Lyapunov functional and Łojasiewicz-Simon inequality with food metric diffusion

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Abstract

Food metric diffusion is a non-autonomous operator which takes into account the effect of spatial heterogeneity of resource distribution to biological migration. Its use in a mathematical biology model has provided a new insight of biological dispersal phenomena (see [3,4]). Surprisingly, we will see in this paper that a Lyapunov functional and a Łojasiewicz-Simon type inequality can be obtained for a solution to a general group reaction-diffusion equations even with this non-autonomous diffusion. We hope the classical mathematical analysis with the usual diffusion to be extended by using these two ingredients.

Résumé

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1. Introduction and biological context

Migration or dispersal is one of key strategies for the survival of biological species and the importance of formulating a realistic dispersal theory under heterogeneous environments has been emphasized by many authors (see, for example, [6, Chapter 5], [7,8]). Various Fokker-Planck equations have been introduced to overcome the limitation of a homogeneous diffusion model. Recently, Choi and Kim [3] introduced a new diffusion equation based on "*food metric diffusion*" to take into account the dynamics of population dispersal under the heterogeneity in resource distribution. If food is the reason for migration of a biological species, the migration distance should be related to food distribution. For a given food distribution $m(x) > 0$ in one space dimension, the food metric measuring the amount of food between two point $a < b$ is defined by

$$d(a,b) = \int_a^b m(x)\,dx. \tag{Food metric}$$

The random walk system with a constant walk length with respect to the food metric (not Euclidean metric) gives a diffusion equation, $u_t = \left(\frac{1}{m}\left(\frac{u}{m}\right)_x\right)_x$ after scaling out constant factors. This diffusion equation has been used to explain the chemotactic traveling wave phenomenon in one space dimension (see [3]) and the interface propagation of the singular limit of a reaction-diffusion equation of bistable type in higher-dimensional spaces (see [4]).

The purpose of this paper is to develop basic mathematical tools, Lyapunov functional and Łojasiewicz-Simon inequality, which are essential ingredients in obtaining an asymptotic analysis of a large class of reaction-diffusion equations when the food metric diffusion is involved. Let Ω be a bounded domain in

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$\mathbf{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$, $m : \overline{\Omega} \to (0, \infty)$ be a given food distribution, and $u(x,t)$ be the population density of an organism. We consider the problem

$$\begin{cases} u_t = \operatorname{div}\left(\dfrac{1}{m}\nabla\left(\dfrac{u}{m}\right)\right) + f(x,u) & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Here $f(x,u)$ is a non-autonomous population dynamics. We will show that the solution of (1) converges to a steady state as $t \to \infty$ and estimate its convergence rate. A function $\varphi \in H_0^1(\Omega)$ is called a steady state (or stationary solution) of (1) if it satisfies

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{m}\nabla\left(\dfrac{\varphi}{m}\right)\right) + f(x,\varphi) = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Note that the food metric diffusion has a spatial heterogeneity in $m = m(x)$. A spatially heterogeneous diffusion usually makes its analysis complicate. However, the food metric diffusion is formulated in a natural way based on random walk and (1) possesses a Lyapunov functional,

$$E(u,m) := \frac{1}{2}\int_\Omega \frac{1}{m(x)}\left|\nabla\left(\frac{u}{m(x)}\right)\right|^2 dx + \int_\Omega \frac{F(x,u)}{m(x)}\, dx, \tag{3}$$

where the potential F satisfies $\frac{\partial}{\partial s}F(x,s) = -f(x,s)$. The existence of a Lyapunov functional will make asymptotic analysis handy when the model equation contains food metric diffusion.

2. Mathematical setting and the main results

One of main tools to show the asymptotic convergence of solutions to a reaction-diffusion equation is a Łojasiewcz-Simon type inequality (see Chill [2, Section 4.1]). We will use the triple $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ to obtain the inequality. We assume the following hypotheses:
(H_1) The food distribution $m \in W^{2,\infty}(\Omega)$ and $m > 0$ in $\overline{\Omega}$.
(H_2) The nonlinearity $f(x,s)$ is analytic in s uniformly in x in the sense that there exists a constant $C = C(\beta)$ for any $\beta > 0$ such that

$$\sup_{x\in\overline{\Omega}, s\in[-\beta,\beta]} \frac{|\partial_s^n f(x,s)|}{n!} < C^{n+1} \quad \text{for all} \quad n \geq 0.$$

(H_3) The associated Nemytskii operator $v \mapsto f(x,v)$ defined on $H_0^1(\Omega)$ is Lipschitz continuous with values in $L^2(\Omega)$, and is continuously differentiable with its derivatives in $H^{-1}(\Omega)$.

