is called a weak super-solution of the Cauchy problem* (5.1) *if, for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n \times [0,T))$,*

$$\iint (-v\phi_t + \nabla v \cdot \nabla \phi - \underline{f}(v)\phi)dxdt \ge \int v_0(x)\phi(x,0)dx.$$

$(iii)_3$ *The function v is called a weak solution of the Cauchy problem* (5.1) *if it is a weak super- and sub-solution at the same time or, equivalently, for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n \times [0,T))$,*

$$\int v_0(x)\phi(x,0)dx \in \iint (-v\phi_t + \nabla v \cdot \nabla \phi - \{f\}(v)\phi)dxdt. \tag{5.3}$$

Remark 5.2. *Discontinuous nonlinearity appears from obstacle problems with a different reason and context, and one can find related solution definition for it (e.g., see* [7]*). However, this definition is valid only when the solution stays on one side of an obstacle. On the other hand, the above definition is valid for the obstacle problems which now allows solution values to cross the discontinuities.*

5.2. **Existence.** We show the existence of a solution using a smooth approximation of the discontinuous reaction function f.

Theorem 5.1 (Existence). *Let $v_0 \in C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ be a non-negative and bounded function. Let f be a function satisfying* (1.3) *and there exist constants $b_1, b_0 > 0$ such that*

$$f(v) \le b_1 v + b_0 \quad \text{for } v > 0. \tag{5.4}$$

Then, there is a weak solution v of (5.1) *globally in time and $v \in C^{1+\alpha,(1+\alpha)/2}_{\mathrm{loc}}(\mathbb{R}^n \times [0,+\infty))$.*

Proof. Step 1. We shall approximate f by a sequence of smooth functions f^ϵ as in Figure 3(b), such that, for $0 < \epsilon < \epsilon_0$ with a fixed $\epsilon_0 > 0$,

(a) $f^\epsilon(0) = f^*$, and
$$f^\epsilon(v) \le 2(b_1 v + b_0) \quad \text{for all } v > 0 \text{ and for all } \epsilon > 0;$$
(b) $f^\epsilon(v) \downarrow f(v)$ as $\epsilon \to 0$ for all $v > 0$, $f(v) \le f^\epsilon(v) \le f(v) + \epsilon$ for $v \notin [0, \epsilon]$;
(c) $f^\epsilon(v) \le f^*$ for $v \in [0, \epsilon]$;
(d) there exists a constant $m(c) > 0$ for any $c > 0$ such that
$$|f^\epsilon(v)| \le m(c) \quad \text{for all } 0 \le v \le c, \ \ 0 < \epsilon < \epsilon_0.$$

Now we consider a regularized problem,

$$\begin{cases} v^\epsilon_t = \Delta v^\epsilon + f^\epsilon(v^\epsilon), & x \in \mathbb{R}^n,\ t > 0, \\ v^\epsilon(x,0) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{5.5}$$

In the following we shall show that (5.5) has a solution v^ϵ, which converges to a solution v of (5.1).

Step 2. Take $c_0 > 0$ such that

$$0 \le v_0(x) \le c_0 \quad \text{for all } x \in \mathbb{R}^n.$$

From conditions (1.3) and (a) we know that $\underline{v} = 0$ is a sub-solution of (5.5) and $\bar{v} = (c_0 + b_0/b_1)e^{2b_1 t} - b_0/b_1$ is a super-solution of (5.5). Using the monotonicity method (see [19, Theorem 3.1] and [21, Lemma 1.2]) we know that (5.5) has a global classical solution v^ϵ satisfying

$$0 \le v^\epsilon(x,t) \le \bar{v}(x,t) \quad \text{for all } x \in \mathbb{R}^n, \ \ t > 0.$$

In particular, for any $T > 0$ it holds that

$$0 \le v^\epsilon(x,t) \le \Big(c_0 + \frac{b_0}{b_1}\Big)e^{2b_1 T} - \frac{b_0}{b_1} =: c(T).$$

Then from condition (d) we have

$$|f^\epsilon(v^\epsilon(x,t))| \le M(T) := m(c(T))$$

for all $x \in \mathbb{R}^n$, $0 \le t \le T$, $0 < \epsilon < \epsilon_0$.

Step 3. Now we derive a local L^2 estimate of ∇v^ϵ. Denote

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}, \quad Q(R,T) = B_R \times (0,T).$$

We shall show that for any $R > 0$ and $T > 0$, there exists a constant $C(R,T) > 0$ which depends only on R, T, n, b_1, b_0, c_0, such that, for all $0 < \epsilon < 1$,

$$\sup_{0<t\le T} \int_{B_R} |v^\epsilon(t)|^2 + \iint_{Q(R,T)} |\nabla v^\epsilon|^2 \le C(R,T)(1 + \|e^{-|x|} v_0\|^2_{L^2(\mathbb{R}^n)}). \quad (5.6)$$

To prove, let $\eta_j(x)$ be a cut-off function with compact support. We multiply the equation (5.5) by $\eta_j^2 v^\epsilon$ and use condition (a) to get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n} |\eta_j v^\epsilon(t)|^2 dx + \iint_{\mathbb{R}^n\times(0,t)} |\nabla(\eta_j v^\epsilon)|^2 dxds \\ \le &\frac{1}{2}\int_{\mathbb{R}^n} |\eta_j v_0|^2 dx + \iint_{\mathbb{R}^n\times(0,t)} \{|v^\epsilon|^2 |\nabla \eta_j|^2 + (2b_1 + b_0)|\eta_j v^\epsilon|^2 + b_0 \eta_j^2\} dxds. \end{aligned}$$

Let η_j approach $\eta = e^{-|x|}$. Then $\nabla \eta_j$ approaches $-\eta \frac{x}{|x|}$. From the above inequality we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n} |\eta v^\epsilon(t)|^2 dx \\ \le &\frac{1}{2}\int_{\mathbb{R}^n} |\eta v_0|^2 dx + (1 + 2b_1 + b_0) \iint_{\mathbb{R}^n\times(0,t)} |\eta v^\epsilon|^2 dxds + b_0 t \int_{\mathbb{R}^n} \eta^2 dx. \end{aligned} \quad (5.7)$$

