A LOGARITHMIC CHEMOTAXIS MODEL FEATURING GLOBAL EXISTENCE AND AGGREGATION

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Abstract. We consider a chemotaxis model which belongs to the class of logarithmic models. It naturally appears when one drops the assumption that microscopic scale organisms measure a macroscopic scale chemical gradient. We show that weak solutions exist globally in time in coefficient regimes that include an aggregation phenomenon in dimensions $n \in \{1, 2, 3\}$ and for large initial data. We also show the instability of constant steady states and provide numerical simulations which illustrate the formation of aggregation patterns.

1. Introduction and conclusion

The purpose of this paper is to develop a chemotaxis theory which is based on a Fokker-Planck type diffusion equation,

$$u_t = \Delta(\gamma_c(v)u), \qquad x \in \Omega,\ t > 0, \tag{1.1}$$

where u is the cell density, v is the chemical concentration, Ω is a smooth bounded domain of $\mathbf{R}^n$ ($n \in \{1, 2, 3\}$), and γ_c is the cell motility given by

$$\gamma_c(v) = \frac{1}{c + v^k}, \quad k > 0,\ c \geq 0. \tag{1.2}$$

The cell density equation contains a diffusion term which writes as the Laplacian of a nonlinear term. We do not assume in the modeling that microscopic scale organisms have to measure a macroscopic scale chemical gradient. However, technically, the model covers both bounded and unbounded chemosensitivity cases. By rewriting the diffusion as

$$\begin{aligned} u_t = \Delta(\gamma_c(v)u) &= \nabla \cdot (\gamma_c(v)\nabla u + \gamma_c'(v)u\nabla v) \\ &= \nabla \cdot \Big(\gamma_c(v)\Big(\nabla u - k\frac{v^k}{c + v^k}\frac{u}{v}\nabla v\Big)\Big), \end{aligned} \tag{1.3}$$

we see that, if $c > 0$ and $k \geq 1$, the coefficient of the advection term is bounded. If $c = 0$, the Fokker-Planck type diffusion gives a logarithmic model type advection,

$$u_t = \nabla \cdot \Big(\gamma_0(v)\Big(\nabla u - k\frac{u}{v}\nabla v\Big)\Big),$$

which has unbounded chemosensitivity as $v \to 0$.

One can consider this equation as a special case of the original model of Keller and Segel [14, (6) and (9)] when the effective body ratio is $\alpha = 0$ (see Appendix B for an extra discussion). In this paper, we focus on the chemotactic self-aggregation phenomenon where the chemical concentration v is assumed to satisfy

$$v_t = \varepsilon\Delta v + u - v, \qquad x \in \Omega,\ t > 0. \tag{1.4}$$

This model equation indicates that the chemical is diffused with a constant diffusivity $\varepsilon > 0$, produced by cells with production rate one, and degraded with degradation rate one.

We study the problem with strictly positive smooth initial values,

$$u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \quad x \in \Omega. \tag{1.5}$$

The domain $\Omega \subset \mathbf{R}^n$ is smooth and bounded. We take the usual zero flux boundary condition,

$$\partial_\nu(\gamma_c(v)u) = 0, \quad \partial_\nu v = 0 \quad \text{on } \partial\Omega,$$

which is equivalent to the homogeneous Neumann boundary condition,

$$\partial_\nu u = 0, \quad \partial_\nu v = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

The global existence of weak solutions to the system (1.1)–(1.6) is obtained in Section 2. Keller and Segel introduced their first chemotaxis equations to explain the initiation of the aggregation phenomenon of slime mold. They considered a logarithmic model [13, (2.3) and (2.4)] and viewed the aggregation as the instability of constant steady states. They showed that the instability occurs when the chemosensitivity is greater than the diffusivity, which corresponds to the model equation (1.3) with $k > 1$ and $c = 0$. However, the global existence of strong or weak solutions for the logarithmic model has been obtained only for the cases when the constant steady state is stable, or in specific cases (low dimension, very weak (renormalized) solutions, etc. see discussion in Appendix C). Our main goal is to obtain the instability and the global existence at the same time. The global existence obtained in this paper, Theorem 2.1, includes cases of multi-space dimensions and both bounded and unbounded chemosensitivity cases $c \geq 0$. More precisely, the global existence is obtained for $0 < k < 7/3$ if $n = 1$, for $0 < k < 2$ if $n = 2$, and for $0 < k < 4/3$ if $n = 3$ without any smallness assumption on the initial data or parameters.

The instability of homogeneous steady states is obtained in Section 3. For the case with $c = 0$, it is shown that constant steady states are unstable if $\varepsilon < \frac{k-1}{\mu_1}$, where $\mu_1 > 0$ is the principal eigenvalue of the Laplace operator $-\Delta$ on Ω (see [35]). The population size is not involved in this unbounded chemosensitivity case. We extend the instability analysis in Theorem 3.1 to include the bounded chemosensitivity case with $c > 0$. If $c > 0$, there is a minimum population size required for aggregation (that is, $\left(\frac{c}{k-1}\right)^{\frac{1}{k}}|\Omega|$). If the average of the population is $\overline{u} > u_1 := \left(\frac{c}{k-1}\right)^{\frac{1}{k}}$ and $\varepsilon < \frac{(k-1)\bar{u}^k - c}{\mu_1(c+\bar{u}^k)}$, then the constant state is unstable.

Numerical simulations for the aggregation phenomenon are given in Section 4. We can observe from numerical simulations that the instability conditions in Theorem 3.1 provide sharp bounds on coefficients and population size for a pattern formation when $c \neq 0$. If the domain size is small and the population size is bigger than a critical one for a given diffusivity $\varepsilon > 0$, we obtain a single hump solution. If the domain size is large, multiple peaks appear in the first stage and then are combined to eventually form a larger single peak. We can observe a similar behavior for the solutions for all space dimensions. Remember that the behavior of solutions for the minimal model (cf. Appendix B) is different when different space dimensions $n \in \{1, 2, 3\}$ are considered. In Appendices B and C we discuss the results of this paper and compare them to the ones in the literature. In particular,

a brief comparison between the original Keller-Segel equations and the ones in this paper is given.

2. Global existence

In this section, we show the existence of global weak solutions to the initial-boundary problem (1.1)–(1.6) for arbitrary (nonnegative) initial data in a suitable functional space. This result of existence holds for arbitrary $\varepsilon > 0$ and $c \geq 0$. When the dimension is $n = 1$, we assume that $0 < k < 7/3$. When the dimension is $n = 2$, we assume that $0 < k < 2$. When the dimension is $n = 3$, we assume that $0 < k < 4/3$. Note that for any dimension $n \in \{1, 2, 3\}$, the considered range of k includes real numbers which are strictly larger than 1, corresponding thus to the regime in which a steady state solution is not always stable. Therefore, an aggregation phenomenon may happen for small ε or for large total population as shown in Theorem 3.1. The ratio of the chemosensitivity over the diffusivity of the model (1.3), which is $k\frac{v^{k-1}}{c+v^k}$, is unbounded for small v and $c = 0$. This unboundedness had been up to now a key difficulty in obtaining the global existence of a logistic type equations when aggregation may occur.

We provide in Appendix C a discussion on the assumptions used in our work, and on the link with other works on the same subject, including the important recent paper [30] by Tao and Winkler.

We will use the following definition of (very) weak solution: for nonnegative initial data (u_0, v_0) in $(L^1(\Omega))^2$, a pair of nonnegative functions (u, v) on $\mathbf{R}_+ \times \Omega$ such that u, v and $\gamma_c(v)\, u$ lie in $L^1([0, T] \times \Omega)$ for all $T > 0$, is a (very) weak solution (global in time) of (1.1)–(1.6) on $\mathbf{R}_+ \times \Omega$ if the equations

$$
\begin{aligned}
&-\int_{\mathbf{R}_+}\int_\Omega u(t,x)\partial_t\psi(t,x)\,dxdt - \int_\Omega u_0(x)\psi(0,x)\,dx \\
&\qquad = \int_{\mathbf{R}_+}\int_\Omega \gamma_c(v(t,x))\, u(t,x)\,\Delta\psi(t,x)\,dxdt, \\
&-\int_{\mathbf{R}_+}\int_\Omega v(t,x)\partial_t\psi(t,x)\,dxdt - \int_\Omega v_0(x)\psi(0,x)\,dx \\
&\qquad = \int_{\mathbf{R}_+}\int_\Omega \varepsilon\, v(t,x)\,\Delta\psi(t,x)\,dxdt \\
&\qquad\quad + \int_{\mathbf{R}_+}\int_\Omega (u(t,x) - v(t,x))\ \psi(t,x)\,dxdt,
\end{aligned}
\tag{2.1}
$$

are satisfied for all compactly supported test function $\psi \in C_c^2(\mathbf{R}_+ \times \overline{\Omega})$ such that $\partial_\nu\psi = 0$ on $\mathbf{R}_+ \times \partial\Omega$.

