On the Rate of Convergence and Asymptotic Profile of Solutions to the Viscous Burgers Equation

YONG-JUNG KIM & WEI-MING NI

ABSTRACT. In this paper we control the first moment of the initial approximations and obtain the order of convergence and the asymptotic profile of a general solution by two explicit "canonical" approximations: a diffusive N-wave and a diffusion wave solution. The order of convergence of both approximations is $O(t^{1/(2r)-3/2})$ in L^r norm, $1 \le r \le \infty$, as $t \to \infty$, which is faster than the well-known classical convergence order $O(t^{1/(2r)-1/2})$ for the inviscid Burgers equations case. A further comparison between the convergence rates of these two approximations and a discussion of the metastability phenomenon of the Burgers equation are also included. The method devised here allows us to obtain convergence up to any order by introducing new canonical solutions and controlling higher moments of the initial approximation.

1. INTRODUCTION

The main purpose of this paper is to understand the behavior of sign-changing solutions to the Cauchy problem of the viscous Burgers equation

$$\begin{cases} u_t + uu_x = \mu u_{xx}, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mu > 0$ is the viscosity constant (or diffusion rate), and, the initial value u_0 is continuous with a compact support and changes sign.

For the inviscid Burgers equation (i.e., $\mu = 0$ in (1.1))

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

it is well known that the two quantities

$$p(t) = -\inf_{x\in\mathbb{R}} \int_{-\infty}^{x} u(y,t)\,dy, \quad q(t) = \sup_{x\in\mathbb{R}} \int_{x}^{\infty} u(y,t)\,dy, \tag{1.3}$$

play important roles. Indeed, they are invariant in time, i.e., $p(t) = p(0)$ $(\equiv p)$ and $q(t) = q(0)$ $(\equiv q)$ for all $t \geq 0$, and the solution of (1.2) converges to an N-wave

$$N_{p,q}(x,t) = \begin{cases} \dfrac{x}{t}, & -\sqrt{2pt} < x < \sqrt{2qt}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

with the invariant positive and negative masses. For the viscous problem (1.1), these quantities are not constant anymore. Moreover, it is well known that, for each fixed time $t > 0$, as $\mu \to 0$ the solution of (1.1) tends to that of (1.2). On the other hand, E. Hopf [8] showed in 1950 that, for $\mu > 0$ fixed, as $t \to \infty$ the solution of (1.1) must converge to the well known diffusion wave of mass $M = -p + q$, which is actually the source solution of (1.1) with initial value $M\delta(x)$, a weighted Dirac-measure. Roughly speaking, the solution of (1.1) quickly evolves into a pattern of several N-waves. Then, after a series of interactions among these N-waves, a single N-wave emerges. This single N-wave lasts for a long time and eventually the positive and negative parts of this single N-wave merge into a single hump. In [11], the metastability of single N-waves for (1.1) was studied and the transition from an approximate N-wave to the final stage of a diffusion wave was made explicit.

In the study of the asymptotic behavior of conservation laws a number of techniques have been developed. We refer to [3], [4], and [16] for inviscid problems, [6], [7], and [17] for the convection-diffusion equations, and [5], [12], and [13] for systems. Diffusion waves and diffusive N-waves for the equal positive and negative masses are introduced in Whitham [20], and the technique is generalized to construct diffusive N-waves with unequal positive and negative masses in Kim and Tzavaras [11].

Observe that $-p(t) + q(t) = \int u_0(x)\,dx \equiv M$ and that, after a translation of the initial value in x-direction, it is convenient to assume that

$$p = -\int_{-\infty}^{0} u_0(y)\,dy, \quad q = \int_{0}^{\infty} u_0(y)\,dy, \tag{1.5}$$

without loss of generality. The optimal decay rate and the convergence to N-waves for the inviscid problems are well studied for more general equations and systems including $u_t + f(u)_x = 0$ (see [5], [13]). Liu and Pierre [16] show the L^r convergence for the power law, $f(u) = |u|^\gamma$, $\gamma > 1$. For the Burgers equation ($\gamma = 2$), their result reads

$$\lim_{t\to\infty} t^{(1/2-1/(2r))}\|u(x,t) - N_{p,q}(x,t)\|_r = 0, \quad 1 \le r < \infty. \tag{1.6}$$

Thus, if L^1-norm is considered, (1.6) gives the convergence, but not the convergence order. Dafermos [4, Chapter XI] proves the pointwise convergence for strictly convex flux $f(u)$ such that

$$\lim_{t\to\infty} t\|u(x,t) - N_{p,q}(x,t)\|_\infty = \text{constant}. \tag{1.7}$$

The optimal convergence rate in L^1-norm has been considered recently by Kim [10], and, for the Burgers case, it reads

$$\lim_{t\to\infty} \sqrt{t}\|u(x,t) - N_{p,q}(x,t)\|_1 = \text{constant}. \tag{1.8}$$

Moreover, under further minor conditions for the initial profile $u_0(x)$, it actually holds that

$$\lim_{t\to\infty} t\|u(x,t) - N_{p,q}(x,t)\|_1 = \text{constant}. \tag{1.9}$$

It would seem interesting to compare the convergence results under the presence of the viscosity, which gives extra regularity to the problem. In this paper, we shall study the evolution of the solutions to (1.1) more closely by finding explicit "*approximate solutions*", or, "*canonical solutions*". In particular, our result yields the profile of the solution to (1.1).

To describe our main result, we first set

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy, \quad U_0(x) = U(x,0) \tag{1.10}$$

for $x \in \mathbb{R}$, $t \ge 0$, and $u_0^+(x) = \max\{u_0(x), 0\}$, $u_0^-(x) = \max\{-u_0(x), 0\}$. Then we define

$$\begin{aligned} \tilde{u}(x,t) &= -2\mu \frac{\tilde{\varphi}(x,t)}{1 + \int_{-\infty}^{x} \tilde{\varphi}(y,t)\,dy}, \\ u^*(x,t) &= -2\mu \frac{\varphi^*(x,t)}{1 + \int_{-\infty}^{x} \varphi^*(y,t)\,dy}, \end{aligned} \tag{1.11}$$

where

$$\tilde{\varphi}(x,t) = \frac{c}{\sqrt{4\pi\mu t}} e^{-(x-\gamma)^2/(4\mu t)},$$
$$\varphi^*(x,t) = \frac{a}{\sqrt{4\pi\mu t}} e^{-(x-\alpha)^2/(4\mu t)} - \frac{b}{\sqrt{4\pi\mu t}} e^{-(x-\beta)^2/(4\mu t)},$$

with

$$(1.12)\qquad \begin{aligned} a &= \frac{1}{2\mu}\int_{-\infty}^{\infty} u_0^-(y) e^{-[1/(2\mu)]U_0(y)}\,dy, \\ b &= \frac{1}{2\mu}\int_{-\infty}^{\infty} u_0^+(y) e^{-[1/(2\mu)]U_0(y)}\,dy, \\ c = a - b &= -\frac{1}{2\mu}\int_{-\infty}^{\infty} u_0(y) e^{-[1/(2\mu)]U_0(y)}\,dy, \end{aligned}$$

and α, β, γ being points in $\mathbb{R}$ such that

$$(1.13)\qquad \begin{aligned} &\int_{-\infty}^{\infty} (x-\alpha) u_0^-(x) e^{-[1/(2\mu)]U_0(x)}\,dx = 0, \\ &\int_{-\infty}^{\infty} (x-\beta) u_0^+(x) e^{-[1/(2\mu)]U_0(x)}\,dx = 0, \\ &\int_{-\infty}^{\infty} (x-\gamma) u_0(x) e^{-[1/(2\mu)]U_0(x)}\,dx = 0. \end{aligned}$$

Since $u_0^\pm e^{-[1/(2\mu)]U_0(x)}$ are positive functions, such points α and β always exist and are unique. On the other hand, such a point γ may not exist and we shall only consider the case where it does exist. For example, if $M \neq 0$, such a γ exists. It will be proved in Section 3 below that there exists a time $T \geq 0$ such that the quantities

$$1 + \int_{-\infty}^{x} \varphi^*(y,t)\,dy, \quad 1 + \int_{-\infty}^{x} \tilde{\varphi}(y,t)\,dy$$

are uniformly bounded below by a positive constant for all $t > T$. Hence, $\tilde{u}(x,t)$ and $u^*(x,t)$ are well-defined by (1.11) for $t \geq T$.

Our main result is the existence of the constants $C_1, C_2 > 0$ and a time $T > 0$ depending on μ, u_0 and $1 \leq r \leq \infty$, such that

$$(1.14)\qquad \begin{aligned} t^{(3/2-1/(2r))}\|u(x,t) - \tilde{u}(x,t)\|_r &\leq C_1, \\ t^{(3/2-1/(2r))}\|u(x,t) - u^*(x,t)\|_r &\leq C_2, \end{aligned}$$

for all $t > T$. The detailed statement and its proof are given in Theorem 3.3.