Denote

$$U(t) = \iint_{\mathbb{R}^n\times(0,t)} |\eta v^\epsilon|^2 dxds, \quad c_1 = \int_{\mathbb{R}^n} |\eta v_0|^2 dx, \quad c_2 = 2b_0 \int_{\mathbb{R}^n} \eta^2 dx.$$

From (5.7) we have

$$U'(t) \le c_3 U(t) + c_1 + c_2 t.$$

where $c_3 := 2(1 + 2b_1 + b_0)$. Let $W(t) := U(t) + \frac{c_2}{c_3} t + \frac{c_1 c_3 + c_2}{c_3^2}$. Then $W'(t) \le c_3 W(t)$ and $W(t) \le W(0) e^{c_3 t}$. This implies

$$U(t) \le c_1 \Big[\Big(1 + \frac{1}{c_3}\Big) e^{c_3 t} - \frac{1}{c_3}\Big] + \frac{c_2}{c_3^2}(e^{c_3 t} - 1).$$

Therefore for any fixed $T > 0$, there exists $C_1, C_0 > 0$ depending only on b_1, b_0, such that

$$\int_{\mathbb{R}^n} |\eta(x) v^\epsilon(x,t)|^2 dx \le C_1 \|\eta v_0\|^2_{L^2(\mathbb{R}^n)} + C_0 \quad \text{for all } 0 < t \le T. \quad (5.8)$$

On the other hand, in a similar fashion as before, we can derive

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n}|\eta_j v^\epsilon(t)|^2dx+\iint_{\mathbb{R}^n\times(0,t)}|\eta_j\nabla v^\epsilon|^2dxds\\ \le&\frac{1}{2}\int_{\mathbb{R}^n}|\eta_j v_0|^2dx+\iint_{\mathbb{R}^n\times(0,t)}\{2|\eta_j v^\epsilon|^2+\frac{1}{2}|\nabla\eta_j|^2|\nabla v^\epsilon|^2+2b_1|\eta_j v^\epsilon|^2+2b_0\eta_j^2v^\epsilon\}dxds.\end{aligned}$$

Let η_j approach $\eta=e^{-|x|}$. Then we have

$$\iint_{\mathbb{R}^n\times(0,t)}|\eta\nabla v^\epsilon|^2dxds\le\int_{\mathbb{R}^n}|\eta v_0|^2dx+4\iint_{\mathbb{R}^n\times(0,t)}\{(1+b_1)|\eta v^\epsilon|^2+b_0\eta^2v^\epsilon\}dxds.$$

From this and (5.8) we have

$$\begin{aligned}&\iint_{\mathbb{R}^n\times(0,T)}|\eta\nabla v^\epsilon|^2dxdt\le\|\eta v_0\|^2_{L^2(\mathbb{R}^n)}+4(1+b_1)(C_1\|\eta v_0\|^2_{L^2(\mathbb{R}^n)}+C_0)+c_2(T)\\ =&(1+4(1+b_1)C_1)\,\|\eta v_0\|^2_{L^2(\mathbb{R}^n)}+4(1+b_1)C_0+c_2(T).\end{aligned}$$

Now (5.6) follows from this and (5.8).

Step 4. Let ψ be the unique bounded solution of the heat equation

$$\begin{cases}\partial_t\psi=\Delta\psi, & x\in\mathbb{R}^n,\ t>0,\\ \psi(x,0)=v_0(x), & x\in\mathbb{R}^n.\end{cases}$$

Let $u^\epsilon=v^\epsilon-\psi$. Then

$$\begin{cases}u^\epsilon_t=\Delta u^\epsilon+f^\epsilon(v^\epsilon), & x\in\mathbb{R}^n,\ t>0,\\ u^\epsilon(x,0)=0, & x\in\mathbb{R}^n.\end{cases}$$

Let $\zeta(x)$ be a smooth cut-off function such that $\zeta(x)=1$ if $|x|<R$, $\zeta(x)=0$ if $|x|>R+1$. Let

$$w^\epsilon(x,t)=\zeta(x)u^\epsilon(x,t).$$

Then

$$\begin{cases}w^\epsilon_t=\Delta w^\epsilon+g^\epsilon, & (x,t)\in Q(R,T),\\ w^\epsilon(x,t)=0, & (x,t)\in\Gamma(R,T),\end{cases}\tag{5.9}$$

where

$$\begin{aligned}&g^\epsilon=\zeta(x)f^\epsilon(v^\epsilon)-u^\epsilon\Delta\zeta-2\nabla\zeta\cdot\nabla u^\epsilon,\\ &\Gamma(R,T)=B_R\times\{0\}\cup\partial B_R\times(0,T).\end{aligned}$$

Using conditions (a), (d) and applying the estimate (5.6) to both v^ϵ and ψ, we have

$$\begin{aligned}\|g^\varepsilon\|_{L^2(Q(R,T))}&\le C\{\|f^\epsilon(v^\epsilon)\|_{L^2(Q(R,T))}+\|u^\epsilon\|_{L^2(Q(R,T))}+\|\nabla u^\epsilon\|_{L^2(Q(R,T))}\}\\ &\le C\{M(T)|Q(R,T)|+\|\nabla v^\epsilon\|_{L^2(Q(R,T))}+\|\psi\|_{L^2(Q(R,T))}+\|\nabla\psi\|_{L^2(Q(R,T))}\}\\ &\le C\{1+\|e^{-|x|}v_0\|_{L^2(\mathbb{R}^n)}\},\end{aligned}$$

where C varies from line to line, and it depends only on R,T,n,b_1,b_0,c_0.