More precisely, we show the following theorem:

Theorem 2.1. *Let Ω be a bounded smooth (C^2) open subset of $\mathbf{R}^n$, for $n \in \{1, 2, 3\}$. We consider $c \geq 0$, $\varepsilon > 0$ and $0 < k < 7/3$ if $n = 1$, $0 < k < 2$ if $n = 2$, $0 < k < 4/3$ if $n = 3$. Let $u_0 := u_0(x) \geq 0$ lying in $L^1(\Omega) \cap H_m^{-1}(\Omega)$ and $v_0 := v_0(x) \geq c_0 > 0$ lying in $W^{1,2}(\Omega)$ if $n = 1$, lying in $W^{1,2-\delta}(\Omega)$ (for all $\delta > 0$) if $n = 2$, and lying in $W^{1,\frac{10-5k}{5-2k}-\delta}(\Omega)$ (for all $\delta > 0$) if $n = 3$. Then, there exists*

a (very) weak (global in time) solution (u, v) such that $u \geq 0$, $v \geq 0$ of the initial-boundary problem (1.1)–(1.6) on $\mathbf{R}_+ \times \Omega$ (and such that $u(0,\cdot) = u_0$, $v(0,\cdot) = v_0$). Moreover u and v satisfy the following bounds (for any $T > 0$ and $\eta > 0$):

- *When $n = 1$, $v, v^{-1} \in L^\infty([0,T] \times \Omega)$, $\partial_x v \in L^4([0,T] \times \Omega)$, $\partial_t v, \partial_{xx} v \in L^2([0,T] \times \Omega)$, $u \in L^2([0,T] \times \Omega)$, $u \in L^\infty([0,T]; L^1(\Omega))$, $u^r \in L^2([0,T]; H^1(\Omega))$ for all $0 < r < 1/2$.*
- *When $n = 2$, $v \in L^{1/\eta}([0,T] \times \Omega) \cap L^\infty([0,T]; L^1(\Omega))$, $v^{-1} \in L^\infty([0,T] \times \Omega)$, $\nabla_x v \in L^{4-\eta}([0,T] \times \Omega)$, $\partial_t v, \nabla_x \nabla_x v \in L^{2-\eta}([0,T] \times \Omega)$, $u \in L^{2-\eta}([0,T] \times \Omega) \cap L^\infty([0,T]; L^1(\Omega))$, $u^r \in L^2([0,T]; H^1(\Omega))$ for all $0 < r < 1/2$.*
- *When $n = 3$, $v \in L^{10-5k-\eta}([0,T] \times \Omega) \cap L^\infty([0,T]; L^1(\Omega))$, $v^{-1} \in L^\infty([0,T] \times \Omega)$, $\nabla_x v \in L^{\frac{10-5k}{3-k}-\eta}([0,T] \times \Omega)$, $\partial_t v, \nabla_x \nabla_x v \in L^{\frac{10-5k}{5-2k}-\eta}([0,T] \times \Omega)$, $u \in L^{\frac{10-5k}{5-2k}-\eta}([0,T] \times \Omega) \cap L^\infty([0,T]; L^1(\Omega))$, $u^r \in L^2([0,T]; H^1(\Omega))$ for all $0 < r < \frac{1}{2}\frac{4-3k}{5-2k}$.*

We list in the remarks below some direct extensions of this theorem.

Remark 2.2 (Weak solutions). When ($n = 1$ or $n = 2$) and when ($n = 3$ and $k < 1$), the solutions obtained are weak solutions, in the sense that ∇v and $\nabla[\gamma_c(v)\, u]$ lie in $L^1([0,T] \times \Omega)$ for all $T > 0$, so that we can use an integration by parts (or Green's identity) in the third terms of the two equations in the very weak formulation (2.1), and replace the set of test functions by $C^1_c(\mathbf{R}_+ \times \overline{\Omega})$. Indeed, writing $\nabla u = \frac{1}{r} u^{1-r}\, \nabla(u^r)$ and using Hölder's inequality, we see that for all $\eta > 0$, ∇u is bounded in $L^{4/3}([0,T] \times \Omega)$ when $n = 1$, in $L^{4/3-\eta}([0,T] \times \Omega)$ when $n = 2$, and in $L^{\frac{10-5k}{8-3k}-\eta}([0,T] \times \Omega)$ when $n = 3$ and $k < 1$. Then, writing $\nabla[\gamma_c(v)\, u] = u\, \nabla\gamma_c(v) + \gamma_c(v)\, \nabla u$, we obtain thanks to Hölder's inequality that for all $\eta > 0$, $\nabla[\gamma_c(v)\, u]$ is bounded in $L^{4/3}([0,T] \times \Omega)$ when $n = 1$, in $L^{4/3-\eta}([0,T] \times \Omega)$ when $n = 2$, and in $L^{\frac{10-5k}{8-3k}-\eta}([0,T] \times \Omega)$ when $n = 3$ and $k < 1$.

Remark 2.3 (Regularity of the initial data). The assumption that the initial datum v_0 lies in $W^{1,p_n}(\Omega)$ (with $p_n = 2$ when $n = 1$, $p_n = 2 - \eta$ when $n = 2$, and $p_n = \frac{10-5k}{5-2k} - \eta$ when $n = 3$, fore some $\eta > 0$) is used in the proof to apply the maximal regularity of the heat equation in the L^{p_n} space to equation (1.4). However this assumption is not optimal when $p_n < 2$: indeed, it suffices to assume the initial datum to be in the fractional Sobolev space $W^{2-2/p_n,p_n}(\Omega)$ (see for example [15]). Therefore, it is possible to replace in the theorem the assumption "$v_0 \in W^{1,2-0}(\Omega)$" by the assumption "$v_0 \in W^{1-\nu,2-\nu}(\Omega)$ for all $\nu > 0$", and the assumption "$v_0 \in W^{1,\frac{10-5k}{5-2k}-\eta}(\Omega)$" (for all $\eta > 0$) by the assumtion "$v_0 \in W^{1-\frac{k}{10-5k}-\nu,\frac{10-5k}{5-2k}-\nu}(\Omega)$ for all $\nu > 0$".

Remark 2.4 ($L \log L$ estimate). When $n = 1$, if we furthermore assume that $u_0 \log u_0$ lies in $L^1(\Omega)$, then we have that for all time $T > 0$, $u \log u \in L^\infty([0,T]; L^1(\Omega))$ and $\sqrt{u} \in L^2([0,T]; H^1(\Omega))$. Indeed, it is a consequence of the following computation (at least at the formal level)

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u \log u\, dx &= \int_\Omega (1 + \log u)\, \Delta\left(\frac{u}{c + v^k}\right) dx = -\int_\Omega \frac{\nabla u}{u} \cdot \nabla\left(\frac{u}{c + v^k}\right) dx \\
&= -\int_\Omega \frac{|\nabla u|^2}{u} \frac{1}{c + v^k}\, dx - \int_\Omega \nabla u \cdot \nabla\left(c + v^k\right)^{-1} dx \\
&\leq -\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} \frac{1}{c + v^k}\, dx + \int_\Omega u\,(c + v^k)\left|\nabla\left(c + v^k\right)^{-1}\right|^2 dx,
\end{aligned}$$

where, after integration on some time interval $(0,T)$, the last term is controlled thanks to the known estimates on u and v.

Proof of Thm. 2.1. We present here the *a priori* estimates related to the problem (1.1)–(1.6). Existence will then be obtained thanks to a suitable approximation process. In all the sequel, $T>0$ will denote any strictly positive time, and C_T will denote a strictly positive constant (changing from line to line) depending on T and on the parameters of the problem (Ω, ε, c, k, n and the initial data).

First step (first estimates): Thanks to an integration w.r.t. $x\in\Omega$, we immediately see that

$$\sup_{t\in[0,T]}||u(t,\cdot)||_{L^1(\Omega)}+\sup_{t\in[0,T]}||v(t,\cdot)||_{L^1(\Omega)}\leq C_T. \tag{2.2}$$

Then, observing that $\partial_t v-\varepsilon\,\Delta v\geq -v$ and using the minimum principle, we get the estimate

$$\inf_{t\in[0,T],x\in\Omega} v(t,x)\geq C_T^{-1}, \tag{2.3}$$

where $C_T^{-1}:=c_0\,\exp(-T)$.

Second step (use of the duality lemma): Observing that

$$\partial_t u-\Delta(A\,u)=0,\qquad \partial_\nu(A\,u)|_{\partial\Omega}=0,$$

where $A:=(c+v^k)^{-1}\geq 0$ lies in $L^1([0,T]\times\Omega)$ thanks to estimate (2.3), a standard duality lemma (cf. [27]) implies that

$$\int_0^T\int_\Omega \frac{u^2}{c+v^k}\,dxdt\leq C_T, \tag{2.4}$$

since the initial datum $u_0\geq 0$ lies in $H_m^{-1}(\Omega)$.