It is well known that the solution $u(x,t)$ decays at the rate $t^{-1/2}$ for t large. (See e.g. [11].) Thus, intuitively speaking, the above result gives good approximations for the solution u of (1.1) with general initial data u_0 by the explicit "*canonical solutions*" $\tilde{u}$ and u^* defined in (1.11). Observe that u^* has the N-wave like structure and it plays the role as a diffusive N-wave.

The other canonical solution $\tilde{u}(x,t)$, which has the structure of a diffusion wave, can be considered as a solution of the Burgers equation with initial value $M\delta(x-y)$. If $M > 0$, $\tilde{u}(x,t)$ is a strictly positive solution and, hence, it does not represent the typical behavior of N-waves. So, it seems surprising that $\tilde{u}(x,t)$ approximate the solution with the same convergence order $O(t^{(1/(2r)-3/2)})$. However, we remark that the effectiveness of these two approximations $\tilde{u}$, u^* is reflected in the size of the two constants C_1 and C_2. For the asymptotic strutures of general systems of conservation laws with viscosity we refer readers to [1, 2, 14, 15, 18].

The convergence order achieved in (1.14) is higher than that of (1.6). If L^1 norm is considered, i.e., $p = 1$, the convergence order for the viscosity problem is higher than the optimal one for the inviscid problem with general initial value, (1.8). This convergence order is achieved for the inviscid problem as in (1.9) only with extra conditions imposed on the initial value.

The technique we are introducing in this article can be applied to obtain a higher convergence order. If the error of the initial approximation has zero moments up to n-th order, then the solution of the heat equation converges with order $O(t^{1/(2r)-(n+2)/2})$ in L^r norm, as in Theorem 2.5.

Our approach is as follows. In Section 3, by the Cole-Hopf transformation, the viscous Burgers equation (1.1) is reduced to the heat equation. So we consider the corresponding result for the heat equation first in Section 2. Canonical solutions for the heat equation are constructed by placing fundamental solutions of the correct size at the correct places, i.e., at the centers of masses. The results for the heat equation are then converted to that of the Burgers equation through the Cole–Hopf transformation in Section 3.

In Section 4, we consider the sensitivity of the Cole-Hopf transformation via the quantities p, q in (1.5) and show how the initial value u_0 is related to the constants α, β, a, b in (1.12) and (1.13). This property of the transformation reflects the metastability phenomenon of the Burgers equation. In the last section, Section 5, we present numerical examples. First we compare approximations by $\tilde{u}(x,t)$ and $u^*(x,t)$ and measure the effectiveness of u^* versus that of $\tilde{u}$ via the comparison between $\tilde{\varphi}$ and φ^*, their counterparts in the heat equation. The sensitivity of the Cole-Hopf transformation is then illustrated in Figure 5.2:small changes in the initial value may cause huge differences in the Cole-Hopf transformation.

2. THE HEAT EQUATION

We first formulate and study two kinds of "canonical solutions" to the following initial value problem for the usual heat equation

$$\begin{cases} \psi_t = \psi_{xx}, & x \in \mathbb{R}, t > 0, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where, for simplicity, ψ_0 is assumed to be continuous and to have a compact support $\subset [-R, R]$ with finite total mass $\int \psi_0(x)\,dx = c$. Note that we have reserved M for the total mass of solutions to the Burgers equation and we let 'c' denote the total mass of solutions to the heat equation. Throughout this entire section, we will only deal with the unique solution of (2.1) which is bounded near $x = \pm\infty$.

Letting $K(x,y,t)$ denote the fundamental solution of the heat equation, we have the following explicit representation of the solution ψ

$$\begin{aligned} \psi(x,t) &= \int_{-\infty}^{\infty} K(x,y,t)\psi_0(y)\,dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} \psi_0(y)\,dy. \end{aligned} \tag{2.2}$$

Lemma 2.1. *Let $\zeta(x,t)$ be the solution of (2.1) with its initial value ζ_0 being continuous and compactly supported. Then, for $1 \le r \le \infty$,*

$$\lim_{t\to\infty} t^{(3/2-1/(2r))} \|\zeta_{xx}\|_r = C_r \left| \int_{-\infty}^{\infty} \zeta_0(x)\,dx \right|, \tag{2.3}$$

where

$$C_r = \begin{cases} \dfrac{1}{\sqrt{4\pi}} \left(2\displaystyle\int_{-\infty}^{\infty} \left| \left[-\frac{1}{2} + \xi^2 \right] e^{-\xi^2} \right|^r d\xi \right)^{1/r} & \text{if } 1 \le r < \infty, \\ \dfrac{1}{4\sqrt{\pi}} & \text{if } r = \infty. \end{cases} \tag{2.4}$$

Proof. Let $\bar{c} = \int_{-\infty}^{\infty} \zeta_0(x)\,dx$. From (2.2), we have

$$\zeta_{xx}(x,t) = \frac{1}{\sqrt{4\pi t^3}} \int_{-\infty}^{\infty} \left[-\frac{1}{2} + \frac{(x-y)^2}{4t} \right] e^{-(x-y)^2/(4t)} \zeta_0(y)\,dy. \tag{2.5}$$

Consider the similarity variables $\xi = x/\sqrt{4t}$, $\xi' = y/\sqrt{4t}$. Then (2.5) is written as

$$t^{3/2}\zeta_{xx}(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[-\frac{1}{2} + (\xi-\xi')^2 \right] e^{-(\xi-\xi')^2} \sqrt{4t}\,\zeta_0(\sqrt{4t}\,\xi')\,d\xi'.$$

Note that $\sqrt{4t}\zeta_0(\sqrt{4t}\xi')$ converges to $\bar{c}\delta(\xi')$, a weighted Dirac-measure, as $t \to \infty$. So we get

$$\begin{aligned}\lim_{t\to\infty} t^{3/2}\zeta_{xx}(x,t) &= \lim_{t\to\infty} \frac{1}{\sqrt{4\pi}}\left(-\frac{1}{2}+\xi^2\right)e^{-\xi^2}\bar{c}\\ &= \lim_{t\to\infty} \frac{1}{\sqrt{4\pi}}\left(-\frac{1}{2}+\frac{x^2}{4t}\right)e^{-x^2/(4t)}\bar{c} = -\frac{1}{4\sqrt{\pi}}\bar{c}\end{aligned}$$

for any $x \in \mathbb{R}$ and, hence, $\lim_{t\to\infty} t^{3/2}\|\zeta_{xx}\|_\infty \ge C_\infty\|\zeta_0\|_1$. Since the support of $\sqrt{4t}\zeta_0(\sqrt{4t}\xi')$ is shrinking to the origin and $|-\frac{1}{2}+\xi^2|e^{-\xi^2} \le \frac{1}{2}$, we have

$$t^{3/2}|\zeta_{xx}(x,t)| \le \frac{1}{\sqrt{4\pi}}\left|\int_{-\infty}^{\infty}\frac{1}{2}\sqrt{4t}\zeta_0(\sqrt{4t}\xi')\,d\xi'\right| = \frac{1}{4\sqrt{\pi}}|\bar{c}|$$

for any $x \in \mathbb{R}$ and, hence, $\lim_{t\to\infty} t^{3/2}\|\zeta_{xx}\|_\infty \le C_\infty\|\zeta_0\|_1$. So the proof is complete for $r = \infty$. Now let $1 \le r < \infty$. Then

$$\begin{aligned}&\|\zeta_{xx}(x,t)\|_r\\ &= \left(\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{4\pi t^3}}\int_{-\infty}^{\infty}\left[-\frac{1}{2}+\frac{(x-y)^2}{4t}\right]e^{-(x-y)^2/(4t)}\zeta_0(y)\,dy\right|^r dx\right)^{1/r}\\ &= t^{(1/(2r)-3/2)}\frac{1}{\sqrt{4\pi}}\\ &\qquad\times\left(2\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\left[-\frac{1}{2}+(\xi-\xi')^2\right]e^{-(\xi-\xi')^2}\zeta_0(\sqrt{4t}\xi')\sqrt{4t}\,d\xi'\right|^r d\xi\right)^{1/r}.\end{aligned}$$

Setting $g_t(\xi) = \sqrt{4t}\zeta_0(\sqrt{4t}\xi)/\bar{c}$ and $f(\xi) = [-\frac{1}{2}+\xi^2]e^{-\xi^2}$, we may write

$$t^{(3/2-1/(2r))}\|\zeta_{xx}\|_r = \frac{|\bar{c}|\sqrt[r]{2}}{\sqrt{4\pi}}\|f * g_t\|_r,$$

where $f * g_t$ is the convolution between two functions. Clearly $\int g_t(\xi)\,d\xi = 1$, and standard arguments imply that $\|f * g_t\|_r \to \|f\|_r$ as $t \to \infty$. Hence, (2.3) holds for $1 \le r < \infty$. (See [19, p. 62]) ◻

The first derivative of $\zeta(x,t)$,

$$\zeta_x(x,t) = \frac{-1}{\sqrt{4\pi t^2}}\int_{-\infty}^{\infty}\frac{(x-y)}{\sqrt{4t}}e^{-(x-y)^2/(4t)}\zeta_0(y)\,dy,$$

can be similarly estimated. The results can also be written in a slightly different version, as follows.