Applying the L^2 estimates of parabolic equations to (5.9) (see for instance [8, p.114, Corollary 2] we get

$$\|w^\epsilon\|_{W_2^{2,1}(Q(R,T))} \le C_1 \|g^\epsilon\|_{L^2(Q(R,T))}, \tag{5.10}$$

where C_1 depends only on R, T, n. The estimate (5.10) with R replaced by $2R$ also holds, from which we get

$$\|u^\epsilon\|_{W_2^{2,1}(Q(R,T))} \le A_1 \|g^\epsilon\|_{L^2(Q(2R,T))}.$$

Then by the Sobolev imbedding theorem we have $|\nabla u^\epsilon| \in L^{p_1}(Q(R,T))$, and

$$\|\nabla u^\epsilon\|_{L^{p_1}(Q(R,T))} \le A_1 \|g^\epsilon\|_{L^2(Q(2R,T))},$$

where $p_1 = 2(n+2)/n$. Therefore $g^\epsilon \in L^{p_1}_{\text{loc}}(\mathbb{R}^n \times (0,T))$, and

$$\|g^\epsilon\|_{L^{p_1}(Q(R,T))} \le B_1 \{\|f^\epsilon(v^\epsilon)\|_{L^{p_1}(Q(R,T))} + \|g^\epsilon\|_{L^2(Q(2R,T))}\},$$

Then we apply the L^p estimate of parabolic equations (see for instance [8, p.113, Theorem 4.2]) to get

$$\|w^\epsilon\|_{W_{p_1}^{2,1}(Q(R,T))} \le C_2' \{\|g^\epsilon\|_{L^{p_1}(Q(R,T))} + \|w^\epsilon\|_{W_2^{2,1}(Q(R,T))}\} \le C_2 \|g^\epsilon\|_{L^{p_1}(Q(R,T))},$$

where C_2', C_2 depend only on n, p, p_1, R, T. It follows that $|\nabla u^\epsilon| \in L^{p_2}(Q(R,T))$, and

$$\|\nabla u^\epsilon\|_{L^{p_2}(Q(R,T))} \le A_2 \|g^\epsilon\|_{L^2(Q(3R,T))},$$

where $p_2 = p_1(n+2)/(n+2-p_1)$, see [8, p.31, Theorem II.2.4]. Therefore $g^\epsilon \in L^{p_2}_{\text{loc}}(\mathbb{R}^n \times (0,T))$, and

$$\|g^\epsilon\|_{L^{p_2}(Q(R,T))} \le B_2 \{\|f^\epsilon(v^\epsilon)\|_{L^{p_1}(Q(R,T))} + \|g^\epsilon\|_{L^2(Q(2R,T))}\},$$

Iterating the above computations in $k = k(n)$ steps, we conclude that there is a $p > n+2$ such that

$$\|w^\epsilon\|_{W_p^{2,1}(Q(R,T))} \le C_k' \{\|g^\epsilon\|_{L^p(Q(R,T))} + \|w^\epsilon\|_{W_2^{2,1}(Q(R,T))}\} \le C_k \|g^\epsilon\|_{L^p(Q(R,T))},$$

and

$$\|g^\epsilon\|_{L^p(Q(R,T))} \le B_{k-1} \{\|f^\epsilon(v^\epsilon)\|_{L^p(Q(R,T))} + \|g^\epsilon\|_{L^2(Q(kR,T))}\},$$

hence

$$\|u^\epsilon\|_{W_p^{2,1}(Q(R,T))} \le C_k \{\|f^\epsilon(v^\epsilon)\|_{L^p(Q(2R,T))} + \|g^\epsilon\|_{L^2(Q((k+1)R,T))}\},$$

where C_k, B_k depend only on n, p, R, T.

Then by Sobolev imbedding theorem we see that $u^\epsilon \in C^{1+\alpha,(1+\alpha)/2}_{\text{loc}}(\mathbb{R}^n \times [0,\infty))$, and

$$\|u^\epsilon\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q(R,T)})} \le C_\alpha \{\|f^\epsilon(v^\epsilon)\|_{L^p(Q(2R,T))} + \|g^\epsilon\|_{L^2(Q((k+1)R,T))}\},$$

where C_α and k depend only on n, α, R, T.

Hence $v^\epsilon = u^\epsilon + \psi \in C^{1+\alpha,(1+\alpha)/2}_{\mathrm{loc}}(\mathbb{R}^n \times [0,\infty))$, and

$$\begin{aligned}\|v^\epsilon\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q(R,T)})} &\leq \|u^\epsilon\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q(R,T)})} + \|\psi\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q(R,T)})} \\ &\leq C_\alpha\{\|f^\epsilon(v^\epsilon)\|_{L^p(Q(2R,T))} + \|g^\epsilon\|_{L^2(Q((k+1)R,T))} + C(T)\|v_0\|_{C^{1+\alpha}(\overline{B_{2R}})}\}.\end{aligned} \tag{5.11}$$

Note the right-hand side has an upper bound which is independent of ϵ.

Step 5. For any $T > 0$, using estimate (5.11), we can take a sequence, which is still denoted by v^ϵ for simplicity, such that, for any $0 < \beta < \alpha$,

$$v^\epsilon \to v \quad \text{in } C^{1+\beta,(1+\beta)/2}(\overline{Q(R,T)}) \quad \text{as } \epsilon \to 0, \tag{5.12}$$

for any $R, T > 0$. Since the estimate for v^ϵ in (5.11) is uniform in ϵ, it follows that

$$v \in C^{1+\alpha,(1+\alpha)/2}_{\mathrm{loc}}(\mathbb{R}^n \times [0,+\infty)).$$

In particular

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}^n.$$

In the following we show that v is a weak solution of (5.1).