Note that (H_2) and (H_3) are the hypotheses used to obtain the Łojasiewicz-Simon inequality for classical diffusion and we are taking the same hypotheses for f. The hypothesis (H_3) holds if f is smooth function with compact support. Problem (1) is uniformly parabolic under (H_1). The existence, uniqueness, and the regularity for the solution of (1) follow from the classical theory (see, for example, [5, pp. 490–491]).
Lemma 2.1 (Lyapunov functional) *Let $m = m(x)$ satisfy (H_1) and u be a smooth solution of* (1). *Then the energy function E decreases monotone in time.*

PROOF. Integration by parts yields that

$$
\begin{aligned}
\frac{d}{dt}E(u(x,t),m(x)) &= \int_\Omega \left[\frac{1}{m}\nabla(\frac{u}{m})\cdot\nabla(\frac{u_t}{m}) - \frac{f(x,u)u_t}{m}\right] dx \\
&= -\int_\Omega \left[\operatorname{div}(\frac{1}{m}\nabla(\frac{u}{m})) + f(x,u)\right]\frac{u_t}{m}\, dx \\
&= -\int_\Omega \frac{u_t^2}{m}\, dx \ \le\ 0.
\end{aligned}
$$

Therefore, the energy E is a deceasing function of time. □

Having a Lyapunov functional is the first step to show the asymptotic convergence. To complete the asymptotic convergence a Łojasiewicz-Simon type inequality for the Lyapunov functional E is needed.

Lemma 2.2 (Łojasiewicz-Simon inequality) *Let m and f satisfy $(H_1),(H_2)$ and (H_3), and $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be a solution of (2). Then there exist $\theta\in(0,\frac{1}{2}]$ and $\sigma>0$ such that, for any $v\in H_0^1(\Omega)$ with $\|v-\varphi\|_{H_0^1(\Omega)}\le\sigma$,*

$$
|E(v)-E(\varphi)|^{1-\theta} \le C\left\|\operatorname{div}\left(\frac{1}{m}\nabla\left(\frac{v}{m}\right)\right)+f(x,v)\right\|_{H^{-1}(\Omega)}, \tag{4}
$$

where the second term in this Łojasiewicz inequality is given by

$$
\left\|\operatorname{div}\left(\frac{1}{m}\nabla\left(\frac{v}{m}\right)\right)+f\right\|_{H^{-1}(\Omega)} := \sup\left\{\left|\int_\Omega -\frac{1}{m}\nabla\left(\frac{v}{m}\right)\nabla\psi + f\psi\right| : \psi\in H_0^1(\Omega), \|\psi\|_{H_0^1(\Omega)}=1\right\}.
$$

PROOF. Consider a functional defined on $H_0^1(\Omega)$ by

$$
\hat{E}(\hat{v}) = \frac{1}{2}\hat{a}(\hat{v},\hat{v}) + \int_\Omega G(x,\hat{v})\,dx, \ \text{with}\ G(x,s) := \frac{F(x,m(x)s)}{m(x)},
$$

where the bilinear form $\hat{a}: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbf{R}$ is given by

$$
\hat{a}(w_1,w_2) := \int_\Omega \frac{\nabla w_1\cdot\nabla w_2}{m(x)}\,dx.
$$

We will prove the Łojasiewicz inequality for $\hat{E}$ and then deduce the result for E. The proof of Łojasiewicz inequality for $\hat{E}$ is based on [2, Section 4.1] and [2, Corollary 4.6] where we need to check the conditions (H_2) and (H_3) for $g(x,s) := -\partial_s G(x,s) = f(x,m(x)s)$ and the coercivity of $\hat{a}$.

It is easy to see that $\hat{a}$ is continuous, symmetric, and coercive, which are used in [2, Page 587, Section 4.1]. In order to verify (H_2) and (H_3) for g, we use the following inequalities

$$
C_1\|v\|_{H_0^1(\Omega)} \le \|\hat{v}\|_{H_0^1(\Omega)} \le C_2\|v\|_{H_0^1(\Omega)} \tag{5}
$$

for all $v\in H_0^1(\Omega), \hat{v} := \frac{v}{m}$ and some constants $C_1, C_2 > 0$ (see Remark 1).