Lemma 2.2. *Let $\zeta(x,t)$ be the solution of (2.1) with its initial value ζ_0 being continuous and compactly supported. Then, for each $1 \le r \le \infty$, there exists a constant $C > 0$, depending on ζ_0 and r, such that*

$$\|\zeta_x\|_r < Ct^{(1/(2r)-1)} \qquad \text{for all } t > 0, \tag{2.6}$$

and

$$\|\zeta_{xx}\|_r < Ct^{(1/(2r)-3/2)} \quad \text{for all } t > 0. \tag{2.7}$$

Now we construct canonical solutions for the original problem (2.1). A point $y \in \mathbb{R}$ is called a center of mass for a function ψ_0 if it satisfies

$$\int_{-\infty}^{\infty} (x-y)\psi_0(x)\,dx = 0. \tag{2.8}$$

Such a point exists uniquely if the total mass of ψ_0 is not zero, i.e., $c \neq 0$. In general it is possible that $y \notin [-R, R]$. Therefore we set

$$R_1 = \min\{y, -R\}, \quad R_2 = \max\{y, R\}.$$

If the initial value has zero total mass, then either there is no center of mass or every point is a center of mass. *In the following discussion we assume $y \in \mathbb{R}$ is a center of mass for the given initial value ψ_0.*

The first kind of canonical solution of the original problem (2.1) is given by

$$\tilde{\psi}(x,t) = \frac{c}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)} = cK(x,y,t). \tag{2.9}$$

That is, $\tilde{\psi}$ is the solution of the heat equation (2.1) with initial value $\tilde{\psi}(x,0) = c\delta_y$, where δ_y denote the Dirac measure at $x = y$. The solution $\tilde{\psi}$ is sometimes referred to as a "*canonical solution*" for (2.1) if ψ_0 has finite total mass c with y as its center of mass. We are interested in comparing the asymptotic behaviors of ψ and $\tilde{\psi}$ as $t \to \infty$, noting that the behavior of $\tilde{\psi}(x,t)$ is rather explicit.

To this end, we first let $\tilde{\rho}(x,t)$ denote the solution of the heat equation (2.1) with initial value

$$\tilde{\rho}(x,0) = \tilde{\rho}_0(x) = \int_{-\infty}^{x} \psi_0(y)\,dy - cH_y(x), \tag{2.10}$$

where

$$H_y(x) = \begin{cases} 0, & x < y, \\ 1, & x > y. \end{cases}$$

We may easily check that $\operatorname{supp} \tilde{\rho}_0 \subset [R_1, R_2]$. Next, we let $\tilde{\zeta}(x,t)$ be the solution of the heat equation (2.1) with initial value

$$\tilde{\zeta}_0(x) = \int_{-\infty}^{x} \tilde{\rho}_0(y)\,dy = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} \psi_0(z)\,dz - cH_y(y) \right) dy. \tag{2.11}$$

Then it is not hard to see that $\operatorname{supp}\tilde{\zeta}_0 \subset [R_1, R_2]$. For $x \le R_1$, it is clear that $\tilde{\zeta}_0(x) = 0$. For $x \ge R_2$, we have $\tilde{\zeta}_0(x) = \tilde{\zeta}_0(R_2)$, and

$$\begin{aligned}
\tilde{\zeta}_0(R_2) &= \int_{R_1}^{R_2}\int_{R_1}^{x} \psi(y)\,dy\,dx - c\int_{R_1}^{R_2} H_\gamma(x)\,dx \\
&= \int_{R_1}^{R_2}\int_{y}^{R_2} \psi(y)\,dx\,dy - c(R_2-\gamma) \\
&= \int_{R_1}^{R_2} [(R_2-\gamma)+(\gamma-y)]\psi(y)\,dy - c(R_2-\gamma) = 0,
\end{aligned}$$

in view of (2.8). On the other hand, $\tilde{\zeta}_x = \tilde{\rho}$, $\tilde{\zeta}_{xx} = \tilde{\rho}_x$, and, by (2.10),

$$\begin{aligned}
\tilde{\zeta}_x(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)}\left(\int_{-\infty}^{y}\psi_0(z)\,dz\right)dy \\
&\quad - \frac{c}{\sqrt{4\pi t}}\int_{\gamma}^{\infty} e^{-(x-y)^2/(4t)}\,dy \\
&= \Psi(x,t) - \tilde{\Psi}(x,t),
\end{aligned} \tag{2.12}$$

where

$$\Psi(x,t) = \int_{-\infty}^{x}\psi(y,t)\,dy, \quad \tilde{\Psi}(x,t) = \int_{-\infty}^{x}\tilde{\psi}(y,t)\,dy. \tag{2.13}$$

Thus, for $t > 0$,

$$\tilde{\zeta}_{xx}(x,t) = \psi(x,t) - \tilde{\psi}(x,t). \tag{2.14}$$

Now we compare the solution ψ of the problem (2.1) and our first canonical solution $\tilde{\psi}(x,t)$.

Theorem 2.3. *Let $\psi(x,t)$ be the solution of* (2.1) *with continuous and compactly supported initial value ψ_0. Suppose that there exists a point $\gamma \in \mathbb{R}$ satisfying* (2.8) *and $\tilde{\psi}$, $\tilde{\zeta}_0$, Ψ, $\tilde{\Psi}$ are given by* (2.9), (2.11), *and* (2.13). *Then, for each $1 \le r \le \infty$,*

$$\lim_{t\to\infty} t^{(3/2-1/(2r))}\|\psi(x,t)-\tilde{\psi}(x,t)\|_r = C_r\left|\int_{-\infty}^{\infty}\tilde{\zeta}_0(y)\,dy\right|, \tag{2.15}$$

with C_r given by (2.4), *and there exists $C > 0$, depending on ψ_0 and r, such that*

$$\|\Psi(x,t)-\tilde{\Psi}(x,t)\|_r < Ct^{1/(2r)-1}, \quad \text{for all } t > 0. \tag{2.16}$$

Proof. Let $\tilde{\zeta}(x,t)$ be the solution of the heat equation (2.1) with its initial value $\tilde{\zeta}_0$. We have seen that $\tilde{\zeta}_0$ has a compact support, $\tilde{\zeta}_x = \Psi(x,t) - \tilde{\Psi}(x,t)$ and $\tilde{\zeta}_{xx} = \psi(x,t) - \tilde{\psi}(x,t)$. So we may apply Lemma 2.1 and Lemma 2.2 to conclude (2.15) and (2.16). ◻

Next, we consider the second kind of canonical solutions. First, decompose the initial value,

$$\psi_0(x) = \psi_0^+(x) - \psi_0^-(x), \tag{2.17}$$

where $\psi_0^+(x) = \max\{\psi_0(x), 0\}$, $\psi_0^-(x) = \max\{-\psi_0(x), 0\}$. Then set

$$a = \int_{-\infty}^{\infty} \psi_0^+(x)\,dx, \quad b = \int_{-\infty}^{\infty} \psi_0^-(x)\,dx, \tag{2.18}$$

and denote by α, β, respectively, the unique points such that

$$\int_{-\infty}^{\infty} (x-\alpha)\psi_0^+(x)\,dx = 0 = \int_{-\infty}^{\infty} (x-\beta)\psi_0^-(x)\,dx. \tag{2.19}$$

In this case, since $\psi_0^\pm(x)$ are positive functions, α and β do exist and are unique. Moreover, $\alpha, \beta \in [-R, R]$. The second kind of canonical solution is defined by

$$\begin{aligned} \psi^*(x,t) &= \frac{a}{\sqrt{4\pi t}} e^{-(x-\alpha)^2/(4t)} - \frac{b}{\sqrt{4\pi t}} e^{-(x-\beta)^2/(4t)} \\ &= aK(x,\alpha,t) - bK(x,\beta,t), \end{aligned} \tag{2.20}$$

and $\Psi^*(x,t) = \int_{-\infty}^{x} \psi^*(y,t)\,dy$. Let

$$\zeta_0^*(x) = \int_{-\infty}^{x} \left[\int_{-\infty}^{y} \psi_0(z)\,dz - (aH_\alpha(y) - bH_\beta(y)) \right] dy, \tag{2.21}$$

and $\zeta^*(x,t)$ be the solution of the heat equation (2.1) with initial value ζ_0^*. Then, we can similarly show that $\operatorname{supp} \zeta_0^* \subset [-R, R]$, $\zeta_x^*(x,t) = \Psi(x,t) - \Psi^*(x,t)$ and $\zeta_{xx}^*(x,t) = \psi(x,t) - \psi^*(x,t)$. Therefore the comparison of the asymptotic behavior of $\psi(x,t)$ and $\psi^*(x,t)$ follows similarly.

