CHEMOTACTIC TRAVELING WAVES WITH COMPACT SUPPORT

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Abstract. A logarithmic model type chemotaxis equation is introduced with porous medium diffusion and a population dependent consumption rate. The classical assumption that individual bacterium can sense the chemical gradient is not taken. Instead, the chemotactic term appears by assuming that the migration distance is inversely proportional to the amount of food if food is the reason for migration. The existence and uniqueness of a traveling wave solution of the model are obtained. In particular, solutions have interfaces that divide into constant and non-constant regions. In particular, the profile of the population distribution has compact support. Numerical simulations are provided and compared with analytic results.

1. Introduction

We consider a chemotaxis model,

$$\begin{aligned} &u_t = \Big(\mu(m,u)\Big(u_x - \frac{u}{m}m_x\Big)\Big)_x, \\ &m_t = -\kappa(m,u)u, \end{aligned} \qquad x\in\mathbb{R},\ t>0, \tag{1.1}$$

where u is the population density of the bacteria and m is the resource (or food) density that the bacteria consume. If the diffusivity $\mu(m,u)$ and the consumption rate $\kappa(m,u)$ are constants, (1.1) is a classical logarithmic model type system, which is often taken to explain chemotactic traveling wave phenomena. The purpose of this paper is to develop a chemotaxis model in the form of (1.1), which provides a compactly supported traveling wave solution. Remember that the chemotactic traveling wave presented by Adler (1966) is of a pulse type. It is needed to understand what brings such a compactly supported traveling wave band. Furthermore, having a mathematical theory with such a solution will promote our understanding on this ubiquitous phenomenon. We will see indeed that, by choosing the two parameters, $\mu(m,u)$ and $\kappa(m,u)$, in a physically meaningful way, the population distribution of a traveling wave solution turns out to be compactly supported.

It is natural to expect that, if there is more food, individual bacterium consumes more and, if there is a larger population, individual bacterium consumes less food due to competition. Therefore, the consumption rate $\kappa(m,u)$ should be an increasing function of m and a decreasing function of u. In this paper we take the consumption rate as

$$\kappa(m,u)=\alpha(m)u^{-\delta},\quad 0<\delta<1, \tag{1.2}$$

where $\alpha(m)$ satisfies

$$\alpha(m)>0,\quad \alpha'(m)\ge 0,\quad \text{and}\quad \lim_{m\to 0+}\alpha(m)=0. \tag{1.3}$$

The conditions in (1.3) indicate that an individual organism consumes food ($\alpha(m)>0$), consumes more if there is more food ($\alpha'(m)\ge 0$), and does not consume food if there is no food ($\lim_{m\to 0+}\alpha(m)=0$). If $\alpha(m)=m^r$ with $r>0$, these conditions are satisfied. Note that the most meaningful case is when $r<1$. If $r=1$, the consumption rate implies that each individual consumes two times more if there is two times as much food. However, that is rarely the case and it is reasonable to assume $r < 1$.

For the diffusivity coefficient $\mu(m, u)$, we take

$$\mu(u, m) = \frac{p}{m^2}\Big(\frac{u}{m}\Big)^{p-1}, \quad p > \delta. \tag{1.4}$$

In this case, the first equation of (1.1) turns into

$$u_t = \Big(\frac{1}{m}\Big(\Big(\frac{u}{m}\Big)^p\Big)_x\Big)_x, \quad p > \delta. \tag{1.5}$$

If m is constant, this is a nonlinear diffusion equation. Hence, the equation can be considered as a nonlinear diffusion equation in a food metric space (see Section 6 for a discussion). The characterizing property of the porous medium equation ($p > 1$) is the propagation of an interface, which helps produce a compactly supported traveling wave. This porous medium equation is widely accepted as an appropriate diffusion model for the dispersal of biological organisms.

Under the choice of κ and μ given in (1.2) and (1.4), respectively, the chemotaxis equations in (1.1) are written as

$$\begin{aligned} &u_t = \Big(\frac{1}{m}\Big(\Big(\frac{u}{m}\Big)^p\Big)_x\Big)_x, \\ &m_t = -\alpha(m)u^q, \end{aligned} \qquad 0 < q < 1,\ p + q > 1,\ x \in \mathbb{R},\ t > 0. \tag{1.6}$$

Notice that we are taking $p + q > 1$ here. Since $q = 1 - \delta$ and $0 < \delta < 1$, this condition is equivalent to $p > \delta$. We will show that the traveling wave solution of (1.6) has an interface in front of the wave propagation if $p > 1$. The traveling wave solution also has an interface behind the wave propagation, which is called a tail interface, if $\alpha(m)m^q > C_1 m^{1-\epsilon}$ for small $\epsilon > 0$ and for small $m > 0$. If we set $\alpha(m) = m^r$, the condition becomes $r+q = r+1-\delta < 1$, or $r < \delta$. In terms of κ, we obtain a condition for the existence of a tail interface,

$$\kappa(m, m)^{-1} = O(m^\epsilon) \quad \text{for some } \epsilon > 0 \quad \text{as} \quad m \to 0. \tag{1.7}$$

If a traveling wave solution has a tail interface, m will be zero behind the interface. In this case, extra care is needed since the model equations (1.1) or (1.6) are meaningless if $u \neq 0$. We will see that $u(\xi) = 0$ if $m(\xi) = 0$ and the condition $r < \delta$ is essential for this.