Obviously v satisfies the conditions (i) and (ii) in Definition 5.1 for any $T > 0$. We show that v satisfies $(iii)_3$ in Definition 5.1. Let $\phi \in C^\infty_c(\mathbb{R}^n \times [0,T))$ be a nonnegative test function and denote

$$\begin{aligned}\Sigma(T) =& \{(x,t) \in \mathrm{supp}(\phi) :\ v(x,t) = 0\}, \\ \Omega(T) =& \mathrm{supp}(\phi) \setminus \Sigma(T).\end{aligned}$$

From condition (b) we have

$$f^\epsilon(v^\epsilon(x,t)) \to f(v(x,t)) \quad \text{for every } (x,t) \in \Omega(T). \tag{5.13}$$

Since v^ϵ is a solution in the classical sense, we have

$$\int_{\mathbb{R}^n} v_0(x)\phi(x,0)dx = \iint_{\Omega(T)\cup\Sigma(T)} \{-v^\epsilon\phi_t + \nabla v^\epsilon \cdot \nabla\phi - f^\epsilon(v^\epsilon)\phi\}dxdt. \tag{5.14}$$

Notice that the right side is constant with respect to ϵ and hence the limit exits as $\epsilon \to 0$. Using (5.12) and (5.13) we see that

$$\begin{aligned}\lim_{\epsilon\to 0} \iint_{\Omega(T)} &\{-v^\epsilon\phi_t + \nabla v^\epsilon \cdot \nabla\phi - f^\epsilon(v^\epsilon)\phi\}dxdt \\ &= \iint_{\Omega(T)} \{-v\phi_t + \nabla v \cdot \nabla\phi - f(v)\phi\}dxdt.\end{aligned}$$

Therefore, the integral over $\Sigma(T)$ also converges and we estimate it below.

Let $\eta > 0$ be an arbitrarily given small number. Since $f(v) \to -f_*$ as $v \to 0+$, there exists $\delta > 0$ such that $f(v) \geq -f_* - \eta$ for all $0 < v \leq \delta$. Since $\Sigma(T)$ is compact, v^ϵ are smooth, and $v^\epsilon \to 0$ locally uniformly, there exists $\epsilon_1 < \delta$ such that

$$|v^\epsilon(x,t)| < \delta \quad \text{on } \ \Sigma(T) \quad \text{whenever } \ \epsilon < \epsilon_1.$$

By (b) and (c),

$$-f_* - \eta \le f(v^\epsilon) \le f^\epsilon(v^\epsilon) \le f^\delta(v^\epsilon) \le f^* \quad \text{on} \ \ \Sigma(T) \ \ \text{for} \ \ \epsilon < \epsilon_1.$$

Therefore, we have

$$\lim_{\epsilon\to 0} \iint_{\Sigma(T)} \{-v^\epsilon \phi_t + \nabla v^\epsilon \cdot \nabla\phi - f^\epsilon(v^\epsilon)\phi\} dxdt \tag{5.15}$$

$$\in \iint_{\Sigma(T)} \{-v\phi_t + \nabla v \cdot \nabla \phi - \{f\}_\eta(v)\phi\} dxdt,$$

where the set valued function $\{f\}_\eta(s)$ is similarly defined by $\{f\}_\eta(s) := \{h \in \mathbb{R} : \underline{f}(s) - \eta \le h \le \overline{f}(s)\}$. Since it holds for arbitrary $\eta > 0$, the previous relations, (5.14)-(5.15), imply (5.3). ☐

We have obtained the existence of a weak solution through a subsequential convergence. We will see via a comparison principle that the solution is unique and hence this subsequential convergence is actually a convergence.

Remark 5.3. *A growth condition such as* (5.4) *in Theorem 5.1 is needed for the global existence. For example, if $f(v) = v^p$ with $p > 1$, $v = (1-(p-1)t)^{-1/(p-1)}$ is a solution of* (5.1) *with the initial value $v_0 = 1$ which blows up at $t = 1/(p-1)$.*

Remark 5.4. *Note that the solution obtained in the theorem has a weak derivative with respect to t variable and $\partial_t v \in L^2(Q)$. In such a case, the three relations for the super-, sub-, and solutions in the definition, i.e.,* (5.2)-(5.3)*, are respectively equivalent to*

$$\iint (v_t\phi + \nabla v \cdot \nabla\phi - \overline{f}(v)\phi) dxdt \le 0, \tag{5.16}$$

$$\iint (v_t\phi + \nabla v \cdot \nabla\phi - \underline{f}(v)\phi) dxdt \ge 0, \tag{5.17}$$

and

$$0 \in \iint (v_t\phi + \nabla v \cdot \nabla\phi - \{f\}(v)\phi) dxdt. \tag{5.18}$$

5.3. **Comparison property in $\mathbb{R}^n$.** Next we show a comparison principle for the weak solutions of (5.1) under the assumption that $\partial_t v \in L^2(Q)$ for any compact subset $Q \Subset \mathbb{R}^n \times (0, \infty)$. The comparison principle naturally gives the uniqueness and the nonnegativity of the solution.

Theorem 5.2 (Comparison principle and uniqueness)**.** *Let f satisfy* (1.3)*.*

(*i*) *Let v_1 and v_2 be super- and sub-solutions of* (5.1)*, respectively, and $v_1(x,0) \le v_2(x,0)$. If $\partial_t v_1, \partial_t v_2 \in L^2(Q)$ for any compact subset $Q \Subset \mathbb{R}^n \times (0,\infty)$, then $v_1(x,t) \le v_2(x,t)$ for all $t > 0$ and $x \in \mathbb{R}^n$.*

(*ii*) *The weak solution of* (5.1) *is unique among functions such that $\partial_t v \in L^2(Q)$ for any compact subset $Q \Subset \mathbb{R}^n \times (0,\infty)$.*

(*iii*) *If v is such a weak solution with $v(x,0)\ge 0$, then $v(x,t)\ge 0$ for all $t>0$ and $x\in\mathbb{R}^n$.*

Proof. The uniqueness in (*ii*) follows from the comparison property (*i*) immediately. Since $\overline{f}(0)=f^*\ge 0$, $v_1=0$ is a sub-solution of (5.1) with $v_1(x,0)=0$ and (*iii*) follows from (*i*) with $v_2=v$. Hence, it is enough to show the comparison property (*i*).