Theorem 2.4. *Let $\psi(x,t)$ be the solution of (2.1) with continuous and compactly supported initial value ψ_0 which changes sign. Let a, b, α, β, $\psi^*(x,t)$, $\Psi^*(x,t)$, and $\zeta_0^*(x)$ be given by (2.17)-(2.21). Then, for each $1 \le r \le \infty$,*

$$\lim_{t\to\infty} t^{(3/2-1/(2r))} \|\psi(x,t) - \psi^*(x,t)\|_r = C_r \left| \int_{-\infty}^{\infty} \zeta_0^*(y)\,dy \right|, \tag{2.22}$$

with C_r given by (2.4), and there exists $C > 0$, depending on ψ_0 and r, such that

$$\|\Psi(x,t) - \Psi^*(x,t)\|_r < Ct^{1/(2r)-1} \quad \text{for all } t > 0. \tag{2.23}$$

We remark that the method used in establishing Theorem 2.3 and Theorem 2.4 also yields the following result, which may be of independent interest.

Theorem 2.5. *Let $\psi(x,t)$ be the solution of* (2.1) *with continuous and compactly supported initial value ψ_0. Suppose that*

$$\int_{-\infty}^{\infty} \psi_0(x)\,dx = \int_{-\infty}^{\infty} x\psi_0(x)\,dx = \cdots = \int_{-\infty}^{\infty} x^n\psi_0(x)\,dx = 0,$$

for some integer $n \geq 0$; then, for each $1 \leq r \leq \infty$, there exists a constant $C > 0$ depending on ψ_0 and r, such that

$$\|\psi(x,t)\|_r \leq Ct^{(1/(2r)-(n+2)/2)} \quad \text{for all } t > 0.$$

Finally, to conclude this section we consider the following heat equation with constant diffusion $\mu > 0$:

$$\text{(2.24)} \qquad \begin{aligned} \varphi_t &= \mu\varphi_{xx} && \text{in } \mathbb{R}\times(0,\infty),\\ \varphi(x,0) &= \varphi_0(x) && \text{in } \mathbb{R}. \end{aligned}$$

A simple change of variables $\psi(x,s) = \varphi(x,t)$, where $s = \mu t$, implies that $\psi_s = \varphi_t t_s = (1/\mu)\varphi_t = \varphi_{xx} = \psi_{xx}$, i.e., ψ is a solution of (2.1). So the asymptotic behavior of the solutions of (2.24) corresponding to Theorems 2.3 and 2.4 can be easily derived by replacing t with μt.

First, decompose the initial value φ_0 and write

$$\text{(2.25)} \qquad \varphi_0(x) = \varphi_0^+(x) - \varphi_0^-(x),$$

where $\varphi_0^+(x) = \max\{\varphi_0(x),0\}, \varphi_0^-(x) = \max\{-\varphi_0(x),0\}$. Then, set

$$\text{(2.26)} \qquad c = \int_{-\infty}^{\infty} \varphi_0(x)\,dx, \quad a = \int_{-\infty}^{\infty} \varphi_0^+(x)\,dx, \quad b = \int_{-\infty}^{\infty} \varphi_0^-(x)\,dx,$$

and denote γ, α, β, respectively, the unique points such that

$$\text{(2.27)} \qquad \begin{aligned} \int_{-\infty}^{\infty} (x-\gamma)\varphi_0(x)\,dx &= \int_{-\infty}^{\infty} (x-\alpha)\varphi_0^+(x)\,dx\\ &= \int_{-\infty}^{\infty} (x-\beta)\varphi_0^-(x)\,dx = 0. \end{aligned}$$

(Again, α and β are guaranteed to exist, while γ is only assumed to exist.) The canonical solutions for the problem (2.24) and their integrals are defined as

$$\text{(2.28)} \qquad \begin{aligned} &\tilde{\varphi}(x,t) = cK(x,\gamma,\mu t), \quad \text{with} \quad \tilde{\Phi}(x,t) = \int_{-\infty}^{x} \tilde{\varphi}(y,t)\,dy,\\ &\varphi^*(x,t) = aK(x,\alpha,\mu t) - bK(x,\beta,\mu t),\\ &\qquad\qquad \text{with} \quad \Phi^*(x,t) = \int_{-\infty}^{x} \varphi^*(y,t)\,dy. \end{aligned}$$

Let

$$
\begin{aligned}
\tilde{\zeta}_0(x) &= \int_{-\infty}^{x}\Big[\int_{-\infty}^{y}\varphi_0(z)\,dz - cH_\gamma(y)\Big]\,dy,\\
\zeta_0^*(x) &= \int_{-\infty}^{x}\Big[\int_{-\infty}^{y}\varphi_0(z)\,dz - (aH_\alpha(y) - bH_\beta(y))\Big]\,dy,
\end{aligned} \tag{2.29}
$$

and $\tilde{\zeta}(x,t)$, $\zeta^*(x,t)$ be, respectively, the solutions of the problem (2.24) with initial value $\tilde{\zeta}_0$, ζ_0^*. Then, we can similarly show that $\operatorname{supp}\zeta_0^* \subset [-R,R]$, $\operatorname{supp}\tilde{\zeta}_0 \subset [R_1,R_2]$ with $R_1 = \min\{\gamma,-R\}$ and $R_2 = \max\{\gamma,R\}$, $\tilde{\zeta}_x(x,t) = \Phi(x,t)-\tilde{\Phi}(x,t)$, $\zeta_x^*(x,t) = \Phi(x,t)-\Phi^*(x,t)$, $\tilde{\zeta}_{xx}(x,t) = \varphi(x,t)-\tilde{\varphi}(x,t)$ and that $\zeta_{xx}^*(x,t) = \varphi(x,t)-\varphi^*(x,t)$. So we may easily convert Theorems 2.3 and 2.4 for the problem (2.1) to the following result.

Theorem 2.6. *Let $\varphi(x,t)$ be the solution of* (2.24) *with φ_0 being continuous and compactly supported. Suppose further that φ_0 changes sign and γ in* (2.27) *exists. Let Φ, $\tilde{\varphi}$, $\tilde{\Phi}$, $\tilde{\zeta}_0$, φ^*, Φ^*, ζ_0^* be given by* (2.25)-(2.29). *Then, for each $1 \le r \le \infty$,*

$$
\begin{aligned}
\lim_{t\to\infty} t^{(3/2-1/(2r))}\|\varphi(x,t)-\tilde{\varphi}(x,t)\|_r &= C_r^\mu\Big|\int_{-\infty}^{\infty}\tilde{\zeta}_0(y)\,dy\Big|,\\
\lim_{t\to\infty} t^{(3/2-1/(2r))}\|\varphi(x,t)-\varphi^*(x,t)\|_r &= C_r^\mu\Big|\int_{-\infty}^{\infty}\zeta_0^*(y)\,dy\Big|,
\end{aligned} \tag{2.30}
$$

with

$$
C_r^\mu = \begin{cases}
\dfrac{1}{\sqrt{4\pi\mu^3}}\Big(2\sqrt{\mu}\displaystyle\int_{-\infty}^{\infty}\Big|\Big[-\frac{1}{2}+\xi^2\Big]e^{-\xi^2}\Big|^r\,d\xi\Big)^{1/r} & \text{if } 1 \le r < \infty,\\[2ex]
\dfrac{1}{4\sqrt{\pi\mu^3}} & \text{if } r = \infty,
\end{cases} \tag{2.31}
$$

and there exists $C > 0$, depending on φ_0, μ, and r, such that

$$
\begin{aligned}
\|\Phi(x,t)-\tilde{\Phi}(x,t)\|_r &< Ct^{1/(2r)-1} \quad \text{for } t>0,\\
\|\Phi(x,t)-\Phi^*(x,t)\|_{r*} &< Ct^{1/(2r)-1} \quad \text{for } t>0.
\end{aligned} \tag{2.32}
$$

It is clear from (2.30) that, to measure the effectiveness of the canonical solution φ^* versus that of $\tilde{\varphi}$, we need to compare the total masses of $\tilde{\zeta}_0$ and ζ_0^*. In the following example we consider the initial value

$$
\varphi_0(x) = \begin{cases} A, & -1 < x < 0,\\ -B, & 0 < x < 1,\\ 0, & \text{otherwise.} \end{cases} \tag{2.33}
$$

We first remark that, although we have only considered continuous initial value so far, the discontinuity of φ_0 in (2.33) can be overcome by continuous approximations φ_0^ε with $\|\varphi_0^\varepsilon - \varphi_0\|_1 < \varepsilon$, since the total masses of $\tilde{\zeta}_0$ and ζ_0^* have continuous dependence in the initial value φ_0. We assume $A > B > 0$ for convenience. Then we can easily check that the center of mass of the initial value φ_0 is $\gamma = (A+B)/[2(B-A)]$. Clearly, $\gamma < 0$, and γ could be smaller than -1. So

$$\int_{-\infty}^{\infty} \tilde{\zeta}_0(x)\,dx = \int_{-1}^{1}\int_{-1}^{x}\int_{-1}^{y} \varphi_0(z)\,dz\,dy\,dx - \int_{\gamma}^{1}\int_{\gamma}^{x}(A-B)H_\gamma(y)\,dy\,dx.$$

The quantity $\int_{-\infty}^{\infty} \zeta_0^*(x)\,dx$ is given similarly. Straightforward computations give

$$\int_{-\infty}^{\infty} \tilde{\zeta}_0(x)\,dx = \frac{A^2 - 14AB + B^2}{24(A-B)}, \qquad \int_{-\infty}^{\infty} \zeta_0^*(x)\,dx = \frac{A-B}{24},$$

and, hence, the ratio of the coefficients is

$$\frac{\left|\int_{-\infty}^{\infty} \tilde{\zeta}_0(x)\,dx\right|}{\left|\int_{-\infty}^{\infty} \zeta_0^*(x)\,dx\right|} = \left|1 - \frac{12AB}{(A-B)^2}\right|. \tag{2.34}$$

If $B = 0$, these two canonical solutions are identical, and the ratio becomes 1 as expected. We can also clearly see that, as $B \to A$, the ratio diverges to ∞. Hence, we may conclude that the approximation by φ^* is more effective if the negative and positive masses have similar sizes.