We can formally explain why the condition $r < \delta$ is essential to obtain a tail interface. If $r < \delta$, the consumption rate κ changes more sensitively with respect to m than u when $u, m \to 0$ as a traveling wave passes by. Suppose m goes to zero faster than u. Then, the consumption rate κ decreases and if κ decreases, m never goes to zero and hence there will be no interface. Therefore, u should go to zero faster than m and hence $\mathrm{supp}(u) \subset \mathrm{supp}(m)$. Suppose that u reaches zero first. If m is not zero at this moment, m will be never zero since there is no longer any consumption. This contradicts the boundary condition $m_- = 0$. Therefore, u and m should have tail interface at the same position if they have any at all.

In conclusion, we will see that the traveling wave for the bacteria population is compactly supported if $p > 1$ and (1.7) is satisfied. The case with $p = 1$ and $q = 1$ ($\delta = 0$) is written by

$$\begin{aligned} &u_t = \Big(\frac{1}{m}\Big(\frac{u}{m}\Big)_x\Big)_x = \Big(\frac{1}{m^2}\Big(u_x - \frac{u}{m}m_x\Big)\Big)_x, \\ &m_t = -\alpha(m)u, \end{aligned} \qquad x \in \mathbb{R},\ t > 0. \tag{1.8}$$

In this case, there is no interface and the traveling waves (u, m) are strictly positive on $\mathbb{R}$ (see Choi and Kim (2015)).

Chemotactic motion is a ubiquitous migration strategy of microscopic scale organisms such as bacteria, amoebae, and individual cells of a multicellular organisms, where cells move toward or against the gradient of chemical substances. The self-organization of slime molds (see Bonner (1967)) and the traveling wave phenomenon of E-coli (see Adler (1966, 1969); Adler and Dahl (1967); Adler and Templeton (1967)) are two exemplary systems of chemotaxis phenomena. Keller and Segel (1970, 1971a,b) proposed mathematical models for chemotaxis and the Keller-Segel equations have been intensively studied to understand the stability, existence, diffusion and relaxation limits, and chemotactic motility. Various mathematical models are introduced by Keller and Odell (1975); Lapidus and Schiller (1978); Horstmann and Stevens (2004); Hillen and Painter (2009b); Xue et al. (2011); Li and Wang (2012). The reader is referred to previous works by Keller and Odell (1975); Erban and Othmer (2004); Wang and Hillen (2008); Hillen and Painter (2009a); Li and Wang (2009); Lui and Wang (2010); Jin et al. (2013) for other related topics and general references.

In the derivation of traditional chemotaxis models, microscopic scale organisms are assumed to sense a macroscopic scale gradient of chemical substances. As mentioned by Keller and Segel (1971a), the fact that a microscopic scale cell senses the macroscopic scale chemical gradient it is a mysterious assumption. However, there are relatively less efforts to develop a chemotaxis theory without this mysterious assumption. Recently, Yoon and Kim (2015, 2017) showed that chemotactic traveling waves and cell aggregation phenomena can be obtained by taking Fokker-Planck type diffusions, developed by Cho and Kim (2013). Desvillettes et al. (2018) showed the global existence of a chemotaxis model that includes aggregation and singular chemosensitivity. Adopting a food metric is another option to develop chemotaxis theory without the assumption of sensing the gradient. Choi and Kim (2015, 2017) proved that chemotactic phenomena can be obtained by assuming a food metric diffusion, and Hilhorst et al. (2018) showed the effect of food metric diffusion on the propagation toward food. These models are based on behavioral changes depending on environmental conditions without any direction information.

Recently, chemotaxis models with diffusion in porous media have received attention in both the experimental and theoretical viewpoints. For example, Olson et al. (2004) consider chemotaxis phenomena in a packed column. Similar column experiments are studied by Long and R.M. (2009); Wang and R.M. (2009). Valds-Parada et al. (2009) derived an effective theoretical chemotaxis model in porous media using Darcy-scale representation (also see Avesani et al. (2016)). Recent researches indicate that the nonlinear effect of porous media should not be ignored. For example, Byrne and Owen (2004); Burger et al. (2006); Hillen and Painter (2001) introduce nonlinear diffusion equations to avoid overcrowding phenomena of chemotaxis models. We refer to papers by Di Francesco et al. (2010); Tao and Winkler (2012) for more mathematical results of chemotaxis models with porous medium diffusion. A traveling wave solution to a porous medium type equation was studied by De Pablo and Sanchez (1998). See also (Funaki et al. (2006)) for an interface problem of the chemotaxis-Fisher equation. The extension of linear food metric diffusion in (1.8) to porous medium diffusion in (1.5) will provide extra freedom in the choice of mathematical models.