Third step (estimate of the r.h.s of the equation for v): We consider $m\in]1,2[$ (the constants C_T will depend upon m in the sequel). We observe that, using Hölder's inequality and estimate (2.4):

$$\int_0^T\int_\Omega |u-v|^m\,dxdt\leq C_T\left(\int_0^T\int_\Omega\left(\frac{u}{\sqrt{c+v^k}}\right)^m(c+v^k)^{m/2}\,dxdt+\int_0^T\int_\Omega v^m\,dxdt\right)$$

$$\leq C_T\left(1+\left[\int_0^T\int_\Omega v^{\frac{km}{2-m}}\,dxdt\right]^{1-m/2}+\int_0^T\int_\Omega v^m\,dxdt\right). \tag{2.5}$$

Fourth step (use of the properties of the heat kernel relative to the L^p spaces): For any $\tilde m\in[1,\frac{m\,(n/2+1)}{n/2+1-m}[$ when $m\leq 1+n/2$ and $\tilde m=\infty$ when $m>1+n/2$, we know (cf. for example [1]) that

$$||v||_{L^{\tilde m}([0,T]\times\Omega)}\leq C_T\left(||\partial_t v-\varepsilon\,\Delta v||_{L^m([0,T]\times\Omega)}+||v(0,\cdot)||_{W^{1,m}(\Omega)}\right).$$

Observing (when $\tilde m>m$) that for any $\delta>0$, there exists $C_\delta>0$ such that

$$\int_0^T\int_\Omega v^m\,dxdt\leq C_\delta+\delta\int_0^T\int_\Omega v^{\tilde m}\,dxdt,$$

and using estimate (2.5), we end up with the estimate

$$||v||_{L^{\tilde{m}}([0,T]\times\Omega)} \leq C_T \left(1 + ||v(0,\cdot)||_{W^{2,m}(\Omega)} + \left(\int_0^T \int_\Omega v^{\frac{km}{2-m}}\, dxdt\right)^{\frac{2-m}{2m}}\right), \tag{2.6}$$

which holds if $m \in]1,2[$ and $\tilde{m} \in]m, \frac{m\,(n/2+1)}{n/2+1-m}[$ when $m \leq 1+n/2$, and if $m \in]1,2[$ and $\tilde{m} = \infty$ when $m > 1+n/2$.

When $n = 1$, using the bounds (2.2) and (2.6), we see that for $m \in]3/2, 2[$,

$$||v||_{L^\infty([0,T]\times\Omega)} \leq C_T \left(1 + ||v||^{(k/2-\frac{2-m}{2m})^+}_{L^\infty([0,T]\times\Omega)}\right).$$

Remembering that in that case, we assumed that $k < 7/3$, we see that by taking $m > 3/2$ close enough to $3/2$, we obtain the final estimate

$$||v||_{L^\infty([0,T]\times\Omega)} \leq C_T. \tag{2.7}$$

When $n = 2$, we assumed that $k \in]0,2[$. Then, we can take $m \in]2-k, 2[$ and $\tilde{m} = \frac{km}{2-m}$, and use the bound (2.6) in order to get

$$||v||_{L^{\frac{km}{2-m}}([0,T]\times\Omega)} \leq C_T \left(1 + ||v||^{k/2}_{L^{\frac{km}{2-m}}([0,T]\times\Omega)}\right),$$

so that

$$||v||_{L^{\frac{km}{2-m}}([0,T]\times\Omega)} \leq C_T.$$

Then, selecting $m < 2$ close to 2, we see that for all $q \in [1,\infty[$,

$$||v||_{L^q([0,T]\times\Omega)} \leq C_T. \tag{2.8}$$

When $n = 3$, using the bound (2.6), an interpolation and the bound (2.2), we obtain

$$\begin{aligned} ||v||_{L^{\tilde{m}}([0,T]\times\Omega)} &\leq C_T \left(1 + ||v||^{\frac{km}{2-m} - \frac{\left(\frac{km}{2-m}-1\right)_+}{1-\frac{1}{\tilde{m}}}}_{L^1([0,T]\times\Omega)} ||v||^{\frac{\left(\frac{km}{2-m}-1\right)_+}{1-\frac{1}{\tilde{m}}}}_{L^{\tilde{m}}([0,T]\times\Omega)}\right)^{\frac{2-m}{2m}} \\ &\leq C_T \left(1 + ||v||^{\frac{\tilde{m}}{\tilde{m}-1}\left(\frac{k}{2}-\frac{2-m}{2m}\right)_+}_{L^{\tilde{m}}([0,T]\times\Omega)}\right), \end{aligned}$$

for all $\tilde{m} \in]\max(1, \frac{km}{2-m}), \frac{5m}{5-2m}[$, which is non empty when we take $m \in]1, \frac{10-5k}{5-2k}[$. Note that indeed $\frac{10-5k}{5-2k} > 1$ since by assumption $k < 5/3$. Now selecting $\tilde{m} < \frac{5m}{5-2m}$ close to $\frac{5m}{5-2m}$, we have for some small $\nu > 0$,

$$||v||_{L^{\tilde{m}}([0,T]\times\Omega)} \leq C_T \left(1 + ||v||^{\left(\frac{5m}{7m-5}+\nu\right)\left(\frac{k}{2}-\frac{2-m}{2m}\right)_+}_{L^{\tilde{m}}([0,T]\times\Omega)}\right).$$

Recalling that by assumption $k < 9/5$, we see that when $\nu > 0$ is small enough, the exponent in the right-hand-side satisfies $\left(\frac{5m}{7m-5}+\nu\right)\left(\frac{k}{2}-\frac{2-m}{2m}\right)_+ < 1$, so that for $\tilde{m} < \frac{5m}{5-2m}$ close enough to $\frac{5m}{5-2m}$, we have

$$||v||_{L^{\tilde{m}}([0,T]\times\Omega)} \leq C_T.$$

Finally, selecting $m < \frac{10-5k}{5-2k}$ close to $\frac{10-5k}{5-2k}$ (therefore $\tilde{m} < \frac{5m}{5-2m} < 10-5k$ is close to $10-5k$), we end up with, for all $q \in [1, 10-5k[$,

$$||v||_{L^q([0,T]\times\Omega)} \leq C_T. \tag{2.9}$$

Note that until now we only used the fact that $k < 5/3$. The assumption that $k < 4/3$ will be used in the next step.

Fifth step (use of a multiplicator for u): We first consider the case when $n = 1$. Thanks to estimates (2.4) and (2.7), we see that

$$||u||_{L^2([0,T]\times\Omega)} \leq C_T. \tag{2.10}$$

Then

$$||\partial_t v - \varepsilon\,\Delta v||_{L^2([0,T]\times\Omega)} \leq C_T\,(||u||_{L^2([0,T]\times\Omega)} + ||v||_{L^2([0,T]\times\Omega)}).$$

Using estimates (2.7), (2.10), and the maximal regularity of the heat equation, we obtain the bound

$$||\partial_t v||_{L^2([0,T]\times\Omega)} + ||\partial_{xx} v||_{L^2([0,T]\times\Omega)} \leq C_T. \tag{2.11}$$

We interpolate then between the bounds (2.7) and (2.11), and see that

$$||\partial_x v||_{L^4([0,T]\times\Omega)} \leq C_T. \tag{2.12}$$

We now use the equation for u, and compute, for any $q \in]0,1[$,

$$\frac{d}{dt}\int_\Omega \frac{u^q}{q}\,dx = (1-q)\int_\Omega u^{q-2}\big|\partial_x u\big|^2\,(c+v^k)^{-1}\,dx - k(1-q)\int_\Omega u^{q-1}\,\partial_x u\,\partial_x v\,\frac{v^{k-1}}{(c+v^k)^2}\,dx.$$

Using Young's inequality and estimate (2.3), we see that

$$k\,(1-q)\left|\int_\Omega u^{q-1}\,\partial_x u\,\partial_x v\,\frac{v^{k-1}}{(c+v^k)^2}\,dx\right| \leq \frac{1}{2}\,(1-q)\int_\Omega u^{q-2}\big|\partial_x u\big|^2\,(c+v^k)^{-1}\,dx$$
$$+ C_T\left(\int_\Omega |\partial_x v|^4\,dx + \int_\Omega u^{2q}\,dx\right).$$

Then, integrating w.r.t. time on $[0,T]$, thanks to the bounds (2.2), (2.10) and (2.12),

$$\int_0^T\int_\Omega u^{q-2}\big|\partial_x u\big|^2\,(c+v^k)^{-1}\,dx \leq C_T,$$

so that, using again estimate (2.3), we end up with the bound,

$$\int_0^T\int_\Omega \big|\partial_x(u^r)\big|^2\,dxdt \leq C_T, \tag{2.13}$$

for all $r \in]0,1/2[$.