Since $\partial_t v_1, \partial_t v_2 \in L^2(Q)$ for any compact subset $Q\Subset\mathbb{R}^n\times(0,\infty)$, we can take alternative relations, (5.16), (5.17), and (5.18) in the proof (see Remark 5.4). Subtracting (5.17) from (5.16) gives that, for a nonnegative test function $\phi(x,t)$,

$$\iint\Big((v_1-v_2)_t\phi+\nabla(v_1-v_2)\cdot\nabla\phi-(\overline{f}(v_1)-\underline{f}(v_2))\phi\Big)dxdt\le 0. \quad (5.19)$$

Remember that $\overline{f}$ and $\underline{f}$ are Lipschitzian on $\mathbb{R}^+$ and $\mathbb{R}^-$, have a *decreasing* discontinuity at $v=0$, are identical for $v\neq 0$, and $\overline{f}\ge\underline{f}$. Therefore, there exists an upper bound $c>0$ (but not a lower bound) such that

$$\frac{\overline{f}(v_1)-\underline{f}(v_2)}{v_1-v_2}\le c,\quad v_1,v_2\in\mathbb{R},\quad v_1\neq v_2.$$

This implies

$$\int_0^t\int(\overline{f}(v_1)-\underline{f}(v_2))w\eta\,dxdt\le\int_0^t\int cw^2\eta\,dxdt,$$

where $\eta(x)$ is a non-negative smooth function with a compact support and $w=(v_1-v_2)^+=\max\{v_1-v_2,0\}$. Substitute $\phi(x,t)=w(x,t)\eta(x)\chi_{[0,T]}(s)$ into (5.19)[2] and obtain

$$\int_0^T\int\left(w_tw\eta+|\nabla w|^2\eta+w\nabla w\cdot\nabla\eta-cw^2\eta\right)dxdt\le 0.$$

Since $w(x,0)\equiv 0$, we have

$$\int_0^T\int ww_t\eta\,dxdt=\frac12\int_0^T\Big(\frac{d}{dt}\int w^2\eta dx\Big)dt$$
$$=\frac12\int w^2(x,T)\eta(x)dx-\lim_{t\to0^+}\frac12\int w^2(x,t)\eta(x)dx=\frac12\int w^2(x,T)\eta(x)dx.$$

Therefore,

$$\frac12\int w^2(x,T)\eta(x)dx+\int_0^T\int\left(|\nabla w|^2\eta+w\nabla w\cdot\nabla\eta\right)dxdt\le c\int_0^T\int w^2\eta\,dxdt.$$

Now we take $\eta=\eta_j(x)\in C_c^\infty(\mathbb{R}^n)$ such that $0\le\eta_j\le e^{-|x|}$, $|\eta_{j,x_i}|\le e^{-|x_i|}$, $\eta_j(x)\to e^{-|x|}$ and $\eta_{j,x_i}\to-\frac{x_i}{|x|}e^{-|x|}$. By the condition (*ii*) in Definition

[2]Even if $w(x,t)\eta(x)\chi_{[0,T]}(s)$ is not smooth, we can do this using classical approximation arguments.

5.1, $w^2 e^{-|x|}, |\nabla w|^2 e^{-|x|} \in L^1(\mathbb{R}^n)$ so the Lebesgue dominated convergence theorem gives

$$\frac{1}{2}\int w^2(x,T)e^{-|x|}dx + \int_0^T \int \Big\{|\nabla w|^2 e^{-|x|} - \sum_{i=1}^n w w_{x_i} \frac{x_i}{|x|} e^{-|x|}\Big\} dxdt \leq c\int_0^T \int w^2 e^{-|x|} dxdt.$$

Since

$$\Big|\int_0^T \int w w_{x_i} \frac{x_i}{|x|} e^{-|x|} dxdt\Big| \leq \int_0^T \int \frac{1}{2}(w^2 + w_{x_i}^2) e^{-|x|} dx,$$

we get

$$\frac{1}{2}\int w^2(x,T)e^{-|x|}dx + \frac{1}{2}\int_0^T \int |\nabla w|^2 e^{-|x|} dxdt \leq (c + \frac{n}{2}) \int_0^T \int w^2 e^{-|x|} dxdt.$$

Since $T > 0$ is arbitrary, by Grönwall's inequality, we have $\int w^2(x,t)e^{-|x|}dx = 0$ for all $T > 0$. Therefore $w = 0$ and hence $v_1 \leq v_2$ as claimed. $\square$

Remember that the existence of a solution has been obtained in Theorem 5.1 when v_0 and ∇v_0 are bounded and continuous, and the obtained solution satisfies $\partial_t v \in L^2(Q)$ for every compact set $Q \Subset \mathbb{R}^n \times [0,\infty)$. The uniqueness is proved among such functions in Theorem 5.2. However, we do not know yet if there exists a weak solutions such that $\partial_t v \notin L^2(Q)$ or not. We can answer that if we show the comparison property among a larger class of functions or find an counter example. From now on we only consider soltuions such that $\partial_t v \in L^2(Q)$ for every compact set $Q \Subset \mathbb{R}^n \times [0,\infty)$ as if it is a part of definition or an admissibility condition.

6. Criterions for the Allee effect with spatial distribution

In this section we will consider nonnegative solutions in one space dimension and study the extinction and the expansion of a population. The main tool of this section is the comparison property between super- and sub-solutions obtained in Theorem 5.2. Hence, we consider solutions which satisfies

$$\partial_t v \in L^2(Q) \qquad \text{for all compact set} \quad Q \Subset \mathbb{R} \times [0,\infty).$$

In other words, we consider the solutions obtained in Theorem 5.1. We will use the comparison theory to show that the Neumann-Dirichlet solution in Theorem 3.2, denoted by $\Psi_0(x)$, provides a criterion of extinction. Recall that Φ_0 is a steady state with its support $[0, L]$ and

$$\max_{x\in[0,L]} \Psi_0(x) = \Psi_0(L/2) = h > 0.$$

Suppose that $v_0(x) \geq \Psi_0(x)$. Then, by the comparison property, we have $v(x,t) \geq \Psi_0(x)$ for all $t > 0$. Therefore, the solution never goes extinct.