2. NOTATION AND MAIN RESULTS

The traveling wave solution of speed $c \in \mathbb{R}$ is a solution of (1.6) in the form

$$u(x,t) = u(\xi), \quad m(x,t) = m(\xi), \quad \xi = x - ct,$$

where ξ denotes the moving frame variable. Then, the wave profiles u and m satisfy

$$\begin{aligned} &-cu' = \Big(\frac{1}{m}\Big(\Big(\frac{u}{m}\Big)^p\Big)'\Big)', \\ &-cm' = -\alpha(m)u^q, \end{aligned} \quad \xi \in \mathbb{R},\ 0 < q < 1,\ p+q > 1, \tag{2.1}$$

where the ordinary differentiation is with respect to the variable $\xi \in \mathbb{R}$. For the boundary conditions of the traveling wave solutions we take

$$\begin{aligned} &(u^p)'(\xi) \to 0, \quad m'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty, \\ &u(\xi) \to u_\pm \ge 0, \quad m(\xi) \to m_\pm \ge 0 \quad \text{as} \quad \xi \to \pm\infty. \end{aligned} \tag{2.2}$$

Since the problem has the symmetry of spatial orientation, we assume without loss of generality that

$$0 \le m_- \le m_+.$$

There are five free parameters which include four boundary values, $u_\pm, m_\pm$, and the wave speed c. For the existence of a traveling wave, the boundary conditions should satisfy

$$u_+ = m_- = 0, \quad m_+ > 0. \tag{2.3}$$

Then, there are three parameters left, u_-, m_+ and c, which should satisfy (2.4). Therefore, the traveling wave solutions are of a two parameter family of functions. Note that it has been shown that the conditions in (2.3) are necessary for the existence of a traveling wave solution when $p = q = 1$ (see Choi and Kim (2015, 2017)). The proof for the general case of this paper is similar and omitted.

Notice that the diffusion is degenerate if $p > 1$ and the reaction is not necessarily Lipschitz continuous. Hence, the solution is considered in a weak sense.

Definition 2.1. *A pair (u,m) of $H^1(\mathbb{R})$ functions is called a (weak) solution of* (2.1) *if*

$$\operatorname{supp}(u) \subset \operatorname{supp}(m),$$
$$\int \frac{1}{m}\Big(\Big(\frac{u}{m}\Big)^p\Big)'\varphi' + cu\varphi' dx = 0, \quad \textit{and} \quad \int cm\varphi' + \alpha(m)u^q\varphi\, dx = 0,$$

for any test function $\varphi \in C_c^\infty(\mathbb{R})$.

The condition $\operatorname{supp}(u) \subset \operatorname{supp}(m)$ is needed to make the ratio $\frac{u}{m}$ in the equation meaningful. We will see that the resource distribution m of a traveling wave solution is always monotone and hence is called a font type. However, the population distribution u is either of a pulse or a front type.

First, recall that a piecewise classical solution with continuity of solutions and its derivatives appearing in the integral form of the equation is a weak solution:

Lemma 2.1. *A pair (u,m) of $H^1(\mathbb{R})$ functions is a weak solution of* (2.1) *if (u,m) satisfies* (2.1) *piecewise in the classical sense and $\left(\frac{u}{m}\right)^p \in C^1(\mathbb{R})$.*

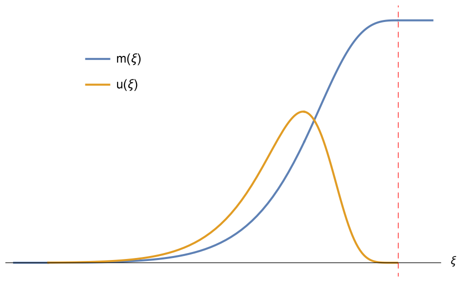
Proof. Since the proof is elementary, we omit it. □

The nonlinear diffusion and non-Lipschitzian consumption can produce an interface (or a free boundary). The two different dynamics will produce two kinds of interfaces.

Definition 2.2. *Let (u,m) be a (weak) traveling wave solution of* (2.1) *satisfying the boundary conditions in* (2.2) *and* (2.3)*.*

(1) *A point $\xi_+ \in \mathbb{R}$ is called a front interface if $[\xi_+,\infty)$ is the maximal interval such that $(u,m)=(u_+,m_+)$ on it.*
(2) *A point $\xi_- \in \mathbb{R}$ is called a tail interface if $(-\infty,\xi_-]$ is the maximal interval such that $(u,m)=(u_-,m_-)$ on it.*

Below we present a schematic graph of a traveling wave solution (u,m). We will see that, if $p>1$, a traveling wave solution contains a front interface, as in the figure. The tail interface is decided by the consumption rate. The population distribution u has compact support if it has both front and tail interfaces.



A schematic graph.

The left is an illustration of a traveling wave solution (u,m). The red dashed line indicates the location of a front interface.

The main result of this paper is the following theorem, which is proved in Section 4.