For $\hat{v}_1,\hat{v}_2\in H_0^1(\Omega)$, there is a constant $L = L(\|\hat{v}_1\|_{H_0^1(\Omega)}, \|\hat{v}_2\|_{H_0^1(\Omega)})$ such that

$$
\begin{aligned}
\|g(x,\hat{v}_1)-g(x,\hat{v}_2)\|_{H_0^1(\Omega)} &= \|f(x,m(x)\hat{v}_1)-f(x,m(x)\hat{v}_2)\|_{H_0^1(\Omega)} \\
&\le L\|m(x)(\hat{v}_1-\hat{v}_2)\|_{H_0^1(\Omega)} \le \frac{L}{C_1}\|\hat{v}_1-\hat{v}_2\|_{H_0^1(\Omega)}.
\end{aligned}
$$

Also for $\hat{v}, h\in H_0^1(\Omega)$, the following identities hold in $H^{-1}(\Omega)$:

$$
g(x,\hat{v}+h) = f(x,m\hat{v}+mh) = f(x,m\hat{v}) + f_s'(x,m\hat{v})mh + o(\|mh\|_{H_0^1(\Omega)}) \tag{6}
$$

$$
= g(x,\hat{v}) + g_s'(x,\hat{v})h + o(\|h\|_{H_0^1(\Omega)}). \tag{7}
$$

Therefore, the Nemytskii operator associated with the function $g(x,s)$ satisfies (H_3). Furthermore, because of the uniform boundedness of m, $g(x,s)$ is analytic with respect to s and uniformly in x (see (H_2)). Thus, in view of [2, Corollary 4.6], there exist $\theta \in (0, \frac{1}{2}]$ and $\hat{\sigma} > 0$ such that, for any $\hat{v} \in H_0^1(\Omega)$ satisfying $\|\hat{v} - \hat{\varphi}\|_{H_0^1(\Omega)} \leq \hat{\sigma}$ with $\hat{\varphi} := \frac{\varphi}{m}$, we have

$$|\hat{E}(\hat{v}) - E(\hat{\varphi})|^{1-\theta} \leq C \left\| \operatorname{div}\left(\frac{\nabla \hat{v}}{m}\right) + g(x, \hat{v}) \right\|_{H^{-1}(\Omega)}.$$

Since $E(v) = \hat{E}(\hat{v})$ for $\hat{v} := \frac{v}{m}$, we obtain the Łojasiewicz-Simon inequality for E with $\sigma := \hat{\sigma}/C_2$. $\square$

Remark 1 For a given function $k \in C^1(\overline{\Omega})$, there exists a constant $C = C(k) > 0$ such that

$$\|kw\|_{H_0^1(\Omega)} \leq C\|w\|_{H_0^1(\Omega)} \quad \text{for all} \quad w \in H_0^1(\Omega).$$

Remark 2 As we see in [2, Corollary 4.6], the analyticity of $f(x, s)$ in a neighborhood of the interval $s \in (\operatorname{essinf}_\Omega \varphi, \operatorname{esssup}_\Omega \varphi)$ is sufficient to prove the Łojasiewicz-Simon inequality.

The asymptotic convergence is obtained by combining the two ingredients of the Lyapunov functional and the Łojasiewicz-Simon inequality. Its proof follows from the same lines as in [1]. We omit it in details.

Theorem 2.3 (Long-time asymptotic behavior of solution) *Let u be a global solution of (1). Assume that the solution orbit $\{u(t) : t \geq 1\}$ is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and precompact in $H_0^1(\Omega)$. Then $u(t)$ converges to $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which is a stationary solution of (2). Furthermore, as $t \to \infty$*

$$\|u(t) - \varphi\|_{L^2(\Omega)} = \begin{cases} O(\exp(-ct)) & \text{for a constant } c > 0 \text{ if } \theta = 1/2, \\ O(t^{\theta/(1-2\theta)}) & \text{if } \theta \in (0, 1/2). \end{cases} \tag{8}$$