In fact, the solution grows and the solution support expands. To show the solution dynamics, we use two properties of the reaction term. For $v > 0$, f is assumed to satisfy the following two conditions:

(1) There exists $c > 0$ such that
$$f(av) - af(v) < -c(1-a)v \quad \text{for all } \ 0 < a < 1, \ 0 < v < h. \tag{6.1}$$
(2) There exists $c > 0$ and $b_0 > 1$ such that
$$f(b_0 v) - b_0 f(v) > c(b_0 - 1) \quad \text{for all } \ 0 < v < h. \tag{6.2}$$

Note that, if $f(v)$ is continuous at $v = 0$ and $f(0) = 0$, then (6.2) is not satisfied. Hence, the discontinuity of $f(v)$ at $v = 0$ plays a key role in these assumptions.

Remark 6.1. *Let us consider cases when the two conditions are satisfied.*

(i) If f is analytic, (6.1) *and* (6.2) *can be written respectively as*
$$(a-1)f(0) > \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{n!}(a^n - a)s^n + c(1-a)s,$$
$$(1-b_0)f(0) > \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{n!}(b_0 - b_0^n)s^n + c(b_0 - 1)s.$$
Therefore, if these strict inequalities hold with $c = 0$, then we can also take a small enough $c > 0$ that satisfies the inequalities.

(ii) If (6.2) *holds for a given $b_0 > 1$, then it holds for all b such that $1 < b \le b_0$.*

(iii) If $f(u) = (u - a^*)\chi_{\{u>0\}}$, the above inequalities hold for all $s > 0$ and $b_0 > 1$.

(iv) If $f(u) = (u - a^*)(1-u)\chi_{\{u>0\}}$, the two hypotheses, (6.1) *and* (6.2)*, hold if $0 < a^* < \frac{23-\sqrt{448}}{9} \cong 0.2038$. Therefore the next Theorem covers this case with $a^* < 0.2038$.*[3]

Recall that, if there is no diffusion, the extinction criterion is given by the critical value a^*, where the solution of the ordinary differential equation (2.1) goes extinct in a finite time if and only if $u(0) < a^*$. However, for the solution of the reaction-diffusion equation (5.1), the extinction is not decided by the initial total population but by the spatial distribution of the initial value. We shall see that the Neumann-Dirichlet solution $\Psi_0(x)$ provides the information of survival, extinction, and expansion of a species.

We show in the following three theorems that, if the initial datum v_0 is strictly less than Ψ_0, the solution of (5.1) goes extinct in a finite time and, if the initial datum is strictly greater than Ψ_0, the solution support expands

[3]On the other hand, numerical simulation shows that the conclusion of next Theorem remains true for all $0 < a^* < \frac{1}{3}$ (see Figure 6). Therefore, the case of $0.2038 < a^* < \frac{1}{3}$ requires a different approach.

and the solution value grows. The proof is completed by finding appropriate super- and sub-solutions which are given in three separate lemmas.

6.1. **Extinction criterion.**

Lemma 6.1 (Shrinking super-solution)**.** *Let f satisfy* (1.3) *and* (6.1) *for a constant $c>0$. Then, $(a-c(1-a)t)\Psi_0(x)$ is a super-solution of* (5.1) *for any $0<a<1$ until $t\le T:=\frac{a}{c(1-a)}$, when it becomes identically zero.*

Proof. Using the relation

$$\underline{f}(\Psi_0(x)) = f(\Psi_0(x)) = -\Psi_{0,xx}(x) \quad \text{for} \quad \Psi_0(x)>0, \tag{6.3}$$

and the condition (6.1), we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(a-c(1-a)t)\Psi_0(x) - \frac{\partial^2}{\partial x^2}(a-c(1-a)t)\Psi_0(x) - \underline{f}((a-c(1-a)t)\Psi_0(x))\\
&\quad = -c(1-a)\Psi_0(x) + (a-c(1-a)t)\underline{f}(\Psi_0) - \underline{f}((a-c(1-a)t)\Psi_0)\\
&\quad \ge -c(1-a)\Psi_0(x) + c(1-a+c(1-a)t)\Psi_0(x)\\
&\quad = c^2(1-a)t\Psi_0(x) \ge 0,
\end{aligned}$$

where the relation (6.1) is valid for $a-c(1-a)t>0$, i.e., $t<T:=\frac{a}{c(1-a)}$. Hence $(a-c(1-a)t)\Psi_0(x)$ is a super-solution of (5.1) until $t\le T:=\frac{a}{c(1-a)}$. □

We first show that Ψ_0 is the extinction criterion. This extinction is completed in a finite time.

Theorem 6.2 (Finite time extinction)**.** *Let v be a global solution of* (5.1) *and the reaction term f satisfy* (1.3) *and* (6.1)*. If there is a constant $0<a<1$ such that*

$$0\le v_0(x)\le a\Psi_0(x) \quad \textit{for all } x\in\mathbb{R},$$

then $v(x,t)\equiv 0$ for all $t\ge T:=\frac{a}{c(1-a)}$.

Proof. Since $(a-c(1-a)t)\Psi_0(x)$ is a super-solution with its initial value $a\Psi_0(x)$,

$$0\le v(x,t)\le (a-c(1-a)t)\Psi_0(x) \quad \text{for all } x\in\mathbb{R},\ 0\le t\le T$$

by the comparison property and Lemma 6.1. Therefore, $v(x,T)\equiv 0$ and, by taking it as an initial value, we have $v(x,t)\equiv 0$ for all $t\ge T$. □

Remark 6.2. *In this theorem and following theorems, we assume the existence of the global solution. Refer the sufficient conditions for its existence in Theorem 5.1.*

6.2. **Expansion and blowup criterions.** Next, we show when the solution support expands by constructing a traveling wave sub-solution.