Theorem 2.1. *Let $0<q<1$, $p+q>1$, and α satisfy* (1.3)*. There exists a nontrivial traveling wave solution satisfying* (2.1), (2.2), *and* (2.3) *if and only if $c>0$ and*

$$u_- = \lim_{m\to 0}\Big(\frac{c^2}{p}\frac{m^{p+2}}{\alpha(m)}\Big)^{\frac{1}{p+q-1}} < \infty. \tag{2.4}$$

The traveling wave solution is unique up to a translation and $m\in C^1(\mathbb{R})$. If $p<2$, $u\in C^1(\mathbb{R})$. Furthermore,

(1) $\int_{\mathbb{R}} u^q d\xi < \infty$ *if and only if* $\int_0^{m_+}\frac{1}{\alpha(\eta)}d\eta<\infty$.
(2) $\int_{\mathbb{R}} u\, d\xi < \infty$ *if* $\int_0^{m_+}\frac{1}{\alpha(\eta)}d\eta<\infty$.
(3) *The traveling wave has a front interface if and only if $p>1$.*
(4) *The traveling wave has a tail interface if $\alpha(m)m^q > m^{1-\epsilon}$ for some $0<\epsilon<1$ when $m>0$ is small.*
(5) *If $0<p\le 1$ and there exist constants $C_1>0$ and $\epsilon>0$ such that*
$$\alpha(m)m^q \le C_1 m,\ \ m\in(0,\epsilon),$$
then $u,m\in C^2(\mathbb{R})$, and $u(\xi)>0$ and $0<m(\xi)<m_+$ for all $\xi\in\mathbb{R}$.

Recall that the solution of a porous medium equation, $u_t=\Delta u^p$, is $C^1(\mathbb{R})$ only if $p\le 2$. The solution of (2.1) also has the same structure. However, the resource distribution m is at least $C^1(\mathbb{R})$ even if it has an interface. Analysis of this problem is based on the properties of α given in (1.3). However, we may obtain a more intuitive idea if we specify the case using a power law $\alpha(m)=m^r$ with $r>0$. In this case, the consumption rate $\kappa(m,u)$ becomes $\kappa(m,u)=m^r u^{-\delta}$ with $0<\delta<1$. Then,

$$\kappa(m,u)u = m^r u^{-\delta}u = m^r u^q = \alpha(m)m^q, \quad q=1-\delta.$$

In this case, one can easily check the conditions in Theorem 2.1 in terms of $r>0$ and obtain the following corollary.

Corollary 2.1. *Suppose that $\alpha(m) = m^r$ with $r > 0$. Then,*

(1) *If $r > p + 2$, there is no (bounded) traveling wave solution.*
(2) *If $r = p + 2$, the population profile is of a front type.*
(3) *If $1 \le r < p + 2$, the population profile is of a pulse type and $\int_{\mathbb{R}} u^q d\xi = \infty$.*
(4) *If $\delta \le r < 1$, traveling waves have $\int_{\mathbb{R}} u^q \, d\xi < \infty$ without tail interface.*
(5) *If $r < \delta$, traveling waves have a tail interface.*

The rest of this paper is organized as follows. In Section 3, we derive a decoupled equation for m only and show the existence and uniqueness of a decoupled problem. The existence of front and tail interfaces is obtained in Theorem 3.1. In Section 4, we return to the system and prove Theorem 2.1. In Section 5, we demonstrate the analytical results via numerical simulations. In Section 6, we discuss the derivation of the chemotaxis model (1.6) in terms of the food metric and compare it to a typical approach of chemotaxis models. We also discuss what the theoretical results of this paper implicate.

3. Existence and uniqueness of a decoupled problem

We first derive a decoupled single equation for m under the boundary conditions in (2.2) and (2.3). The key analysis of this paper is on this decoupled equation.

Remember that we consider parameters that satisfy $p+q > 1$ and $0 < q < 1$. Therefore, we have $p > 1 - q > 0$. Integrate the first equation of (2.1) on an interval (ξ, ζ) and take a limit $\zeta \to \infty$, which gives

$$-c \lim_{\zeta\to\infty} u(\zeta) + cu(\xi) = \lim_{\zeta\to\infty} \Big(\frac{1}{m(\zeta)}\Big(\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^p\Big)'\Big) - \Big(\frac{1}{m(\xi)}\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)'\Big).$$

The first part of the right side is written by

$$\begin{aligned}\lim_{\zeta\to\infty} \frac{1}{m(\zeta)}\Big(\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^p\Big)' &= \lim_{\zeta\to\infty} \frac{p}{m(\zeta)}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^{p-1}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)' \\ &= \lim_{\zeta\to\infty} \frac{p}{m(\zeta)}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^{p-1}\Big(\frac{m'(\zeta)u(\zeta) - m(\zeta)u'(\zeta)}{m^2(\zeta)}\Big).\end{aligned}$$