3. Application

As an concrete example, we may take the logistic case, $f(x, s) = s(m(x) - s)$, or a spatially heterogeneous bistable nonlinearity case $f = s(s - \frac{m(x)}{2})(1 - \frac{s}{m(x)})$. Generally, we may take

$$f(x, s) = s(m(x) - s)h(x, s), \quad h(x, s) = \Pi_{i=1}^{i=n} p(x)(s - k_i(x)), \quad n \geq 1,$$

where $p(x), k_i(x) \in C^\infty(\overline{\Omega}), p(x), k_i(x) \geq 0$ in $\overline{\Omega}$. For simplicity we assume that m is smooth, $u_0 \in L^\infty(\Omega)$, $m > 0$ in $\overline{\Omega}$, and $u_0 \geq 0$ in Ω. Our task is to verify $(H_2), (H_3)$, and the precompactness of the solution orbit in $H_0^1(\Omega)$. First note that the local existence and the uniqueness are classical results. It is easy to check that $w^-(x, t) := 0$ and $w^+(x, t) := Km(x)$ with a large K are sub- and super-solutions for (1), respectively. This implies that the solution is uniformly bounded and exists globally. Also by a standard argument for parabolic equations, we have $u \in C^\infty(\overline{\Omega} \times (0, \infty))$.

For convenience, we give a direct proof for the precompactness of the solution in $H_0^1(\Omega)$. It is sufficient to show that the solution orbit is bounded in $H^2(\Omega)$. Then, the result follows from the compact embedding of $H^2(\Omega)$ into $H^1(\Omega)$. The idea is to show that the right hand side of the equation $\operatorname{div}\left(\frac{1}{m}\nabla\left(\frac{u}{m}\right)\right) = -u_t - f(x, u)$ is bounded in L^2 uniformly on $[1, \infty)$ and to use the theory of elliptic regularity. It is sufficient to prove that $\|u_t(t)\|_{L^2(\Omega)} \leq C$ for all $t \geq 1$, where C is a generic constant.

Lemma 2.1 and the uniform boundedness of the solution imply

$$\int_1^t \int_\Omega \frac{u_t^2}{m} = E(u(1)) - E(u(t)) \leq E(u(1)) - \int_\Omega \frac{F(x, u(t))}{m(x)} \leq C \tag{9}$$

for all $t \geq 1$. On the other hand, we have $u_{tt} = \operatorname{div}\left(\frac{1}{m}\nabla\left(\frac{u_t}{m}\right)\right) + f_s'(x,u)u_t$. Multiplying this equation by $\frac{u_t}{m}$ and integrating it over Ω give

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{u_t^2}{m} = -\int_\Omega \frac{1}{m}|\nabla\left(\frac{u_t}{m}\right)|^2 + \int_\Omega f_s'(x,u)\frac{u_t^2}{m}$$

and hence

$$\int_\Omega \frac{u_t^2(t)}{m} \leq \int_\Omega \frac{u_t^2(1)}{m} + C\int_1^t \int_\Omega \frac{u_t^2}{m} \leq C \text{ for all } t \geq 1,$$

where we have used (9) in the last inequality. This implies $\|u_t(t)\|_{L^2(\Omega)} \leq C$ for all $t \geq 1$.

Note that f does not satisfy (H_3) in multi-space dimensions and we cannot employ the theory directly. We modify it and consider a new energy functional that has the same value as E on solution orbits. Let $c < d$ be such that $c \leq u(x,t) \leq d$ for all $x \in \overline{\Omega}$, $t \geq 0$. Let $\tilde{f}(x,s) = m^2(x)h_1(\frac{s}{m(x)})h(x,s)$, where $h_1(s)$ has compact support and equals to $s(1-s)$ on the interval $(c-1, d+1)$. Since $\partial_s f(x,s)$ is uniformly bounded, (H_3) is satisfied (see [1, Lemma 3.2] for a similar case). Furthermore, $\tilde{f}(x,s)$ is analytical for $s \in (c,d)$. Consequently, Lemma 2.2 and Remark 2 give the Łojasiewicz-Simon inequality for a functional $\tilde{E}$ given by

$$\tilde{E}(u) := \frac{1}{2}\int_\Omega \frac{1}{m(x)}\left|\nabla\left(\frac{u}{m(x)}\right)\right|^2 dx + \int_\Omega \frac{\tilde{F}(x,u)}{m(x)}\, dx, \tag{10}$$

where $\partial_s F(x,s) = -\tilde{f}(x,s)$. Note that we may also choose $F(x,s) := -\int_{(c+d)/2}^s f(x,\bar{s})\, d\bar{s}$ and $\tilde{F}(x,s) := -\int_{(c+d)/2}^s \tilde{f}(x,\bar{s})\, d\bar{s}$. Then, $\tilde{E}$ shares the same value with E on the solution orbits.

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