Lemma 6.3 (Traveling wave sub-solution)**.** *Let f satisfy* (1.3) *and* (6.2) *for some constants $b_0 > 1$ and $c > 0$. Then, for any $1 < b \le b_0$, $b\Psi_0(x+st)$ is a sub-solution of* (5.1) *for all $|s| \le s_0$, $s_0 := \frac{c(b-1)}{b \max |\Psi_0'|}$.*

Proof. Clearly, the inequality in (6.2) holds for all $1 < b \le b_0$ if it holds for a given $b_0 > 1$. From (6.2) and (6.3), we obtain

$$\begin{aligned}
&\partial_t(b\Psi_0(x+st)) - \partial_{xx}(b\Psi_0(x+st)) - f(b\Psi_0(x+st)) \\
&\quad = sb\Psi_0'(x+st) - b\Psi_{0,xx}(x+st) - f(b\Psi_0(x+st)) \\
&\quad = sb\Psi_0'(x+st) + bf(\Psi_0(x+st)) - f(b\Psi_0(x+st)) \\
&\quad < sb\Psi_0'(x+st) - c(b-1),
\end{aligned}$$

which is negative for all x and $t > 0$ if $|s| < s_0 := \frac{c(b-1)}{b \max |\Psi_0'|}$. Since $\Psi_0'(x)$ is continuous, s_0 is defined and positive. Therefore, $b\Psi_0(x+st)$ is a sub-solution of (5.1) if $|s| < s_0$. $\square$

This lemma shows that a traveling wave $b\Psi_0(x+st)$ is a sub-solution if the wave speed $|s|$ is small enough. If the initial value satisfies $v_0(x) \ge b\Psi_0(x)$, then $v(x,t) \ge b\Psi_0(x+st)$ for all small enough s. The size of maximum wave speed s_0 is decided by Ψ_0', c, and $b-1$. Remember that Ψ_0' and $c > 0$ are decided by the choice f and independent of the initial value. However, b depends on the initial value.

Theorem 6.4 (Survival and Expansion)**.** *Let v be a global solution of* (5.1) *with a nonnegative initial value $v_0 \ge 0$ and a reaction term f that satisfies* (1.3)*.*

(*i*) *(survival) Suppose that there exist $b > 1$ and $\gamma \ge 0$ such that*
$$v_0(x) \ge b\Psi_\gamma(x) \quad \textit{for all } x \in \mathbb{R}.$$
Then, $v(x,t) \ge \Psi_\gamma(x)$ for all $t > 0$.

(*ii*) *(expansion) Suppose that f satisfies* (6.2) *for some constant $b_0 > 1, c > 0$ and there exists $b > 1$ such that*
$$v_0(x) \ge b\Psi_0(x) \quad \textit{for all } x \in \mathbb{R}.$$
Then, the support of the solution expands to $\mathbb{R}$ and, for any $x_0 \in \mathbb{R}$,
$$\liminf_{t\to\infty} v(x_0,t) \ge \min(b_0,b)h.$$

Proof. The first part (*i*) is already done by the comparison property and holds for $b \ge 1$. This part is written here for record.

Let us show the second part (*ii*). Let $v(x,t)$ be the solution of (5.1) with the initial value $v_0(x) \ge \min(b_0,b)\Psi_0(x)$. Take s small enough that

$\min(b_0, b)\Psi_0(x+st)$ is the traveling wave type sub-solution of (5.1) given in Lemma 6.3 with its initial value $\min(b_0, b)\Psi_0(x)$. Therefore,

$$\min(b_0, b)\Psi_0(x+st) \le v(x,t) \quad \text{for all } x \text{ and } t \ge 0. \tag{6.4}$$

Let $x_0 \in \mathbb{R}$ be fixed. Then, we may take $t_0 = -\frac{x_0 - L/2}{s} > 0$ by choosing the sign of s as the opposite one of $x_0 - L/2$. Then, by (6.4),

$$\min(b_0, b)\Psi_0(x+st_0) \le v(x,t_0).$$

Since the stationary profile $\min(b_0, b)\Psi_0(x+st_0)$ also a sub-solution with zero traveling wave speed, we have

$$\min(b_0, b)\Psi_0(x+st_0) \le v(x,t) \quad \text{for all } x \text{ and } t \ge t_0.$$

In particular, substitute $x = x_0$ and obtain

$$v(x_0,t) \ge \min(b_0, b)\Psi_0(L/2) = \min(b_0, b)h \quad \text{for all } t \ge t_0,$$

which completes the proof of the second part. □

If the initial population distribution is slightly larger than Ψ_0, i.e, $b-1$ is small, then the local population increases up to a steady state. To show this dynamics we first construct a growing sub-solution.

Lemma 6.5 (Growing sub-solution)**.** *Let f satisfy* (1.3) *and* (6.2) *for some constants $b_0 > 1$ and $c > 0$. Let $1 < b < b_0$ be given and*

$$s(t) = 2b - b_0 + \frac{2(b_0 - b)}{1 + e^{-rt}}.$$

Then, if $0 < r < r_0 := \frac{c(b-1)}{2(b_0-b)h}$, $s(t)\Psi_0(x)$ is a sub-solution of (5.1) *for all $t > 0$ with its initial value $b\Psi_0(x)$.*

Proof. Note that $s(t)$ is a logistic function connecting $s(0) = b$, $\lim_{t\to\infty} s(t) = b_0$, and

$$0 < s'(t) = r\frac{2(b_0 - b)e^{-rt}}{(1+e^{-rt})^2} < 2r(b_0 - b).$$

Therefore, $b < s(t) < b_0$ for all $t > 0$. From (6.2) and (6.3), we obtain

$$\begin{aligned}
&\partial_t(s(t)\Psi_0(x)) - \partial_{xx}(s(t)\Psi_0(x)) - f(s(t)\Psi_0(x)) \\
&\quad = s'(t)\Psi_0(x) - s(t)\Psi_{0,xx}(x) - f(s(t)\Psi_0(x)) \\
&\quad = s'(t)\Psi_0(x) + s(t)f(\Psi_0(x)) - f(s(t)\Psi_0(x)) \\
&\quad \le s'(t)\Psi_0(x) - c(s(t) - 1) \\
&\quad \le s'(t)h - c(b-1).
\end{aligned}$$

Since $s'(t)h - c(b-1)$ is negative for all x, $t > 0$, and $0 < r < r_0$, $s(t)\Psi_0(x)$ is a sub-solution of (5.1) for all $t > 0$ and its initial value $b\Psi_0(x)$. □

Theorem 6.6 (Growth and blowing-up)**.** *Under the same assumptions as in Theorem 6.4,* (*i*) *(expansion and growth) For any* $x_0 \in \mathbb{R}$,

$$\liminf_{t\to\infty} v(x_0,t) \geq b_0 h.$$

(*ii*) *(blow-up) If* (6.2) *holds for all* $b_0 > 1$, *then* $v(x,t) \to +\infty$ *as* $t \to +\infty$ *for all* x.