We claim this limit is zero. Since $m(\zeta) \to m_+ > 0$ as $\zeta \to \infty$, it is enough to consider the numerator. Then, since $p > 0$, $u(\zeta) \to 0$, and $m(\zeta) \to m_+$ as $\zeta \to \infty$, we have

$$\lim_{\zeta\to\infty} \Big(pu^{p-1}m'u - pu^{p-1}mu'\Big) = \lim_{\zeta\to\infty} \Big(pu^p m' - m(u^p)'\Big) = 0.$$

Therefore, we obtain

$$-cu(\xi)m(\xi) = \Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)'.$$

Next, multiply $\big(\frac{u}{m}\big)^{q-1}$ to both sides,

$$-cu(\xi)m(\xi)\Big(\frac{u(\xi)}{m(\xi)}\Big)^{q-1} = \Big(\frac{u(\xi)}{m(\xi)}\Big)^{q-1}\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)',$$

which is simplified by

$$-cu(\xi)^q m(\xi)^{2-q} = \frac{p}{p+q-1}\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^{p+q-1}\Big)'.$$

Since $u^q = c\frac{m'}{\alpha(m)}$ by the second equation of (2.1), after substitution,

$$-c^2\frac{m(\xi)'}{\alpha(m(\xi))}m(\xi)^{2-q} = \frac{p}{p+q-1}\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^{p+q-1}\Big)'.$$

Integrating it over an interval (ξ,∞) leads to

$$-c^2\int_\xi^\infty \frac{m(\tau)'}{\alpha(m(\tau))}m(\tau)^{2-q}d\tau \quad = \quad \frac{p}{p+q-1}\int_\xi^\infty \Big(\Big(\frac{u(\tau)}{m(\tau)}\Big)^{p+q-1}\Big)'d\tau. \tag{3.1}$$

The left side becomes

$$-c^2\int_\xi^\infty \frac{m(\tau)'}{\alpha(m(\tau))}m(\tau)^{2-q}d\tau = -c^2\int_{m(\xi)}^{m_+}\frac{\eta^{2-q}}{\alpha(\eta)}d\eta.$$

We denote the right side by

$$F(x,y) := c^2\int_x^y \frac{\eta^2}{\alpha(\eta)\eta^q}d\eta. \tag{3.2}$$

The integral in the right side of (3.1) is written by

$$\int_\xi^\infty \Big(\Big(\frac{u(\tau)}{m(\tau)}\Big)^{p+q-1}\Big)'d\tau = \lim_{\zeta\to\infty}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^{p+q-1} - \Big(\frac{u(\xi)}{m(\xi)}\Big)^{p+q-1}.$$

Since $p+q-1>0$, the boundary conditions in (2.2) imply

$$\lim_{\zeta\to\infty}\Big(\frac{u(\zeta)}{m(\zeta)}\Big)^{p+q-1} = 0.$$

Therefore, (3.1) is written by

$$\frac{p}{p+q-1}\Big(\frac{u(\xi)}{m(\xi)}\Big)^{p+q-1} = F(m(\xi),m_+). \tag{3.3}$$

Note that $F(x,y)\ge 0$ if $0<x\le y$ by the assumptions for α in (1.3). We can write u in terms of m and F by using (3.3), i.e.,

$$u(\xi) = m(\xi)\Big(\frac{p+q-1}{p}F(m(\xi),m_+)\Big)^{\frac{1}{p+q-1}}. \tag{3.4}$$

Now we substitute (3.4) into the second equation, $cm' = \alpha(m)u^q$, of (2.1) and finally obtain a decoupled equation, which is

$$m' = \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\alpha(m)m^q\Big(F(m,m_+)\Big)^{\frac{q}{p+q-1}}. \tag{3.5}$$

In the rest of this section we analyze the solution of an initial value problem of the decoupled equation,

$$\begin{aligned}&m' = \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\alpha(m)m^q\Big(F(m,m_+)\Big)^{\frac{q}{p+q-1}}, \quad \xi\in\mathbb{R},\\ &m(0) = m_0, \qquad 0<m_0<m_+,\end{aligned} \tag{3.6}$$

where F is given by (3.2) and α satisfies (1.3). We will show the existence and uniqueness of this initial value problem and verify when the traveling wave solution has front and/or tail interfaces.

Proposition 3.1 (Existence)**.** *Let $0 < q < 1$, $p+q > 1$, and α satisfy* (1.3)*. There exists a maximal interval (ξ_-, ξ_+), $-\infty \le \xi_- < 0$ and $0 < \xi_+ \le \infty$, such that* (3.6) *has a solution bounded by $0 < m(\xi) < m_+$ for $\xi_- < \xi < \xi_+$.*

Proof. Let $G(m(\xi))$ be the right side of (3.6), i.e.,

$$G(x) := \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}} \alpha(x) x^q \Big(F(x,m_+)\Big)^{\frac{q}{p+q-1}}.$$

Then,

$$\begin{aligned} G'(x) &= \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}} (\alpha'(x)x^q + q\alpha(x)x^{q-1}) F^{\frac{q}{p+q-1}} \\ &+ \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}} \frac{q}{p+q-1}\alpha(x)x^q F^{\frac{1-p}{p+q-1}} \frac{\partial}{\partial x}F(x,m_+). \end{aligned}$$