3. Reduction to the heat equation

It is well known that the viscous Burgers equation can be transformed to the heat equation by the Cole–Hopf transformation. Setting

$$\Phi(x,t) = e^{-[1/(2\mu)]U(x,t)} - 1, \tag{3.1}$$

where U, U_0 are given by (1.10), we have

$$\begin{aligned} &\Phi_t = \mu\Phi_{xx} && \text{in } \mathbb{R}\times(0,\infty),\\ &\Phi(x,0) = \exp\left(-\frac{1}{2\mu}U_0(x)\right) - 1 \ (\equiv \Phi_0(x)) && \text{in } \mathbb{R}. \end{aligned} \tag{3.2}$$

Simple computation shows

$$u(x,t) = -2\mu\frac{\Phi_x}{\Phi+1}. \tag{3.3}$$

Note that $\varphi(x,t) \equiv \Phi_x(x,t)$ also satisfies the heat equation

$$
(3.4)\qquad
\begin{aligned}
&\varphi_t = \mu\varphi_{xx}, && \text{in } \mathbb{R}\times(0,\infty),\\
&\varphi(x,0) = -\frac{1}{2\mu}u_0(x)[\Phi_0(x)+1]\ (\equiv \varphi_0(x)), && \text{in } \mathbb{R}.
\end{aligned}
$$

Now, from Theorem 2.6, we know that $\varphi(x,t)$ may be approximated by

$$
(3.5)\qquad
\begin{aligned}
\tilde{\varphi}(x,t) &= \frac{c}{\sqrt{4\pi\mu t}}e^{-(x-\gamma)^2/(4\mu t)},\\
\varphi^*(x,t) &= \frac{a}{\sqrt{4\pi\mu t}}e^{-(x-\alpha)^2/(4\mu t)} - \frac{b}{\sqrt{4\pi\mu t}}e^{-(x-\beta)^2/(4\mu t)},
\end{aligned}
$$

where

$$
(3.6)\qquad
\begin{aligned}
a &= \int_{-\infty}^{\infty}\varphi_0^+(x)\,dx = \frac{1}{2\mu}\int_{-\infty}^{\infty}u_0^-(x)e^{[-1/(2\mu)]U_0(x)}\,dx,\\
b &= \int_{-\infty}^{\infty}\varphi_0^-(x)\,dx = \frac{1}{2\mu}\int_{-\infty}^{\infty}u_0^+(x)e^{[-1/(2\mu)]U_0(x)}\,dx,\\
c &= \int_{-\infty}^{\infty}\varphi_0(x)\,dx = -\frac{1}{2\mu}\int_{-\infty}^{\infty}u_0(x)e^{[-1/(2\mu)]U_0(x)}\,dx,
\end{aligned}
$$

and γ, α, β, respectively, are the unique points such that

$$
(3.7)\qquad
\begin{aligned}
&\int_{-\infty}^{\infty}(x-\gamma)u_0(x)e^{[-1/(2\mu)]U_0(x)}\,dx = 0,\\
&\int_{-\infty}^{\infty}(x-\alpha)u_0^-(x)e^{[-1/(2\mu)]U_0(x)}\,dx = 0,\\
&\int_{-\infty}^{\infty}(x-\beta)u_0^+(x)e^{[-1/(2\mu)]U_0(x)}\,dx = 0.
\end{aligned}
$$

(Again, α, β always exist, and γ is assumed to exist.) This, in turn, implies that $u(x,t)$ may be approximated by

$$
(3.8)\qquad \tilde{u}(x,t) = -2\mu\frac{\tilde{\varphi}(x,t)}{\tilde{\Phi}(x,t)+1} \quad\text{and}\quad u^*(x,t) = -2\mu\frac{\varphi^*(x,t)}{\Phi^*(x,t)+1},
$$

where

$$
(3.9)\qquad
\begin{aligned}
\tilde{\Phi}(x,t) &= \int_{-\infty}^{x}\tilde{\varphi}(y,t)\,dy = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{x}ce^{-(y-\gamma)^2/(4\mu t)}\,dy,\\
\Phi^*(x,t) &= \int_{-\infty}^{x}\varphi^*(y,t)\,dy\\
&= \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{x}[ae^{-(y-\alpha)^2/(4\mu t)} - be^{-(y-\beta)^2/(4\mu t)}]\,dy,
\end{aligned}
$$

if we can give suitable positive lower bound for $\Phi(x,t)+1$, $\widetilde{\Phi}(x,t)+1$, and $\Phi^*(x,t)+1$.

Since $\Phi(x,t)+1$ also satisfies the heat equation, by the Maximum Principle, we conclude

$$0 < \min_{x\in\mathbb{R}} e^{[-1/(2\mu)]U_0(x)} \le \Phi(x,t)+1 \le \max_{x\in\mathbb{R}} e^{[-1/(2\mu)]U_0(x)} < \infty. \tag{3.10}$$

So $\Phi(x,t)+1$ is bounded below by a positive constant. Our next observation concerns a, b, and c.

Lemma 3.1. *Let a, b, and c be given by* (3.6), *with* $\int u_0(x)\,dx = M < \infty$. *Then,*

$$a-b+1 = c+1 = e^{[-1/(2\mu)]M} > 0. \tag{3.11}$$

Proof. Since

$$\begin{aligned} a-b &= -\frac{1}{2\mu}\int u_0 e^{[-1/(2\mu)]U_0(x)}\,dx \quad (=c) \\ &= \left[e^{[-1/(2\mu)]U_0(x)}\right]_{-\infty}^{\infty} = e^{[-1/(2\mu)]M} - 1, \end{aligned}$$

we clearly have (3.11). □

Since $c+1>0$, $\widetilde{\Phi}(x,t)+1$ is bounded below by a positive constant. Note that the quantity $a-b+1$ depends only on the total mass M of the initial value $u_0(x)$, and

$$a-b \longrightarrow \begin{cases} -1 & \text{as } M \to \infty, \\ 0 & \text{as } M \to 0, \\ \infty & \text{as } M \to -\infty. \end{cases} \tag{3.12}$$

Finally we consider the lower bound of $\Phi^*(x,t)+1$. In general, $\Phi^*(x,t)+1$ simply does not have a positive lower bound for all $x\in\mathbb{R}^n$ and for all $t\ge 0$. Nevertheless, the following holds and is sufficient for our purposes.

Lemma 3.2. *There exist $T\ge 0$ and $\delta>0$ such that*

$$\Phi^*(x,t)+1 > \delta > 0 \quad \text{for all } t > T. \tag{3.13}$$

Proof. After a translation of the initial value in x-direction, we may assume $\beta=-\alpha$ without loss of generality. Using the similarity variables,

$$\xi' = \frac{y}{\sqrt{4\mu t}}, \quad \xi = \frac{x}{\sqrt{4\mu t}},$$

(3.9) is written as

$$\Phi^*(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x}[ae^{-(\xi'-\alpha/\sqrt{4\mu t})^2} - be^{-(\xi'+\alpha/\sqrt{4\mu t})^2}]\,d\xi'.$$

First, suppose that $\alpha < 0$ (or $\alpha < \beta$). Then it is clear that

$$\begin{aligned}
\min_x \Phi^*(x,t) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}[ae^{-(\xi'-\alpha/\sqrt{4\mu t})^2} - be^{-(\xi'+\alpha/\sqrt{4\mu t})^2}]\,d\xi' \\
&= \frac{1}{\sqrt{\pi}}a\int_{-\infty}^{\infty} e^{-(\xi'-\alpha/\sqrt{4\mu t})^2}\,d\xi' - b\int_{-\infty}^{\infty} e^{-(\xi'+\alpha/\sqrt{4\mu t})^2}\,d\xi' \\
&= a - b > -1,
\end{aligned}$$

so (3.13) holds with $T = 0$.