We then turn to the case when $n = 2$. Thanks to estimates (2.4) and (2.8), we see that

$$||u||_{L^{2-\nu}([0,T]\times\Omega)} \leq C_T, \tag{2.14}$$

for all $\nu > 0$ small enough. Then

$$||\partial_t v - \varepsilon\,\Delta v||_{L^{2-\nu}([0,T]\times\Omega)} \leq C_T\left(||u||_{L^{2-\nu}([0,T]\times\Omega)} + ||v||_{L^{2-\nu}([0,T]\times\Omega)}\right).$$

Using estimates (2.8) and (2.14), and the maximal regularity of the heat equation, we obtain the bound (for $i,j = 1,2$)

$$||\partial_t v||_{L^{2-\nu}([0,T]\times\Omega)} + ||\partial_{x_i x_j} v||_{L^{2-\nu}([0,T]\times\Omega)} \leq C_T, \tag{2.15}$$

for all $\nu > 0$ small enough. We interpolate then between the bounds (2.8) and (2.15), and see that

$$||\nabla v||_{L^{4-\nu}([0,T]\times\Omega)} \leq C_T. \tag{2.16}$$

We now use the equation for u, and compute, for any $q \in]0,1[$,

$$\frac{d}{dt}\int_\Omega \frac{u^q}{q}\,dx = (1-q)\int_\Omega u^{q-2}\,|\nabla u|^2\,(c+v^k)^{-1}\,dx - k\,(1-q)\int_\Omega u^{q-1}\,\nabla u\cdot\nabla v\,\frac{v^{k-1}}{(c+v^k)^2}\,dx.$$

Using Young's inequality and estimate (2.3), we see that

$$k\,(1-q)\left|\int_\Omega u^{q-1}\,\nabla u\cdot\nabla v\,\frac{v^{k-1}}{(c+v^k)^2}\,dx\right| \le \frac{1}{2}\,(1-q)\int_\Omega u^{q-2}\,|\nabla u|^2\,(c+v^k)^{-1}\,dx$$
$$+ C_T\left(\int_\Omega |\nabla v|^{4-\nu_1}\,dx + \int_\Omega u^{2-\nu_2}\,dx\right),$$

for some $\nu_1, \nu_2 > 0$ small enough (and depending on q). Then, integrating w.r.t. time on $[0,T]$, and thanks to the bounds (2.14) and (2.16),

$$\int_\Omega \frac{u^q}{q}(0)\,dx - \int_\Omega \frac{u^q}{q}(T)\,dx + \frac{1}{2}\int_0^T\int_\Omega u^{q-2}\,|\nabla u|^2\,(c+v^k)^{-1}\,dx \le C_T,$$

so that, remembering that $q<1$ and using estimates (2.2), (2.3), we end up with the bound

$$\int_0^T\int_\Omega |\nabla(u^r)|^2\,dxdt \le C_T, \tag{2.17}$$

for all $r \in]0,1/2[$.

We finally treat the case when $n=3$. By estimates (2.4) and (2.9), we have that

$$||u||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)} \le C_T, \tag{2.18}$$

for all $\nu > 0$ small enough. Then

$$||\partial_t v - \varepsilon\,\Delta v||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)} \le C_T\left(||u||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)} + ||v||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)}\right),$$

so that by the maximal regularity of the heat equation, using estimates (2.9) and (2.18), we have (for $i,j \in \{1,2,3\}$)

$$||\partial_t v||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)} + ||\partial_{x_i x_j} v||_{L^{\frac{10-5k}{5-2k}-\nu}([0,T]\times\Omega)} \le C_T, \tag{2.19}$$

for all $\nu > 0$ small enough. Interpolating between this bound and (2.9), we obtain

$$||\,|\nabla v|^2\,||_{L^{\frac{10-5k}{6-2k}-\nu}([0,T]\times\Omega)} \le C_T. \tag{2.20}$$

Note that $\frac{10-5k}{6-2k} > 1$ thanks to the assumption that $k < 4/3$. We compute using the equation for u and the lower bound (2.3), for any $q \in]0,1[$,

$$-\frac{d}{dt}\int_\Omega \frac{u^q}{q}\,dx + (1-q)\int_\Omega u^{q-2}\,|\nabla u|^2\,(c+v^k)^{-1}\,dx \le C_T\int_\Omega |\nabla v|^2 u^q\,dx,$$

so that by Young's inequality, assuming furthermore that $q < \frac{4-3k}{5-2k}$,

$$-\frac{d}{dt}\int_\Omega \frac{u^q}{q}\,dx + (1-q)\int_\Omega u^{q-2}\,|\nabla u|^2\,(c+v^k)^{-1}\,dx$$
$$\le C_T\left(\int_\Omega |\nabla v|^{2\frac{10-5k}{6-2k}-\nu_1}\,dx + \int_\Omega u^{\frac{10-5k}{5-2k}-\nu_2}\,dx\right),$$

for some small $\nu_1, \nu_2 > 0$ depending on q. We now integrate on $[0,T]$, using the bounds (2.18), (2.20), (2.2), to obtain

$$\int_0^T \int_\Omega u^{q-2} \left|\nabla u\right|^2 (c+v^k)^{-1} \, dxdt \leq C_T.$$

Since $q < \frac{4-3k}{5-2k}$, using furthermore the bound (2.3), we have

$$\int_0^T \int_\Omega \left|\nabla (u^r)\right|^2 dxdt \leq C_T, \tag{2.21}$$

for all $r \in]0, \frac{1}{2}\frac{4-3k}{5-2k}[$.

The above *a priori* estimates are applied uniformy w.r.t. $\delta \in]0,1[$, to the approximated system

$$\partial_t u_\delta = \Delta\left(\frac{u_\delta}{c+(\delta+v_\delta)^k}\right), \tag{2.22}$$

$$\partial_t v_\delta = \varepsilon \Delta v_\delta + \frac{u_\delta}{1+\delta\, u_\delta} - v_\delta, \tag{2.23}$$

together with Neumann boundary conditions and smoothed initial data $u_{in,\delta} \geq \delta$, $v_{in,\delta} \geq \delta$ compatible with the Neumann boundary conditions.

The existence of a strong (classical) solution $u_\delta \geq 0$, $v_\delta \geq 0$ to the system above can be obtained thanks to Schauder fixed-point procedure. More precisely, we denote by $L^2_+([0,T]\times\Omega)$ the set of nonnegative functions u in $L^2([0,T]\times\Omega)$, and we consider the application T_1 defined on $L^2_+([0,T]\times\Omega)$ by

$$T_1 : u \mapsto v = T_1(u) = (\partial_t - \varepsilon\Delta + Id)^{-1}\left(\frac{u}{1+\delta u}\right),$$

(to be understood with initial condition $v(0,\cdot) = v_{in,\delta}$ and homogeneous Neumann boundary conditions), and we denote by $C_+^{0_t,1_x,1-0}([0,T]\times\bar\Omega)$ the set of nonnegative functions on $[0,T]\times\bar\Omega$ which are, and whose first space derivatives are, β-Hölder continuous for any $0<\beta<1$, and we consider the application T_2 defined on $C_+^{0_t,1_x,1-0}([0,T]\times\bar\Omega)$ by

$$T_2 : v \mapsto \tilde u = T_2(v) = \left(\partial_t - \nabla\cdot\left[\frac{1}{c+(\delta+v)^k}\nabla - \frac{k(\delta+v)^{k-1}\nabla v}{(c+(\delta+v)^k)^2}\, Id\right]\right)^{-1}(0),$$

(to be understood with initial condition $\tilde u(0,\cdot) = u_{in,\delta}$ and homogeneous Neumann boundary conditions). For any $u \in L^2_+([0,T]\times\Omega)$, by the minimum principle $v \geq 0$, and by the maximal regularity of the heat equation, $v = T_1(u)$ is in $C_+^{0_t,1_x,1-0}([0,T]\times\bar\Omega)$ with bounds which do not depend upon u (and also in $L^{1/\eta}([0,T];W^{2,1/\eta}(\Omega))\cap W^{1,1/\eta}([0,T]\times\Omega)$ for all $\eta>0$). In particular the composition $T_2\circ T_1$ is defined, and, using the minimum principle, $\tilde u$ is nonnegative. Furthermore, by the maximal regularity of linear parabolic equations (see [15]), we see that $\tilde u = T_2\circ T_1(u)$ is in $L^{1/\eta}([0,T];W^{2,1/\eta}(\Omega))\cap W^{1,1/\eta}([0,T]\times\Omega)$ for all $\eta>0$, with bounds which do not depend upon u (and also in $C_+^{0_t,1_x,1-0}([0,T]\times\bar\Omega)$). Thanks to a Sobolev embedding, we conclude that the image of $L^2_+([0,T]\times\Omega)$ by $T_2\circ T_1$ is relatively compact in $L^2_+([0,T]\times\Omega)$. To check the continuity of $T_2\circ T_1$,

we compute for any u_1, $u_2 \in L^2_+([0,T]\times\Omega)$, noting $v_i := T_1(u_i)$ and $\tilde{u}_i := T_2(v_i)$, $i = 1, 2$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{(\tilde{u}_1-\tilde{u}_2)^2}{2}\,dx \\
&\quad= -\int_\Omega \nabla(\tilde{u}_1-\tilde{u}_2)\cdot\left(\frac{1}{c+(\delta+v_1)^k}\nabla\tilde{u}_1 - \frac{1}{c+(\delta+v_2)^k}\nabla\tilde{u}_2\right)dx \\
&\qquad -\int_\Omega \nabla(\tilde{u}_1-\tilde{u}_2)\cdot\left(\frac{k(\delta+v_2)^{k-1}\nabla v_2}{(c+(\delta+v_2)^k)^2}\,\tilde{u}_2 - \frac{k(\delta+v_1)^{k-1}\nabla v_1}{(c+(\delta+v_1)^k)^2}\,\tilde{u}_1\right)dx,
\end{aligned}$$

so that integrating w.r.t. time on $(0,T)$ and using the initial condition,

$$\begin{aligned}
&\int_\Omega \frac{(\tilde{u}_1-\tilde{u}_2)^2}{2}(T)\,dx + \int_0^T\int_\Omega |\nabla(\tilde{u}_1-\tilde{u}_2)|^2\frac{1}{c+(\delta+v_1)^k}\,dxdt \\
&\quad= -\int_0^T\int_\Omega \nabla(\tilde{u}_1-\tilde{u}_2)\cdot\nabla\tilde{u}_2\left(\frac{1}{c+(\delta+v_1)^k} - \frac{1}{c+(\delta+v_2)^k}\right)dxdt \\
&\qquad -\int_0^T\int_\Omega \nabla(\tilde{u}_1-\tilde{u}_2)\cdot\nabla\left(v_2-v_1\right)\frac{k(\delta+v_1)^{k-1}}{(c+(\delta+v_1)^k)^2}\,\tilde{u}_1\,dxdt \\
&\qquad -\int_0^T\int_\Omega \nabla(\tilde{u}_1-\tilde{u}_2)\cdot\nabla v_2\left(\frac{k(\delta+v_2)^{k-1}}{(c+(\delta+v_2)^k)^2} - \frac{k(\delta+v_1)^{k-1}}{(c+(\delta+v_1)^k)^2}\right)\tilde{u}_1\,dx \\
&\qquad -\int_0^T\int_\Omega \nabla\left(\tilde{u}_1-\tilde{u}_2\right)\cdot\nabla v_2\frac{k(\delta+v_2)^{k-1}}{(c+(\delta+v_2)^k)^2}\left(\tilde{u}_2-\tilde{u}_1\right)dxdt.
\end{aligned}$$