$F(x,m_+)$ is positive for $x < m_+$, $\alpha(x)$ is differentiable for $x > 0$, and the partial derivative $\partial F(x,m_+)/\partial x$ exists for $x > 0$. This implies that G is differentiable for $0 < x < m_+$. By the Cauchy-Lipschitz theorem for an ordinary differential equation and the Picard iteration method, there exists such a maximal interval. □

If $p = q = 1$, then $G(x)$ is uniformly Lipschitz with two steady states $x = 0$ and $x = m_+$. This allowed Choi and Kim (2015) to show the global existence ($\xi_- = -\infty$ and $\xi_+ = \infty$), the uniqueness of the initial value problem (3.6), and that $0 < m(\xi) < m_+$ for all $\xi \in \mathbb{R}$. However, if $p > 1$, then $G(x)$ is not Lipschitz continuous at the zero of $F(x,m_+)$, i.e., at $x = m_+$. The Lipschitz continuity of $G(x)$ can be also broken at $x = 0$ if $\alpha(x)x^q$ is not Lipschitz at $x = 0$. The corresponding condition is the existence of constants $0 < \beta < 1$ and $\epsilon > 0$ such that

$$\alpha(m)m^q > m^\beta, \ m \in (0,\epsilon). \tag{3.7}$$

For example, if $\alpha(m) = m^r$, this condition turns into $r + q < 1$. If we set $q = 1 - \delta$ for the $\delta > 0$ in (1.2), then the condition becomes $r < \delta$.

If $p > 1$, we will see that $m(\xi)$ approaches m_+ as $\xi \to \xi_+ < \infty$, which implies that the traveling wave profile $u(\xi)$ has a front interface. Similarly, if (3.7) is satisfied, we will see that $m(\xi)$ approaches zero as $\xi \to \xi_- > -\infty$. Therefore, by the relation for u in (3.4) and the property of F,

$$u(\xi) = 0, \quad \text{for} \quad \xi \notin (\xi_-, \xi_+).$$

The proof for the case without interface is similar to the case with $p = q = 1$. Hence, we will omit the proof of such cases and will concentrate on the cases when both interfaces appear, i.e., $\xi_-, \xi_+ \in \mathbb{R}$. However, statements hold even for the cases that $\xi_\pm = \pm\infty$.

Lemma 3.1. *Let $m(\xi)$ be a solution of* (3.6)*. Then, $m(\xi)$ is monotone increasing on $\{\xi \in \mathbb{R} : 0 < m(\xi) < m_+\}$.*

Proof. Since $F(x,m_+) > 0$ for all $x < m_+$ and $k(x)x^q > 0$ for all $x > 0$, we have $G(x) > 0$ for all $0 < x < m_+$. Therefore, $m'(\xi) = G(m(\xi)) > 0$ on $\{\xi \in \mathbb{R} : 0 < m(\xi) < m_+\}$, i.e., $m(\xi)$ is an increasing function on $\{\xi \in \mathbb{R} : 0 < m(\xi) < m_+\}$. □

In the next two propositions, we consider two cases that $m(\xi)$ has interfaces.

Proposition 3.2. *Let $0 < q < 1$, $p + q > 1$, and $m(\xi)$ be the solution of* (3.6)*. Suppose* (3.7) *holds for some $\beta \in (0,1)$. Then, $-\infty < \xi_- < 0$ and*

$$\lim_{\xi \searrow \xi_-} m(\xi) = 0.$$

Proof. We assume $\xi_- = -\infty$ and derive a contradiction. Then,

$$m(\xi) > 0, \text{ for all } \xi < 0.$$

By Lemma 3.1, $m(\xi)$ is monotone increasing. Since there is no steady state between 0 and m_+,

$$\lim_{\xi\to-\infty} m(\xi) = 0.$$

Divide the decoupled equation (3.6) by $\alpha(m)m^q$ and obtain

$$\frac{1}{\alpha(m)m^q}m' = \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\Big(F(m,m_+)\Big)^{\frac{q}{p+q-1}}.$$

Integrate both sides over $(\xi, 0)$ with $\xi < 0$ and obtain

$$\int_\xi^0 \frac{m'(\tau)d\tau}{\alpha(m(\tau))m(\tau)^q} = \int_\xi^0 \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\Big(F(m(\tau),m_+)\Big)^{\frac{q}{p+q-1}}d\tau. \tag{3.8}$$

If we take $\xi \to -\infty$, the left side becomes finite:

$$\begin{aligned}\lim_{\xi\to-\infty}\int_\xi^0 \frac{m'(\tau)d\tau}{\alpha(m)m(\tau)^q} &= \int_0^{m(0)}\frac{dm}{\alpha(m)m^q} = \int_0^\epsilon \frac{dm}{\alpha(m)m^q} + \int_\epsilon^{m(0)}\frac{dm}{\alpha(m)m^q}\\ &\le \int_0^\epsilon \frac{dm}{m^\beta} + \int_\epsilon^{m(0)}\frac{dm}{\alpha(m)m^q} < \infty.\end{aligned}$$