Next, suppose that $\alpha > 0$ (or $\beta < \alpha$). Then, a simple computation shows

$$\varphi^*(x,t) = 0 \Longleftrightarrow x = \frac{\mu t}{\alpha}\ln\left(\frac{b}{a}\right).$$

Using the similarity variable, we get

$$\varphi^*(\xi,t) = 0 \Longleftrightarrow \xi = \frac{\sqrt{\mu t}}{2\alpha}\ln\left(\frac{b}{a}\right) \quad (\equiv \xi(t)),$$

where

$$\lim_{t\to\infty}\xi(t) = \begin{cases} \infty & \text{if } b > a, \\ 0 & \text{if } b = a, \\ -\infty & \text{if } b < a. \end{cases}$$

It is clear that

$$\min_x \Phi^*(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\xi(t)}[ae^{-(\xi-\alpha/\sqrt{4\mu t})^2} - be^{-(\xi+\alpha/\sqrt{4\mu t})^2}]\,d\xi.$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{t\to\infty}\min_x \Phi^*(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\xi(\infty)}(a-b)e^{-\xi^2}\,d\xi = \begin{cases} a-b & \text{if } b > a, \\ 0 & \text{if } b \le a. \end{cases}$$

Since $a - b + 1 > 0$, there exists $T \ge 0$ such that (3.13) holds. □

Now we can estimate the error when the solution $u(x,t)$ is approximated by $u^*(x,t)$ or $\tilde{u}(x,t)$, and prove our main result.

Theorem 3.3. *Let $u(x,t)$ be the solution of* (1.1)*, with its initial value u_0 being continuous and compactly supported. Suppose that $\int u_0(x)\,dx = M \in \mathbb{R}$, u_0 changes sign, and $1 \le r \le \infty$.*

(i) *Then there exist $T \ge 0$, depending on u_0, μ, such that $u^*(x,t)$ is well defined for $t > T$, and $C_2 > 0$, depending on u_0, μ, p, such that*

$$\|u(x,t) - u^*(x,t)\|_r < C_2 t^{(1/(2r)-3/2)}, \quad t > T. \tag{3.14}$$

(ii) *Suppose further that $M \ne 0$. Then $\tilde{u}(x,t)$ is well defined, and there exists $C_1 > 0$, depending on u_0, μ, p, such that*

$$\|u(x,t) - \tilde{u}(x,t)\|_r < C_1 t^{(1/(2r)-3/2)}, \quad t > 0. \tag{3.15}$$

Proof. The existence of such a time $T \ge 0$ has been established in Lemma 3.2. The difference between $u(x,t)$ and $u^*(x,t)$ is estimated by

$$\begin{aligned} |u(x,t) - u^*(x,t)| &= \left| 2\mu \left[\frac{\varphi^*(x,t)}{\Phi^*(x,t)+1} - \frac{\varphi(x,t)}{\Phi(x,t)+1} \right] \right| \\ &\le \frac{2\mu}{(\Phi(x,t)+1)(\Phi^*(x,t)+1)} (|\varphi^* - \varphi|(1+|\Phi|) + |\varphi|\,|\Phi - \Phi^*|). \end{aligned} \tag{3.16}$$

From (3.10) and Lemma 3.2, it follows that the terms above can be bounded by a constant multiple of $t^{-3/2}$ in view of Theorem 2.6 and the fact that

$$|\varphi(x,t)| \le Ct^{-1/2} \tag{3.17}$$

for t large, where the constant involved here depends on μ and the initial value u_0. Thus, for t large,

$$\|u(x,t) - u^*(x,t)\|_\infty \le Ct^{-3/2}$$

uniformly in $x \in \mathbb{R}$. So (3.14) holds for $r = \infty$.

Fix $1 \le r < \infty$. Then we have

$$\|u(x,t) - u^*(x,t)\|_r \le C_0(\|\varphi^* - \varphi\|_r(1 + \|\Phi\|_\infty) + \|\varphi\|_\infty \|\Phi - \Phi^*\|_r), \tag{3.18}$$

with $C_0 = 2\mu/[(\min\Phi + 1)(\min\Phi^* + 1)]$. From (3.17) and Theorem 2.6, (3.14) follows for $1 \le r < \infty$, so (i) is complete. Part (ii) can be proved similarly. □

4. Cole-Hopf Transformation and Metastability

The heat equation in (2.24) is invariant under the change of variable $y = -x$. On the other hand, the Burgers equation in (1.1) is not. Nevertheless, the Burgers equation can be transformed to the heat equation by the Cole-Hopf transformation. Therefore it seems natural to expect that the transformation be sensitive on such a change of variables. The sensitivity is reflected in quantities p, q in (1.5) under the change of variables. For example, if the initial value is given by

$$u_0(x) = \begin{cases} \sin x, & -\pi < x < 0, \\ \dfrac{1}{2}\sin x, & 0 < x < \pi, \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

then $p = 2$ and $q = 1$. After the change of variable, the transformed initial data $\bar{u}_0(x) = u_0(-x)$ satisfies $p = 1$, $q = 0$.

In this section we wish to relate the metastability phenomenon of the canonical solution $u^*(x,t)$ to the initial value $u_0(x)$, via the Cole-Hopf Transformation and the reflection $x \to -x$. First, we consider an initial value with a single sign-change, and we may assume that the sign change occurs at the origin $x = 0$. Suppose that

$$\begin{aligned} &u_0(x) \le 0 \quad \text{for } x < 0, \qquad p = -\int_{-\infty}^0 u_0(x)\,dx > 0, \quad \text{and} \\ &u_0(x) \ge 0 \quad \text{for } x > 0, \qquad q = \int_0^\infty u_0(x)\,dx > 0. \end{aligned}$$

Then we can easily check that

$$\begin{aligned} a &= -\frac{1}{2\mu}\int_{-\infty}^0 u_0 e^{[-1/(2\mu)]U_0(x)}\,dx = e^{p/(2\mu)} - 1, \\ b &= \frac{1}{2\mu}\int_0^\infty u_0 e^{[-1/(2\mu)]U_0(x)}\,dx = e^{p/(2\mu)} - e^{-M/(2\mu)}. \end{aligned}$$

The centers α, β given in (3.7) are clearly ordered by $\alpha < 0 < \beta$, and we may say that $\varphi^*(x,t)$ is the solution of (2.24) with its initial value $\varphi_0^*(x) = a\delta_\alpha(x) - b\delta_\beta(x)$. The weights a, b increase exponentially as $\mu \to 0$, and this exponential growth implies the metastability of the Burgers equation translated to the heat equation.

From (3.9), we have $\Phi^*(0,0) = a$ and $\Phi^*(\infty,0) = a - b$, and it is clear from (3.8) that the corresponding initial value for u^* is also a summation of Dirac-δ functions centered at α and β. Since

$$\begin{aligned} \int_{-\infty}^0 u^*(x,0)\,dx &= -2\mu\int_{-\infty}^0 \frac{\varphi^*}{\Phi^*+1}\,dx = -2\mu[\ln(\Phi^*+1)]_{-\infty}^0 = -p, \\ \int_0^\infty u^*(x,0)\,dx &= -2\mu\int_0^\infty \frac{\varphi^*}{\Phi^*+1}\,dx = -2\mu[\ln(\Phi^*+1)]_0^\infty = M + p = q, \end{aligned}$$

we may conclude that $u^*(x,t)$ is the solution of the Burgers equation (1.1) with its initial value $-p\delta_\alpha(x) + q\delta_\beta(x)$.

If $u_0(x) \geq 0$ for $x < 0$ and $u_0(x) \leq 0$ for $x > 0$, then $p = 0$ or $q = 0$. Suppose that $M > 0$. Then $p = 0$, $q = M$, and

$$a = -\frac{1}{2\mu}\int_0^\infty u_0 e^{[-1/(2\mu)]U_0(x)}\,dx = e^{-M/(2\mu)} - e^{-A/(2\mu)},$$
$$b = \frac{1}{2\mu}\int_{-\infty}^0 u_0 e^{[-1/(2\mu)]U_0(x)}\,dx = 1 - e^{-A/(2\mu)},$$

where $A = \int_{-\infty}^0 u_0(x)\,dx > 0$. In this case the weights a, b are uniformly bounded, regardless of the size of $\mu > 0$ and the metastable phenomenon is not observed.