Using Young's inequality, the smoothness of $\tilde{u}_1$, $\tilde{u}_2$, v_1 and v_2, and the Lipschitz continuity of the functions $w \mapsto \frac{1}{c+(\delta+w)^k}$ and $w \mapsto \frac{k(\delta+w)^{k-1}}{(c+(\delta+w)^k)^2}$, we end up with

$$\begin{aligned}
&\int_\Omega \frac{(\tilde{u}_1-\tilde{u}_2)^2}{2}(T)\,dx + \frac{1}{2}\int_0^T\int_\Omega |\nabla(\tilde{u}_1-\tilde{u}_2)|^2\,dx \\
&\quad\le 4C_T\left\{\int_0^T\int_\Omega \left(v_1-v_2\right)^2\,dx + \int_0^T\int_\Omega |\nabla\left(v_2-v_1\right)|^2\,dx + \int_0^T\int_\Omega \left(\tilde{u}_2-\tilde{u}_1\right)^2\,dxdt\right\}.
\end{aligned}$$

Applying the maximal regularity for linear parabolic equations to the equation $\partial_t(v_1-v_2) - \varepsilon\Delta(v_1-v_2) + (v_1-v_2) = \frac{u_1}{1+\delta u_1} - \frac{u_2}{1+\delta u_2}$, we see that

$$||v_1-v_2||_{L^2([0,T]\times\Omega)} + ||\nabla(v_1-v_2)||_{L^2([0,T]\times\Omega)} \le C_T||u_1-u_2||_{L^2([0,T]\times\Omega)},$$

so that we can conclude thanks to a Gronwall lemma that

$$\int_\Omega (\tilde{u}_1-\tilde{u}_2)^2(T)\,dx \le C_T\int_0^T\int_\Omega \left(u_1-u_2\right)^2\,dxdt,$$

which implies the continuity of $T_2\circ T_1$ on $L^2_+([0,T]\times\Omega)$. Considering $T_2\circ T_1$ on the closure of the convex hull of $T_2\circ T_1(L^2_+([0,T]\times\Omega))$, we can now apply Schauder's theorem, which gives the existence of a strong nonnegative solution (u_δ, v_δ) in $(L^{1/\eta}([0,T];W^{2,1/\eta}(\Omega))\cap W^{1,1/\eta}([0,T]\times\Omega))^2$ for all $\eta > 0$, to the approximated system (2.22).

We now show how to pass to the limit in the system (2.22) when $\delta \to 0$. Let p_n, $q_n > 1$ be defined by

$$(p_n, q_n) = (2,\infty) \text{ if } n \in \{1, 2\}, \quad (p_n, q_n) = ((10-5k)/(5-2k), 10-5k) \text{ if } n = 3.$$

According to the *a priori* estimates shown above, we know that we can extract from $(u_\delta, v_\delta)_{\delta>0}$ a subsequence still denoted by $(u_\delta, v_\delta)_{\delta>0}$ converging towards (u, v) weakly in $L^{p_n-\eta}([0,T]\times\Omega)\times L^{q_n-\eta}([0,T]\times\Omega)$ for all $\eta>0$, with $u, v \geq 0$. In particular, the second and third terms appearing in the second equation of the weak formulation (2.1) pass to the limit weakly.

Since we know moreover that $(v_\delta)_{\delta>0}$ is bounded in $W^{1,p_n-\eta}([0,T]\times\Omega)$ for all $\eta>0$, we see that $(v_\delta)_{\delta>0}$ also converges a.e. towards v. As a consequence, $(\delta+c+v_\delta^k)^{-1}$ converges a.e. towards $(c+v^k)^{-1}$ (and is bounded in $L^\infty([0,T]\times\Omega)$). Then we can pass to the weak limit in the product $(\delta+c+v_\delta^k)^{-1}\,u_\delta$ and get its convergence towards $(c+v^k)^{-1}\,u$ in $L^{p_n-\eta}([0,T]\times\Omega)$, for all $\eta>0$.

The two remaining terms appearing in the weak formulation (2.1) also clearly pass to the limit weakly thanks to the regularity of the initial data. Therefore, (u, v) is a (very) weak solution of (1.1)–(1.6) on $[0,T]\times\Omega$. A standard Cantor's diagonal argument allows to extend it into a global in time (very) weak solution on $\mathbf{R}_+\times\Omega$.

Finally, the estimates on the solution (u, v) are obtained by passing to the limit in the uniform estimates on (u_δ, v_δ) using Fatou's lemma. □

3. Instability of constant steady states

In this section we find the instability condition of a constant steady state solution to the Neumann boundary value problem, (1.1)–(1.6). The condition depends on the size of cell population, the size of chemical diffusivity, and the sensitivity of motility. However, the condition does not depend upon the size of cell diffusion. To see it more clearly, we introduce an extra parameter $\varepsilon_u>0$ and set the motility function as

$$\gamma_c(v)=\frac{\varepsilon_u}{c+v^k}.$$

The new parameter ε_u controls the diffusion size of cell population. Notice that this parameter is not involved in the instability condition of the following theorem.

Theorem 3.1 (Instability of a constant steady state)**.** *Let $c\geq 0$, $k>1$, $\bar u=\bar v>0$, Ω be smooth and bounded, and $\mu_1>0$ be the principal eigenvalue of the Laplace operator $-\Delta$ on Ω. Suppose that $\bar u>u_1:=(\frac{c}{k-1})^{\frac{1}{k}}$. Then, $\varepsilon_1(\bar u):=\frac{(k-1)\bar v^k-c}{\mu_1(c+\bar v^k)}>0$ and, if $0<\varepsilon<\varepsilon_1(\bar u)$, the constant state $(u,v)=(\bar u,\bar v)$ is a linearly unstable steady state solution of (1.1)–(1.6). If $\varepsilon>\varepsilon_1(\bar u)$, then $(\bar u,\bar v)$ is linearly asymptotically stable.*

Proof. The case when $c=0$ has been shown in [34] and we consider the other case ($c>0$). The positivity of ε_1 is clear. Denote $u=\bar u+u_1$, $v=\bar v+v_1$ and consider a linearized problem,

$$\frac{\partial}{\partial t}\begin{pmatrix}u_1\\ v_1\end{pmatrix}=A(\bar u,\bar v)\begin{pmatrix}u_1\\ v_1\end{pmatrix}, \tag{3.1}$$

where

$$A(\bar u,\bar v)=\begin{pmatrix}\gamma_c(\bar v)\Delta & \gamma_c'(\bar v)\bar u\Delta\\ 1 & \varepsilon\Delta-1\end{pmatrix}.$$

Let $\{\mu,\phi\}$ be an eigenpair of $-\Delta$ in Ω with the homogeneous Neumann boundary condition. If $\{\lambda,\mathbf{c}\}$ is an eigenpair of a related matrix

$$B(\bar u,\bar v)=\begin{pmatrix}-\gamma_c(\bar v)\mu & -\gamma_c'(\bar v)\bar u\mu\\ 1 & -\varepsilon\mu-1\end{pmatrix},$$

then $(u_1, v_1) = \mathbf{c}\phi e^{\lambda t}$ is a non-trivial solution of (3.1). Therefore, the local stability of the constant steady state is determined by the sign of the eigenvalues of $B(\bar{u}, \bar{v})$. The characteristic equation for an eigenvalue of $B(\bar{u}, \bar{v})$ is

$$\lambda^2 + (\mu(\gamma_c(\bar{v}) + \varepsilon) + 1)\lambda + \mu\gamma_c(\bar{v})(\varepsilon\mu + 1) + \gamma_c'(\bar{v})\bar{u}\mu = 0.$$

The (complex) eigenvalues are

$$\lambda = \frac{-P \pm \sqrt{P^2 - 4Q}}{2},$$

where

$$\begin{aligned} P &:= \mu(\gamma_c(\bar{v}) + \varepsilon) + 1, \\ Q &:= \mu\gamma_c(\bar{v})(\varepsilon\mu + 1) + \gamma_c'(\bar{v})\bar{u}\mu. \end{aligned}$$