Here, we are using the assumption $\beta < 1$. On the other hand, the right side of (3.8) goes to infinity as $\xi \to -\infty$:

$$\begin{aligned}&\lim_{\xi\to-\infty}\int_\xi^0 \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\Big(F(m(\tau),m_+)\Big)^{\frac{q}{p+q-1}}d\tau\\ &\ge \lim_{\xi\to-\infty}\frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\Big(F(m(0),m_+)\Big)^{\frac{q}{p+q-1}}\int_\xi^0 d\tau\\ &= \lim_{\xi\to-\infty}\frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\Big(F(m(0),m_+)\Big)^{\frac{q}{p+q-1}}|\xi| = \infty.\end{aligned}$$

Here, we used the fact that $m(\cdot)$ is monotone increasing and $F(\cdot, m_+)$ is monotone decreasing. We have derived a contradiction and hence ξ_- is finite. If $\lim_{\xi\searrow\xi_-} m(\xi) > 0$ the domain (ξ_-, ξ_+) is not maximal anymore. Therefore, $\lim_{\xi\searrow\xi_-} m(\xi) = 0$. □

Next we verify that $m(\xi)$ approaches m_+ at a finite point if $p > 1$.

Proposition 3.3. *Let* $0 < q < 1$, $p + q > 1$, $p > 1$, *and* $m(\xi)$ *be the solution of* (3.6)*. Then,* $0 < \xi_+ < \infty$ *and*

$$\lim_{\xi\nearrow\xi_+} m(\xi) = m_+.$$

Proof. The idea of the proof is similar. We assume $\xi_+ = \infty$ and derive a contradiction.

Divide (3.6) by $\big(F(m(\xi),m_+)\big)^{\frac{q}{p+q-1}}$ and obtain

$$\frac{m'(\xi)}{\big(F(m(\xi),m_+)\big)^{\frac{q}{p+q-1}}} = \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\alpha(m(\xi))m(\xi)^q.$$

For any $\xi > 0$, we integrate the both sides on $[0, \xi]$ and obtain

$$\int_0^\xi \frac{m'(\tau)}{\big(F(m(\tau), m_+)\big)^{\frac{q}{p+q-1}}} d\tau = \frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}} \int_0^\xi \alpha(m(\tau)) m(\tau)^q d\tau. \tag{3.9}$$

Since $\alpha(x)$ is an increasing function and $q > 0$, $\alpha(x)x^q$ is also an increasing function. The monotonicity of $m(\xi)$ in Lemma 3.1 implies that

$$\alpha(m(\xi))m(\xi)^q \geq \alpha(m_0)m_0^q, \ \xi \in [0, \infty).$$

Therefore, the right side of (3.9) diverges as $\xi \to \infty$:

$$\lim_{\xi\to\infty} \int_0^\xi \alpha(m(\tau))m(\tau)^q d\tau \geq \lim_{\xi\to\infty} \xi\alpha(m_0)m_0^q = \infty.$$

Next, we estimate the left hand side of (3.9). Since $\alpha(m)$ is an increasing function and $m(\xi)$ is also an increasing function, for $\tau \geq 0$,

$$\begin{aligned} F(m(\tau), m_+) &= c^2 \int_{m(\tau)}^{m_+} \frac{\eta^{2-q}}{\alpha(\eta)} d\eta \geq \frac{c^2}{\alpha(m_+)} \int_{m(\tau)}^{m_+} \eta^{2-q} d\eta \\ &\geq c^2 \frac{m_0^{2-q}}{\alpha(m_+)} \int_{m(\tau)}^{m_+} d\eta = c^2 \frac{m_0^{2-q}}{\alpha(m_+)} (m_+ - m(\tau)). \end{aligned}$$

Therefore, the left side is estimated by

$$\int_0^\xi \frac{m'(\tau)d\tau}{\big(F(m(\tau), m_+)\big)^{\frac{q}{p+q-1}}} \leq \int_{m_0}^{m(\xi)} \Big(c^2 \frac{m_0^{2-q}}{\alpha(m_+)}(m_+ - m)\Big)^{\frac{-q}{p+q-1}} dm.$$

Note that we have $0 < \frac{q}{p+q-1} < 1$. Therefore,

$$\lim_{\xi\to\infty} \int_0^\xi \frac{m'(\tau)d\tau}{\big(F(m(\tau), m_+)\big)^{\frac{q}{p+q-1}}} \leq \int_{m_0}^{m_+} \Big(c^2 \frac{m_0^{2-q}}{\alpha(m_+)}(m_+ - m)\Big)^{\frac{-q}{p+q-1}} dm < \infty.$$

This contradiction implies that $\xi_+ < \infty$. If $\lim_{\xi \nearrow \xi_+} m(\xi) < m_+$, the domain (ξ_-, ξ_+) is not maximal anymore. Therefore, $\lim_{\xi \nearrow \xi_+} m(\xi) = m_+$. $\square$