Next, we consider an initial value with finite number of sign-changes. Again, first let

$$u_0(x) \leq 0 \quad \text{on} \bigcup(z_{2k-1}, z_{2k}), \qquad u_0(x) \geq 0 \quad \text{on} \bigcup(z_{2k}, z_{2k+1}),$$

where $k = 0, 1, \ldots, n$ with convention that $z_{-1} = -\infty, z_{2n+1} = \infty$, and

$$U_0(z_k) = \int_{-\infty}^{z_k} u_0(x)\,dx = -p_k. \tag{4.2}$$

Note that, to keep the consistency with (1.5), we consider p_k with the negative sign. After a translation of the initial value we may assume $z_{2m} = 0$, $0 \leq m \leq n$, and

$$p_{2m} = -\int_{-\infty}^0 u_0(x)\,dx = -\inf_{x\in\mathbb{R}}\int_{-\infty}^x u_0(y)\,dy = p$$

without loss of generality, where p is the invariant variable given by (1.5). Consider

$$\begin{aligned} a &= \frac{1}{2\mu}\sum_{k=0}^{n}\int_{z_{2k-1}}^{z_{2k}} u_0^- e^{[-1/(2\mu)]U_0(x)}\,dx \\ &= \sum_{k=0}^{n}[e^{-U_0(x)/(2\mu)}]_{z_{2k-1}}^{z_{2k}} = \sum_{\ell=0}^{2n}(-1)^\ell e^{p_\ell/(2\mu)} - 1. \end{aligned} \tag{4.3}$$

Suppose that $p_k < p$ for all $k \neq 2m$. Then we can easily see that the summation has a dominant term $e^{p_{2m}/(2\mu)}$ for a small enough $\mu \ll 1$ and, hence, we have $a = d(\mu)e^{p/(2\mu)} - 1$ for some $1 \leq d(\mu) \leq n$ such that $d(\mu) \to 1$ as $\mu \to 0$. Thus, for μ sufficiently small,

$$\int_{-\infty}^0 u^*(x,0)\,dx = -2\mu[\ln(e^{p/(2\mu)}) + \ln(d(\mu))] = -(p + 2\mu\ln(d)) \simeq -p.$$

Similarly, the other side of the weighted total mass

$$
\begin{aligned}
(4.4)\qquad b &= \frac{1}{2\mu}\sum_{k=0}^{n}\int_{z_{2k}}^{z_{2k+1}} u_0^+ e^{-U_0(x)/(2\mu)}\,dx \\
&= -\sum_{k=0}^{n}\left[e^{-U_0(x)/(2\mu)}\right]_{z_{2k}}^{z_{2k+1}} = \sum_{\ell=0}^{2n}(-1)^{\ell} e^{p_\ell/(2\mu)} - e^{-M/(2\mu)}
\end{aligned}
$$

has the dominant term $e^{p_{2m}/(2\mu)}$ for a small enough $\mu \ll 1$ and, moreover, $\int_0^\infty u^*(x,0)\,dx \simeq q$. In this case the corresponding initial value for u^* is not exactly $-p\delta_\alpha(x)+q\delta_\beta(x)$, but $u^*(x,0) \to -p\delta_\alpha(x)+q\delta_\beta(x)$ as $\mu \to 0$. Since the negative mass of $u_0^- e^{-U_0/(2\mu)}$ and the positive mass of $u_0^+ e^{-U_0/(2\mu)}$ are dominated by the components on (z_{2m-1}, z_{2m}) and (z_{2m}, z_{2m+1}), respectively, we can easily see that its centers of mass are ordered by $\alpha < 0 < \beta$ for μ sufficiently small.

5. NUMERICAL EXAMPLES

In this section we consider several numerical examples that support and explain the theories and observations in the previous two sections. In the first set of examples we compare diffusive N-waves $u^*(x,t)$ and diffusion waves $\tilde{u}(x,t)$. These examples demonstrate that u^* is a better approximation than $\tilde{u}$ is under suitable circumstances. The second set of examples are designed to illustrate the sensitivity of the Cole-Hopf transformation on the change of variables $x \to -x$, which reflects the metastability of the Burgers equation. Throughout this section we continue to use the functions u, $\tilde{u}$, u^*, φ, Φ, $\tilde{\varphi}$, $\tilde{\Phi}$, φ^*, Φ^* and the constants a, b, c, α, β, γ given by (3.1)-(3.9).

***5.1. Comparison between* $\tilde{\mathbf{u}}$ *and* $\mathbf{u}^*$.** If the inviscid problem is considered ($\mu = 0$), the quantities p, q in (1.5) are invariant variables, and the solution converges to the N-wave $N_{p,q}(x,t)$. So, if $p > 0$, $q > 0$, and $0 < \mu \ll 1$, it seems natural to expect that $u^*(x,t)$ be a better approximation than $\tilde{u}(x,t)$.

Let $u(x,t)$ be the solution of the Burgers equation (1.1) with its initial value u_0 given by (4.1) and $\varphi(x,t)$ be the solution of (2.24) with its initial value

$$
(5.1)\qquad \varphi_0(x) = \begin{cases} -\dfrac{\sin x}{2\mu} e^{(1+\cos x)/(2\mu)}, & -\pi < x < 0, \\ -\dfrac{\sin x}{4\mu} e^{(3+\cos x)/(4\mu)}, & 0 < x < \pi, \\ 0, & \text{otherwise}, \end{cases}
$$

which is the Cole-Hopf transformation (3.4) of u_0.

Theorem 3.3 implies that the approximation with the diffusion wave like solution $\tilde{u}(x,t)$ and the N-wave like one $u^*(x,t)$ have the same convergence order of $O(t^{1/(2r)-3/2})$ in L^r-norm, $1 \le r \le \infty$. Considering the metastable N-wave like state of order $O(e^{1/\mu})$, which persists for a long time, it seems surprising that the advantage of the diffusive N-wave over the diffusion wave does not make any difference in the convergence order. So we need to compare constants C_1, C_2 in order to measure the effectiveness of u^* over $\tilde{u}$.

Relations (3.3), (3.8) indicate that $\|\varphi(x,t) - \tilde{\varphi}(x,t)\|_r$ and $\|\varphi(x,t) - \varphi^*(x,t)\|_r$ can be compared instead of $\|u(x,t) - \tilde{u}(x,t)\|_r$ and $\|u(x,t) - u^*(x,t)\|_r$. Let $\tilde{\zeta}_0$, ζ_0^* be given by (2.29). It is clear that these functions depend on $\mu > 0$ and, hence, the ratio

$$R(\mu) = \frac{\left| \int_{-\infty}^{\infty} \tilde{\zeta}_0(x)\, dx \right|}{\left| \int_{-\infty}^{\infty} \zeta_0^*(x)\, dx \right|} \tag{5.2}$$

is a function of the viscosity constant $\mu > 0$. Considering (2.30), we see that this ratio measures the the effectiveness of φ^* over $\tilde{\varphi}$ for a large time $t > 0$.

TABLE 5.1. The initial value u_0 for the Burgers equation is given by (4.1), which has $p = 2$, $q = 1$. Its Cole-Hopf transformation φ_0, given by (5.1), depends on μ. Constants a, b, α, β, $R(\mu)$ defined by (3.6),(3.7) and (5.2) are approximated in the table numerically increasing $1/\mu$ by 4. In the example we observe that $\alpha \uparrow 0$, $\beta \downarrow 0$, $a/b \to 0$, and $R(\mu) \to \infty$ as $\mu \to 0$.

μ: viscosity	$\frac{1}{\mu}$	α: center of mass for φ_0^+	β: center of mass for φ_0^-	a: total mass of φ_0^+	b: total mass of φ_0^-	$R(\mu)$: the ratio (5.2)
0.1667	6	-7.57e-01	1.05e+00	4.02429e+02	3.83343e+02	4.85e+02
0.1000	10	-5.76e-01	8.33e-01	2.20255e+04	2.18781e+04	2.43e+03
0.0714	14	-4.83e-01	6.98e-01	1.20260e+06	1.20151e+06	1.80e+04
0.0556	18	-4.24e-01	6.10e-01	6.56600e+07	6.56519e+07	1.40e+05
0.0455	22	-3.82e-01	5.48e-01	3.58491e+09	3.58485e+09	1.08e+06
0.0385	26	-3.51e-01	5.02e-01	1.95730e+11	1.95729e+11	8.29e+06
0.0333	30	-3.26e-01	4.66e-01	1.06865e+13	1.06865e+13	6.37e+07
0.0294	34	-3.06e-01	4.37e-01	5.83462e+14	5.83462e+14	5.65e+08

In Table 5.1, constants α, β, a, b, $R(\mu)$ are listed for given values of viscosity constant μ. In the table we can clearly observe that a, b, $R(\mu)$ increase exponentially as $\mu \to 0$. The centers of mass α, β converge to zero as expected. We can also see that $a/b \to 1$ as $\mu \to 0$, and this is similar to the situation in (2.34) where the ratio becomes huge. Finally, we may say that

$$\|\varphi(x,t) - \tilde{\varphi}(x,t)\|_r \simeq 5.65 \times 10^8 \|\varphi(x,t) - \varphi^*(x,t)\|_r \quad \text{for } \mu = 0.0294. \tag{5.3}$$

This estimate, together with the arguments in the proof of Theorem 3.3, gives the scale of effectiveness of approximations by diffusive N-waves versus that by diffusion waves.