Since P is positive, the steady state solution is unstable if there exists an eigenvalue μ such that $Q = Q(\mu) < 0$. Notice that $\mu \mapsto Q(\mu)/\mu = \gamma_c\varepsilon\mu + \gamma_c + \gamma_c'\bar{u}$ is monotone increasing. Hence, $\mu \mapsto Q(\mu)/\mu$ has its minimum value when $\mu = \mu_1$. The minimum value is computed by

$$\frac{Q(\mu_1)}{\mu_1} = \gamma_c\varepsilon\mu_1 + \gamma_c - \frac{\varepsilon_u k\bar{v}^{k-1}}{(c + \bar{v}^k)^2}\bar{u} = \gamma_c(\bar{v})\Big(\varepsilon\mu_1 + 1 - \frac{k\bar{v}^k}{c + \bar{v}^k}\Big).$$

We therefore find the instability condition,

$$\frac{Q(\mu_1)}{\mu_1\gamma_c(\bar{v})} = \varepsilon\mu_1 + 1 - \frac{k\bar{v}^k}{c + \bar{v}^k} < 0 \iff \varepsilon < \frac{(k-1)\bar{u}^k - c}{\mu_1(c + \bar{u}^k)} = \varepsilon_1(\bar{u}).$$

Since $\bar{v} = \bar{u} > u_1$, we have $\varepsilon_1(\bar{u}) > 0$. In conclusion, the constant state $(u, v) = (\bar{u}, \bar{v})$ is a linearly unstable steady state solution of (1.1)–(1.6) if $\varepsilon < \varepsilon_1(\bar{u})$, and a linearly asymptotically stable one if $\varepsilon > \varepsilon_1(\bar{u})$. □

We expect an aggregation phenomenon when constant steady states are unstable. Since the total cell population of our problem is preserved, the constant steady state solution $(\bar{u}, \bar{v})$ related to the initial value problem is

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx, \quad \bar{v} = \bar{u},$$

where $|\Omega|$ is the size of the bounded domain Ω. Therefore, the first instability condition, $\bar{u} > u_1 := (\frac{c}{k-1})^{\frac{1}{k}}$, requires that the initial population size should be bigger than a critical size,

$$\int_\Omega u_0(x)dx > |\Omega|\Big(\frac{c}{k-1}\Big)^{\frac{1}{k}}. \tag{3.2}$$

This condition indicates that, if $c = 0$, then the aggregation phenomenon may happen for any size of population. However, if $c > 0$, there is a minimum size of population to guarantee population aggregation. This population size is independent of the diffusivity size ε_u. It depends on the sensitivity of γ_c which is decided by $c > 0$ and $k > 1$. If $k \to 1$ or $c \to \infty$, this critical population size increases to infinity.

The second instability condition $\varepsilon < \varepsilon_1$ requires small chemical diffusivity size. The critical diffusivity $\varepsilon_1 := \frac{(k-1)\bar{u}^k - c}{\mu_1(c + \bar{u}^k)}$ depends on k, c, μ_1, and $\bar{u}$. If c approaches zero, the critical diffusivity ε_1 tends to $\frac{k-1}{\mu_1}$. In this case the population size is not involved. If $k \to 1$, ε_1 tends to a negative value (or zero). Hence, it is not

possible to have an aggregation phenomenon for any population size and chemical diffusivity. The domain size and shape is involved through the principal eigenvalue $\mu_1 = \mu_1(\Omega)$.

4. Numerical simulation

In this section, we test the instability of constant steady states and the convergence to a nonconstant steady state using numerical simulations. As we shall see, the results of the numerical computations agree with the theoretical analysis performed in the paper. Our numerical simulations are coherent with the result stating that the solutions globally exist and do not blow up.

We consider first in this study the following problem:

$$\begin{cases} u_t = \Delta(\gamma_c(v)u), \\ v_t = \varepsilon\Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0, \quad \text{on } \partial\Omega, \\ u(\mathbf{x},0) = \bar{u}, \\ v(\mathbf{x},0) = \bar{v}(1 + \chi_{\{|\mathbf{x}|<0.01\}}), \end{cases} \tag{4.1}$$

where $\bar{u} = \bar{v} > 0$ are constants. The initial cell population distribution is thus homogeneous. The chemical concentration is a perturbation centered at the origin of an homogeneous steady state. These data correspond to cells close to point 0 which start to secrete the chemoattractant, so that the aggregation phenomenon is initiated. The motility function γ_c is chosen in such a way that $k = 2$ and $c = 1$, i.e.,

$$\gamma_c(v) = \frac{1}{1+v^2}.$$

In this case (cf. Thm. 3.1), the critical population density is

$$u_1 = \Big(\frac{c}{k-1}\Big)^{\frac{1}{k}} = 1.$$

Therefore (still cf. Thm. 3.1), if $\bar{u} > 1$ and $\varepsilon > 0$ is small enough, we expect an aggregation phenomenon to develop. For numerical computations we have taken an explicit finite difference scheme centered in space and forward in time (see Appendix for a matlab code used in the paper).

4.1. **Aggregation in a small domain.**

In this subsection, we take the unit ball as the computation domain, i.e.,

$$\Omega = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1\}.$$

We recall that according to the formulas in Thm. 3.1), $\varepsilon_1(\bar{u}) = \frac{1}{\mu_1}\frac{\bar{u}^2-1}{\bar{u}^2+1}$. Selecting $\varepsilon = \frac{3}{5\mu_1}$ (which corresponds to the critical diffusivity $\varepsilon_1(\bar{u})$ when $\bar{u} = 2$) and observing that $\bar{u} \mapsto \frac{\bar{u}^2-1}{\bar{u}^2+1}$ is an increasing function, we expect that the initial (slighty perturbed) homogeneous steady state will give rise to an evolution towards an inhomogeneous state when $\bar{u} > 2$. Since ε is of the same order as $1/\mu_1$, we also expect that this inhomogeneous state presents typically one single hump, at least for t close to 0. Note that the principal eigenvalue μ_1 of the Laplace operator $-\Delta$ on the unit ball depends on the space dimension. We present in the two following subsections the cases when the dimension are $n = 1$, and then $n = 2$.

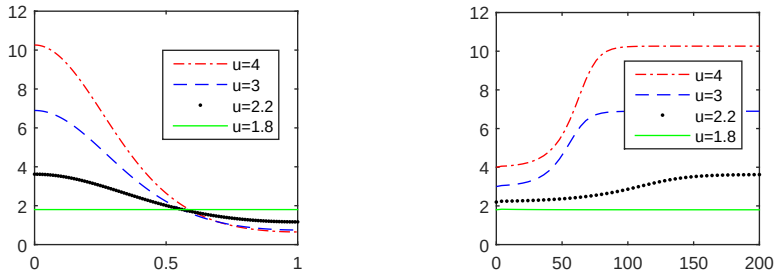
4.1.1. *One space dimension.* We first consider one space dimension $n = 1$. The principal eigenvalue of $-\Delta$ in the domain of the unit ball is then $\mu_1 = \pi^2 \cong 9.8696$. The critical chemical diffusivity is then

$$\varepsilon = \frac{3}{5\mu_1} \cong 0.0608.$$

We performed numerical simulation in four different cases, corresponding to

$$\bar{u} = 4,\ 3,\ 2.2,\ \text{and } 1.8.$$

We recall that the first three cases (with $\bar{u} = 4, 3$, and 2.2) are expected to lead to the development of nonconstant steady states. We numerically computed the solution at time $t = 200$ and displayed the obtained profile for u in Figure 4.1(a). Since the solution is even, only the part with $x > 0$ is presented. We indeed observe that the solution converges to a nonconstant steady state in the first three cases and to a constant one in the last case (when $\bar{u} = 1.8$).



(a) Cell density profile u at $t = 200$. (b) Maximum value of u w.r.t. time.

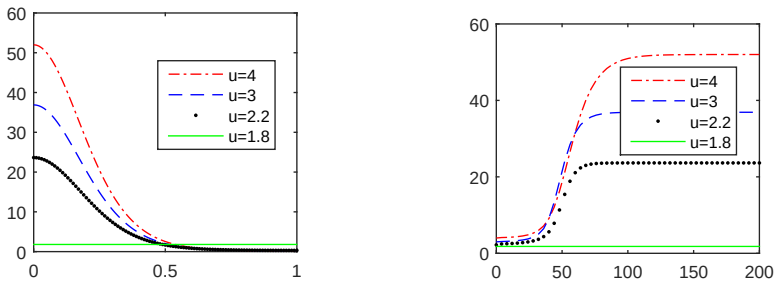
FIGURE 4.1. Cell density distribution at time 200 and sigmoidal-like saturation profile for the maximum of the cell density (one space dimension).

The observed maximum of $u(t, \cdot)$ is at the origin for all $t > 0$. It is shown in Figure 4.1(b) how this maximum value increases and saturates in the four cases. We can observe that the growth is sigmoidal-like.