Proof. The second part (*ii*) is obvious from the first part and we prove the first part (*i*). If $b > b_0$, then Theorem 6.4(*ii*) implies it. Let $1 < b < b_0$. Then, for any $\bar{b}$ between b and b_0, there exists t_0 such that $s(t_0) = \bar{b}$. Therefore, since $s(t)\Psi_0(x)$ in Lemma 6.5 is a sub-solution,

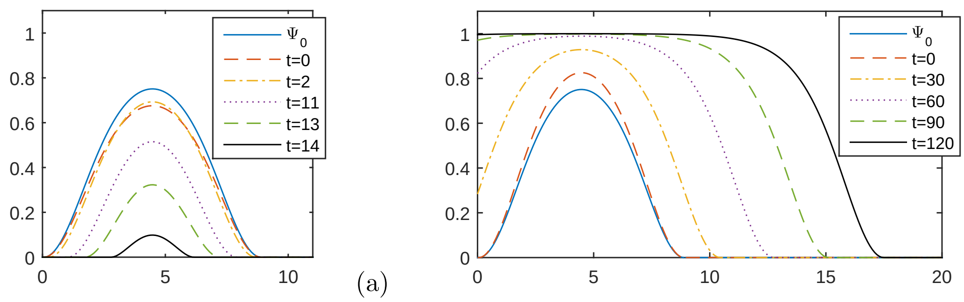
$$\bar{b}\Phi_0(x) \leq v(x,t_0) \quad \text{for all} \quad x \in \mathbb{R}.$$

Now we apply Theorem 6.4(*ii*) and conclude that

$$\liminf_{t\to\infty} v(x_0,t) \geq \bar{b}h \quad \text{for any} \quad \bar{b} < b_0.$$

Therefore, (*i*) is obtained. □

Two sets of numerical simulations of population evolution are given in Figure 6. The extinction phenomenon with $v_0 < \Psi_0$ is given in Figure 6(a). We can observe that the solution decreases slowly when it is close to the Neumann-Dirichlet solution Ψ_0. However, as soon as the solution becomes considerably smaller than Ψ_0, the extinction process is accelerated and finished in a finite time. The expansion phenomenon with $v_0 > \Psi_0$ is given in Figure 6(b). We can observe that the solution converges to the stable steady state $v = 1$ for any fixed x. As soon as the solution form a moving front that connects $v = 0$ and $v = 1$, the moving front moves with a constant speed.



(a) Extinction ($v_0 = 0.9\Psi_0$). (b) Expansion ($v_0 = 1.1\Psi_0$).

FIGURE 6. Evolution of parabolic problem (5.1) when $f(u) = (u - 0.3)(1 - u)\chi_{\{u>0\}}$.

7. Discussion

Extinction or death is one of key events of biological organisms. Having such dynamics correctly in a mathematical model may produce many interesting phenomena such as finite time extinction, propagation of a free boundary, pattern formation, and etc. The Allee effect, which is the phenomenon that the population goes extinct if the initial population size is smaller than a critical size, is one of such dynamics. Population extinction always appears in a finite time and a Lipschitzian reaction function does not give that by the Cauchy-Lipschitz theorem. Furthermore, if one takes a smooth bistable nonlinearity, the support of the population expands even under an extinction process (see Figure 1(a)), which never happens in reality.

A possible choice of such a population dynamics is

$$\frac{du}{dt} = (u - a^*)(1 - u)\chi_{\{u>0\}}\big(\equiv (-u^2 + (1 + a^*)u - a^*)\chi_{\{u>0\}}\big).$$

The characteristic function $\chi_{\{u>0\}}$ is multiplied since the population dynamics should stop if there is no population left. This model is a second order approximation of population dynamics and has the same sign as the logistic equation, i.e., positive first order and negative second order terms. This population dynamics is discontinuous at a stable steady state $u = 0$ and gives an example of nonlinearity with discontinuity considered in the Poisson and the reaction-diffusion equations,

$$-\Delta u = f(u) \quad \text{or} \quad v_t - \Delta v = f(v).$$

Therefore, to study phenomena related to finite time extinction, a development of mathematical theory to handle discontinuous reaction term is a key ingredient. Indeed, super-, sub-, and solutions have been defined for general discontinuous nonlinearity in Definition 3.1 for an elliptic case and in Definition 5.1 for a parabolic case when the left and right side limits of f exist. The existence and comparison theorems, Theorems 5.1 and 5.2, are proved when f satisfies (1.3). This is the case that the discontinuity f is only at one point $u = 0$, which is a stable steady state. It is not clear how far we can extend the theory. We expect that, if the discontinuity points of f has no cluster point and any of them is not an unstable steady state, one may guess the theory would hold. However, if f is discontinuous at an unstable steady states, then the problem is more challenging and the theory of this paper will fail.

To study the role of spatial distribution in an extinction event, we have restricted the problem to one space dimension and considered nonnegative solutions. For the elliptic problem case, we have constructed the Dirichlet-Neumann solution Ψ_0. This is a compactly supported solution that satisfies the Dirichlet and Neumann boundary condition at the same time. It has been shown in Theorems 6.2 and 6.4 that Ψ_0 gives a criterion for the initial distribution that determines the extinction or the survival of the population. It seems that Ψ_0 is the most effective population distribution for survival. In other words, one may conjecture that any other distribution with the same total population size cannot be a sub-solution.

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