If one of the two invariant variables in (1.5) is zero, $p = 0$ or $q = 0$, then the solution $u(x,t)$ of the inviscid problem (1.2) evolves into a single hump structure. So the metastability of the Burgers equation (1.1) is not observed, and we cannot say that the approximation by the diffusive N-wave u^* is better than the one by the diffusion wave $\tilde{u}$. In the following we see what happens in this case through a numerical example.

TABLE 5.2. The initial value u_0 for the Burgers equation is given by (5.4), which has $p = 2.4$, $q = 0$. Its Cole-Hopf transformation φ_0, given by (5.5), depends on μ. Constants a, b, α, β, $R(\mu)$ defined by (3.6), (3.7), and (5.2) are approximated in the table numerically increasing $1/\mu$ by 6. In the example we observe that $\alpha \uparrow 2\pi$, $\beta \downarrow 0$, $a/b \to \infty$, and $R(\mu) \to 1$ as $\mu \to 0$.

μ: viscosity	$\frac{1}{\mu}$	α: center of mass for φ_0^+	β: center of mass for φ_0^-	a: total mass of φ_0^+	b: total mass of φ_0^-	$R(\mu)$: the ratio (5.2)
0.1667	6	4.13e+00	7.57e-01	1.74086e+03	4.02429e+02	5.57e-01
0.0833	12	5.28e+00	5.24e-01	1.95683e+06	1.62754e+05	3.30e-01
0.0556	18	5.73e+00	4.24e-01	2.46870e+09	6.56600e+07	1.86e-01
0.0417	24	5.90e+00	3.66e-01	3.24519e+12	2.64891e+10	1.08e-01
0.0333	30	5.97e+00	3.26e-01	4.32192e+15	1.06865e+13	4.66e+00
0.0278	36	6.01e+00	2.98e-01	5.77891e+18	4.31123e+15	1.57e+00
0.0238	42	6.03e+00	2.75e-01	7.73641e+21	1.73927e+18	1.14e+00
0.0208	48	6.05e+00	2.57e-01	1.03608e+25	7.01674e+20	1.04e+00
0.0185	54	6.06e+00	2.42e-01	1.38769e+28	2.83075e+23	1.01e+00
0.0167	60	6.07e+00	2.30e-01	1.85868e+31	1.14201e+26	1.00e+00
0.0152	66	6.08e+00	2.19e-01	2.48957e+34	4.60719e+28	1.00e+00
0.0139	72	6.09e+00	2.10e-01	3.33460e+37	1.85867e+31	1.00e+00
0.0128	78	6.10e+00	2.01e-01	4.46646e+40	7.49842e+33	1.00e+00
0.0119	84	6.11e+00	1.94e-01	5.98251e+43	3.02508e+36	1.00e+00

Let $u(x,t)$ be the solution of the Burgers equation (1.1) with its initial value

$$u_0(x) = \begin{cases} \sin x, & -\pi < x < \pi, \\ 1.2 \sin x, & \pi < x < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \tag{5.4}$$

and $\varphi(x,t)$ be the solution of (2.24) with its initial value

$$\varphi_0(x) = \begin{cases} -\dfrac{\sin x}{2\mu} e^{(1+\cos x)/(2\mu)}, & -\pi < x < \pi, \\ -\dfrac{1.2\sin x}{2\mu} e^{1.2(1+\cos x)/(2\mu)}, & \pi < x < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \tag{5.5}$$

which is the Cole-Hopf transformation (3.4) of u_0. Then, we can easily check that $p = 2.4$, $q = 0$, and $\int_{-\infty}^{2\pi} u(x)\,dx = -p$. In Table 5.2 centers of mass α, β, positive and negative masses a, b, and the ratio $R(\mu)$ in (5.2) are compared with different viscosity constant $\mu > 0$.

In the table we can clearly observe that $\alpha \uparrow 2\pi$ and $\beta \downarrow 0$ as $\mu \to 0$. So it is the case of $\beta < \alpha$ and, hence, $u^*(x,t)$ is defined only after certain amount of time $t > T$. On the other hand a/b increases exponentially as $\mu \to 0$ and, hence, we may guess such a time T becomes smaller (see the proof of Lemma 3.2). We can also observe that $R(\mu) \to 1$ as $\mu \to 0$. This implies that, for small viscosity constant μ, two different approximations of $\tilde{u}$ and u^* are almost equivalent.

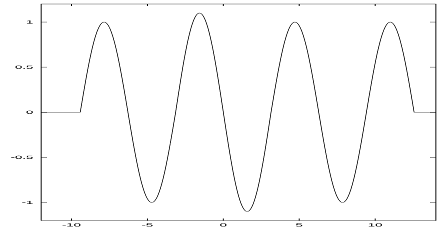
5.2. Change of variables $\mathbf{x} \to -\mathbf{x}$. The following examples further illustrate the dependence of Cole-Hopf transformations, and therefore the metastability property of the Burgers equation (1.1), on the quantities p, q in (1.5) where initial values have several sign-changes. It is easy to check that of initial value

$$u_0(x) = \begin{cases} -\sin x, & -3\pi < x < -\pi,\ \pi < x < 4\pi, \\ -1.1\sin x, & -\pi < x < \pi, \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

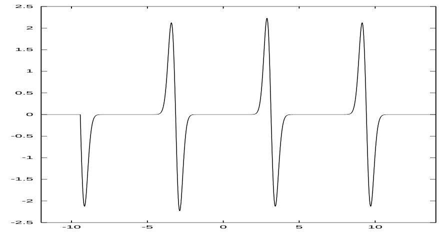
has $p = 0$, $q = M = 2$, and its reflection

$$\bar{u}_0(x) = u_0(-x) = \begin{cases} \sin x, & -4\pi < x < -\pi,\ \pi < x < 3\pi, \\ 1.1\sin x, & -\pi < x < \pi, \\ 0, & \text{otherwise} \end{cases} \tag{5.7}$$

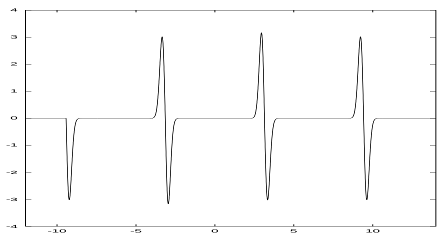
has $p = 0.2$, $q = 2.2$, $M = 2$. The graphs of these initial values and their Cole-Hopf transformations are given in Figure 5.1 for $\mu = 0.04$ and $\mu = 0.02$. It is hard to see directly from the structure of the initial values (a), (d) if the metastable phenomenon will be observed or not. In the Burgers equation, since the correlation between the convection and diffusion terms plays the main role in the phenomenon, we have to check the quantities p, q to decide it. On the other hand, the heat equation has the diffusion term only, and the corresponding property should be reflected in the initial value. We can clearly see that, as $\mu \to 0$, two dominant humps of similar sizes appear (see (e), (f)). Under the presence



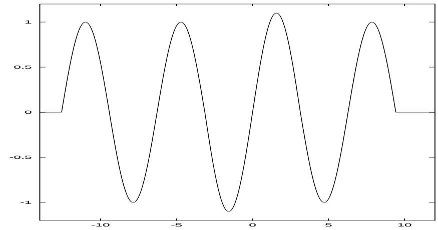
(a) Initial value (5.6), $p = 0, q = 2$



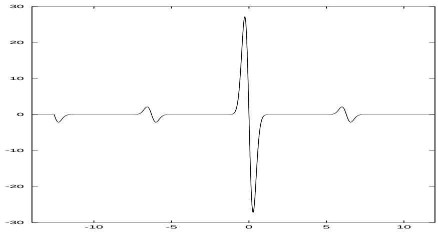
(b) Transformation of (5.6), $\mu = 0.04$



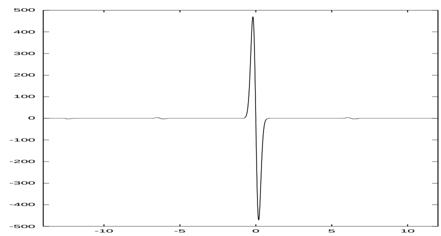
(c) Transformation of (5.6), $\mu = 0.02$



(d) Initial value (5.7), $p = 0.2, q = 2.2$



(e) Transformation of (5.7), $\mu = 0.04$



(f) Transformation of (5.7), $\mu = 0.02$

FIGURE 5.1. In this example we can clearly observe that the Cole-Hopf transformation is very sensitive on the quantities p, q in (1.5) and, for $p > 0$, $q > 0$, $\mu \to 0$ limit of the transformation gives two huge humps of similar sizes, which represents the metastability of the Burgers equation.

of the diffusion, the size of these humps decreases in time, even though it takes exceptionally long time to get one of them sufficiently small. This reflects the long lasting two-hump structure of a diffusive N-wave of the Burgers equation.

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YONG-JUNG KIM & WEI-MING NI:
School of Mathematics
University of Minnesota
Minneapolis, MN 55455, U. S. A.
E-MAIL, Yong-Jung Kim: yongkim@math.umn.edu
E-MAIL, Wei-Ming Ni: ni@math.umn.edu

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