4.1.2. *Two space dimensions.* We next consider as a domain the two dimensional unit disk. The eigenvalues are given by the relation $J_0'(\sqrt{\mu}) = 0$, where J_0 is the Bessel function of the first kind of order zero. The first zero is $\sqrt{\mu_1}$. Approximately, $\mu_1 \cong 14.6819$ (see [35, Remark 4.1]). The critical chemical diffusivity is then

$$\varepsilon = \frac{3}{5\mu_1} \cong 0.0409.$$

We performed numerical simulations for the same four cases as in dimension 1, that is cases corresponding to $\bar{u} = 4,\ 3,\ 2.2$, and 1.8. We also numerically computed the solution at time $t = 200$ and displayed the profile in Figure 4.2(a). Since the solution is radially symmetric, the profile is presented only along any half straight line starting at point 0. We once again observe the convergence to a nonconstant steady state except in the last case (when $\bar{u} = 1.8$). We can also observe a sigmoidal-like growth of the maximum value of the solution in Figure 4.2(b).



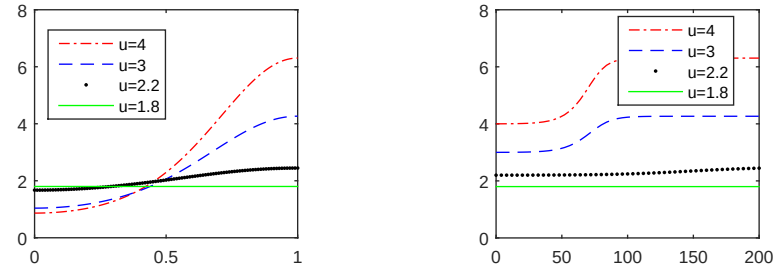
(a) Cell density profile u at $t = 200$. (b) Maximum value of u w.r.t. time.

FIGURE 4.2. Cell density distribution at time 200 and sigmoidal-like saturation profile for the maximum of the cell density (two space dimension).

Nonconstant steady states are not unique and can have a nontrivial basin of attraction. Consider an alternative initial value

$$v(\mathbf{x}, 0) = \bar{v}(1 + \chi_{\{0.09<|\mathbf{x}|\leq 1\}}),$$

in the problem (4.1), corresponding to a case in which the cells along the perimeter of the domain start to produce chemicals. We can observe (cf. Figure 4.3) a profile (and a sigmoidal growth of the maximal value of $u(t, \cdot)$) quite different from the one observed in in Figure 4.2.



(a) Cell density profile u at $t = 200$. (b) Maximum value of u w.r.t. time.

FIGURE 4.3. Cell density distribution at time 200 and sigmoidal-like saturation profile for the maximum of the cell density (one space dimension). The aggregation signal started from the perimeter of the domain $x^2 + y^2 = 1$.

4.2. **Aggregation in a large domain.** In this subsection, we take the same values of the diffusivity as in subsection 4.1 (that is, $\varepsilon \cong 0.608$ in dimension 1 and $\varepsilon \cong 0.0409$ in dimension 2), but we compute on a much larger domain. We hope in this way to observe solutions in which several humps develop (when the time is close to 0), representing multiple points of aggregation of the cells.

As in the previous subsection, we provide computations in dimension 1 as well as in dimension 2.

4.2.1. *One space dimension.* We now take

$$\Omega = \{x \in \mathbf{R} : |x| < 10\}.$$

The domain of computation is thus ten times larger than the domain considered in subsection 4.1, and the principal eigenvalue μ_1 of the Laplace operator $-\Delta$ becomes 100 time smaller, i.e., $\mu_1 = \frac{\pi^2}{100}$. We take the same diffusivity in Subsubsection 4.1.1, that is $\varepsilon \cong 0.0608$. This diffusivity is the critical one when $\bar{u} \cong 1.006$. Therefore, we may expect an aggregation phenomenon when $\bar{u} > 1.007$.

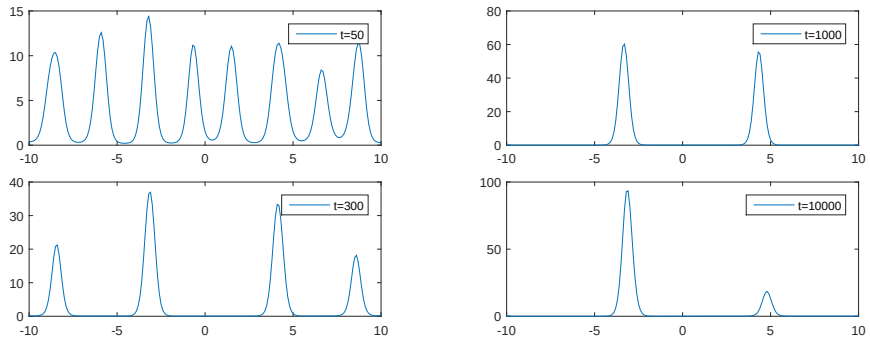


FIGURE 4.4. Cell density distribution at times $t = 50, 300, 10^3$ and 10^4. Initial values corresponding to (4.2) are used.

We performed the numerical computation with initial values corresponding to

$$\bar{u} = \bar{v} = 4, \quad u(x,0) = \bar{u}, \quad v(x,0) = \bar{v}(0.5 + X), \quad -10 < x < 10, \tag{4.2}$$

where $0 < X < 1$ is a random variable with uniform distribution. In Figure 4.4, solution profiles are given for $t = 50, 300, 10^3$, and 10^4. Since the constant steady state $u = 4$ is (linearly) unstable (the critical $\bar{u}$ is $\bar{u} \cong 1.006$), an aggregation phenomenon develops. At time $t = 50$, eight cell aggregation points have appeared. Later, the eight humps progressively merge and bigger humps appear. At time $t = 300$, four such humps have survived. Later, for $t > 10^4$, only one single hump is left. The merging process happens on a larger time scale than the process of the initial formation of the humps.

4.2.2. *Two space dimensions.* We take in this subsubsection a square domain

$$\Omega = \{(x,y) \in \mathbf{R}^2 : 0 < x, y < 10\}.$$

We take the same diffusivity as in Subsubsection 4.1.2, that is $\varepsilon \cong 0.0409$. This diffusivity is the critical one when $\bar{u} \cong 1.006$.

We perform numerical computations with initial values corresponding to

$$\bar{u} = \bar{v} = 4, \quad u(x,y,0) = \bar{u}, \quad v(x,y,0) = \bar{v}(0.5 + X), \quad 0 < x, y < 10, \tag{4.3}$$

where $0 < X < 1$ is a random variable with uniform distribution, since the constant state $u = 4$ is unstable (the critical $\bar{u}$ is $\bar{u} \cong 1.006$), and an aggregation phenomenon develops. In Figure 4.5, the values of u (represented by colours) are given at various times ($t = 25$, 50, 75, 100, 200, 300, 500, 1000, 2000).

We can observe that the uniformly distributed cells start to get aggregated, and a pattern consisting of a few spots already has appeared at time $t = 50$. The

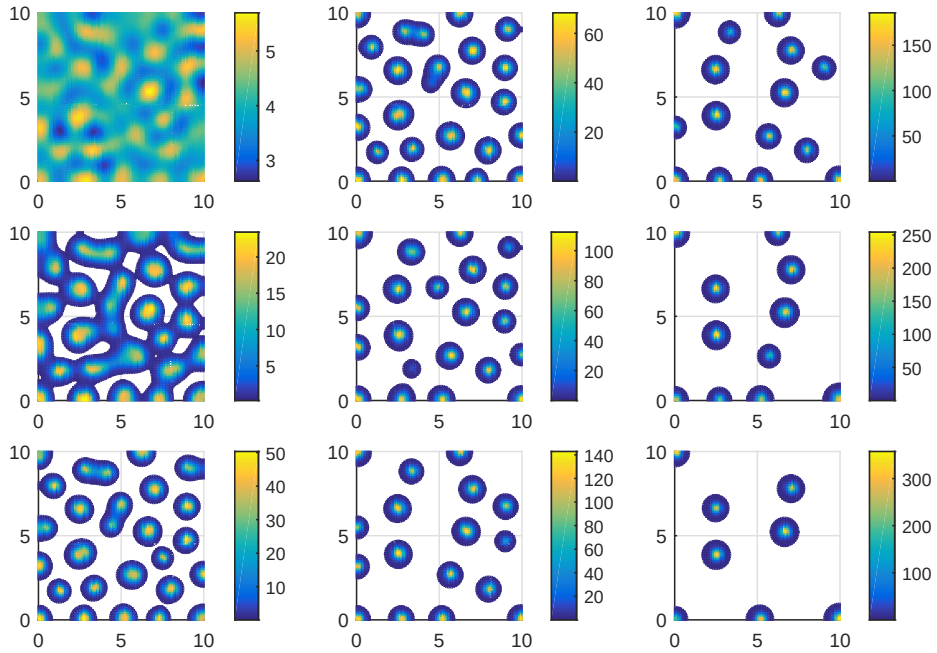


FIGURE 4.5. Cell density distribution at nine different times. Initial values corresponding to (4.3) are used. White regions are associated with zones where $u < 0.5$.

maximum of u reaches to 50 at $t = 75$, which is about the same value as what was observed on radially symmetric solutions in Figure 4.2. The support size of these spots is also similar to the one of those radial solutions. We can also observe later that the aggregation spot patterns merged together and form bigger spots. However, this merging process becomes slower and slower.

APPENDIX A. COMPUTATION CODE

An explicit finite difference scheme centered in space and forward in time has been used for computation. The numerical simulations are performed using matlab. The matlab code used for the computation of Fig. 4.5 is given in the following:

```
%% model parameters
ubar=4;
eps=3/5/14.6819;
c=1;
k=2;
%% computation parameters
L=10;           % computation domain is [0,L]^2
dx=0.1;         % space mesh size
T=2000;           % time domain is [0,T].
dt=dx*dx/4;     % time step size
NX=floor(L/dx)+1;   % number of space grids
NT=floor(T/dt)+1;   % number of time grids
```

```
%% variables and initial value
x=0:dx:L;
u=ubar*ones(NX,NX);         % homogeneous initial value
v=u.*(0.8+0.4*rand(NX,NX)); % random initial value
%% computation
for i=1:NT
    p=u./(c+v.^k);                  % this is gamma times u
    tmp=u+dt*(4*del2(p,dx));        % equation for cell density
    v=v+dt*(eps*4*del2(v,dx)-v+u);  % equation for chemical concentration
    u=tmp;                          % update cell density
    % Neumann boundary condition
    u(1,:)=u(2,:);u(:,1)=u(:,2);u(NX,:)=u(NX-1,:);u(:,NX)=u(:,NX-1);
    v(1,:)=v(2,:);v(:,1)=v(:,2);v(NX,:)=v(NX-1,:);v(:,NX)=v(:,NX-1);
end
%% solution display
M=max(max(u));
mesh(x,x,u);axis([0 L 0 L 0.5 M]);view([0 0 1]);
```

Appendix B. Comparison to Keller-Segel equations

This paper is a part of an effort to show that the chemotaxis phenomenon can be explained without the traditional chemotaxis assumption that a microscopic scale individual cell measures a macroscopic scale chemical gradient (see [2,3,34,35]). The motility (or departing probability) $\gamma_c(v)$ of the diffusion in (1.1) depends only on the chemical concentration. In other words, each cell changes its motility according to the concentration of chemoattractant, but not according to its gradient. Nevertheless, such a migration strategy produces a chemotactic migration phenomenon and can be modeled by a logarithmic type advection term, as written in (1.3).

Models like (1.1), (1.4) have been used recently in the Physics litterature [4]. In [4], the authors use this model to study the formation of periodic stripes in (engineered) bacteria's colonies observed in [19]. For the sake of simplicity, they use for the cell motility a (smoothed version of) a step function, taking only two values. Beyond simplicity, their choice is motivated by biological measurements showing that the motily almost vanishes beyond a critical value of the chemical concentration [19].

From the point of view of analysis, the fact that the diffusion term can be written as a Laplacian of some quantity allows us to apply duality arguments and to obtain global existence. We also obtain the instability of constant steady states for both bounded ($c \neq 0$) and unbounded ($c = 0$) motility cases. The purpose of this section is to show the connection between the model that we study and classical Keller-Segel equations.

The mathematical study of chemotactic migration has been started by the publication of three papers of Keller and Segel. In the first paper, based on macroscopic arguments they proposed an abstract form for the diffusion-advection equation,

$$u_t = \nabla \cdot \Big(\mu(u,v)\nabla u - \chi(u,v)\nabla v\Big) = \nabla \cdot \left[\mu(u,v)\Big(\nabla u - \frac{\chi(u,v)}{\mu(u,v)}\nabla v\Big)\right], \tag{B.1}$$

where u and v are cell and chemical densities respectively (see [13, (3.5)]). This equation contains two terms: the Fickian diffusion (related to μ) and the chemotactic advection (related to χ). Two typical formulas for μ, χ are

$$\mu(u,v)=\mu_0, \qquad \chi(u,v)=\chi_0 u, \qquad \text{(minimal model)}$$

$$\mu(u,v)=\mu_0, \qquad \chi(u,v)=\chi_0\frac{u}{v}, \qquad \text{(logarithmic model)}$$

where μ_0 and χ_0 are constants. The corresponding chemotaxis equations are respectively called minimal and logarithmic models. Keller and Segel took the logarithmic model [13, (2.3)] and showed that the constant steady state is unstable if $\frac{\chi_0}{\mu_0}>1$ (see [13, (4.16)] for the original statement). However, in dimension $n=3$, until now the global existence of weak or strong solutions for the logarithmic model has been shown in general only for cases without aggregation, i.e., for $\frac{\chi_0}{\mu_0}<1$ (see [8] and the discussion in next section).

Keller and Segel derived their new model in their second paper [14] entitled "*Model for Chemotaxis*". Starting from a microscopic model featuring a random walk, they derived the specified chemotaxis equation for the cell density,

$$u_t=\nabla\cdot(\mu(v)\nabla u-\chi(v)u\nabla v), \tag{B.2}$$

where μ is a function of v only and $\chi(u,v)$ in (B.1) is now $\chi(v)u$. A new contribution of the model is the explicit relation between the diffusivity μ and the chemosensitivity χ, given by

$$\chi(v)=(\alpha-1)\mu'(v), \tag{B.3}$$

where α represents the ratio of effective body length (i.e., distance between receptors) over the walk length. If an organism cannot measure the gradient of the chemoattractant, which is the case we are interested in, we may set $\alpha=0$, so that $\chi(v)=-\mu'(v)$. This is the case of special interest mentioned by S. Corrsin as commented in [14, p. 228]. Then, (B.2) writes

$$u_t=\Delta(\mu(v)u), \tag{B.4}$$

which is identical to (1.1), when one identifies γ_c with μ. The difference is that μ is assumed to be an increasing function of v in the works of Keller and Segel, whereas we assume in this work that γ_c is a decreasing function of v.

Appendix C. Discussion on global existence

The purpose of Keller and Segel in introducing their first chemotaxis model was to explain the aggregation phenomenon. Its initiation was viewed as the instability of a constant steady state [13]. For the completeness of the study, it is also important to analyze the well-posedness, global existence, asymptotic convergence to a nonconstant steady state, etc. However, there are many obstacles in obtaining those. For example, the asymptotic convergence to a nonconstant steady state is difficult to obtain, and the main difficulty is in the non-uniqueness of steady states. Instead, the existence of nonconstant steady states has been one of key issues in the elliptic problem viewpoint (see [18, 22–24, 31]).

The global existence of the evolutionary problem has been one of the key issues. One of the challenges consists in obtaining the aggregation and the global existence together. Solutions of the minimal model are global and bounded in one space dimension (see [25]). However, in two space dimensions, the solution of the minimal

model can blow up in a finite-time if the initial population's mass is bigger than a critical size whereas, in three (or more) space dimensions, the solution can blow up with any population size ([7, 10, 12, 20, 21, 26, 28, 33], see [11, 26] for a review).

On the other hand the blow up of the logarithmic model is not reported for dimensions $n \leq 2$. In one space dimension, solutions to the logarithmic model are bounded and exist globally in time. In dimension $n \geq 2$, classical bounded solutions have been obtained when $\frac{\chi_0}{\mu_0} < \sqrt{\frac{2}{n}}$ (see [5, 32]), a condition which is not compatible with the necessary condition of aggregation $\frac{\chi_0}{\mu_0} > 1$. Note that the value $\sqrt{\frac{2}{n}}$ is not optimal as shown in [16] (see also [6] for a result in a perturbative framework close to the elliptic-parabolic case). When it comes to weak solutions, global existence has been obtained in [32] under the condition that $\frac{\chi_0}{\mu_0} < \sqrt{\frac{n+2}{3n-4}}$, which now includes an aggregation regime in dimension $n = 2$. The case of radial solutions also admits weak solution for some aggregation regimes, see [29]. In dimension $n = 3$, to consider regimes of aggregation with global solutions, one possible direction is to resort to a generalized form of solution, as studied in the recent work [17]. Note that in dimension $n = 3$, the solution may blow up in finite time if $\frac{\chi_0}{\mu_0} > 6$ (see [9, Section 6.1.1]). Coming back to our system (1.1)–(1.4), the condition $0 < k < 4/3$ for the global existence in three dimensions does not appear to be simply a technical issue. We do not know the optimal value of the parameter k for the global existence.

Finally, we mention the recent work [30], in which the authors study a related model: they consider the system (1.1), (1.4) in dimension $n \geq 2$, with the cell motility $\gamma_c(v)$ replaced by a smooth function $\gamma(v)$ satisfying $0 < \bar{\gamma}^{-1} \leq \gamma(v) \leq \bar{\gamma}$ for some positive constant $\bar{\gamma}$. They obtain the existence of weak solutions (but do not consider the patterns related to the system), and, under the supplementary assumption that the domain is convex, they obtain strong solutions in dimension 2 (and in dimension $n \geq 3$ under some smallness assumption on the initial data). In their proof, the starting point for showing the existence is to obtain an L^2 estimate for u (Lemma 3.1 in [30]), thanks to a duality technique. Our own use of duality lemmas is reminiscent of the one used in [30]. Note that in dimension $n = 2$ or 3, our result gives the existence of weak solutions for which no L^∞ bound for the concentration v is known. The cell motility $\gamma_c(v)$ in our model is therefore *a priori* not lower bounded (away from zero), so that existence does not directly follow from the estimates of [30] in that case. In dimension 1, our estimates show that v is bounded between two strictly positive constants, so that the estimates of [30] can be used (more precisely, the extension of these estimates in dimension 1).

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