Theorem 3.1. *Let $0 < q < 1$, $p + q > 1$, and α satisfy (1.3). There exits a unique solution of (3.6) which is $C^1(\mathbb{R})$. The solution satisfies the same boundary conditions for m in (2.2) and (2.3). Furthermore,*

1. *If (3.7) holds for some $0 < \beta < 1$, there exists $\xi_- \in (-\infty, 0)$ such that $m(\xi) = 0$ for all $\xi \leq \xi_-$.*
2. *If $\alpha(m)m^q$ is Lipschitz continuous at $m = 0$, $m(\xi) > 0$ for all $\xi \in \mathbb{R}$.*
3. *If $p > 1$, there exists $\xi_+ \in \mathbb{R}_+$ such that $m(\xi) = m_+$ for $\xi \geq \xi_+$.*
4. *If $p \leq 1$, $m(\xi) < m_+$ for all $\xi \in \mathbb{R}$.*

Proof. By Proposition 3.1 and Lemma 3.1, the solution $m(\xi)$ to (3.6) exists on (ξ_-, ξ_+). Here, if there is no interface of the problem, then $\xi_\pm = \pm\infty$ and hence there exists a global solution. Suppose that $\xi_\pm \in \mathbb{R}$, i.e., there exists an interface. Then, we extend m to $\mathbb{R}$ using the same name by

$$m(\xi) = \begin{cases} 0, & \text{if } \xi < \xi_-, \\ m_+, & \text{if } \xi > \xi_+, \\ m(\xi), & \text{otherwise.} \end{cases} \tag{3.10}$$

By relation (3.5), we have m is differentiable at $\xi_\pm$ from the interior of the interval (ξ_-, ξ_+) and $m'(\xi_\pm) = 0$. Therefore, m is $C^1(\mathbb{R})$ and hence is a weak solution. Now we show the boundary conditions. Since (ξ_-, ξ_+) is the maximal domain of the solution, Proposition 3.1, $m(\xi_-) = 0$ if $\xi_- > -\infty$, and $m(\xi_+) = 0$ if $\xi_+ < \infty$. Therefore, the boundary conditions are satisfied. Suppose that $\xi_\pm = \pm\infty$. Then, since $m = 0$ and $m = m_+$ are the only steady state of the ordinary differential equation (3.6), we have $\lim_{\xi\to\infty} m(\xi) = m_+$ and $\lim_{\xi\to-\infty} m(\xi) = 0$.

Next we show the last four statements. If (3.7) holds for some $0 < \beta < 1$, then $\xi_- > -\infty$ by Proposition 3.2. If $p > 1$, then $\xi_+ < \infty$ by Proposition 3.3. Therefore, we have (1) and (3). The proofs for (2) and (4) are similar to the case $p = q = 1$ (Choi and Kim, 2015, Theorem 4.3), which are omitted in this paper. □

4. Existence and uniqueness of the system

Now we return to the system (2.1) with boundary conditions (2.2) and (2.3). The solution of the decoupled problem (3.6) is now used to construct the population density function.

Proposition 4.1. *Let* $0 < q < 1$, $p + q > 1$, *and* α *satisfy* (1.3). (*i*) *If* $c = 0$, *there is no nontrivial traveling wave solution of* (2.1). (*ii*) *If* $c > 0$, *the solution* (u, m) *of* (2.1) *with the boundary conditions* (2.2) *and* (2.3) *satisfies* (3.4) *and* (3.6). (*iii*) *Conversely, the solutions* m *of* (3.6) *and* u *given by* (3.4) *satisfy* (2.1) *and the boundary conditions* (2.2) *and* (2.3) *if the limit* u_- *is given by* (2.4).

Proof. Let $c = 0$. Then, the second equation of (2.1) implies that

$$m(\xi) = 0 \quad \text{or} \quad u(\xi) = 0.$$

Since a weak solution satisfies $\mathrm{supp}(u) \subset \mathrm{supp}(m)$ by definition, we have $u(\xi) = 0$ for all $\xi \in \mathbb{R}$. Hence, there is no nontrivial traveling wave solution if $c = 0$. The second part has been obtained in Section 3 when we derive (3.4) and (3.5).

We now show the third part. Let m be the solution of (3.6) extended to $\mathbb{R}$ by (3.10) and u be given by (3.4), i.e.,

$$u(\xi) = m(\xi)\left(\frac{p+q-1}{p} F(m(\xi), m_+)\right)^{\frac{1}{p+q-1}}.$$

Then, (3.5) is equivalent to the second equation of (2.1). The first equation is obtained by undoing the derivation process. First, the formula for u is written by

$$\left(\frac{u(\xi)}{m(\xi)}\right)^p = \left(\frac{p+q-1}{p} F(m(\xi), m_+)\right)^{\frac{p}{p+q-1}}. \tag{4.1}$$

Differentiate the equation and obtain

$$\left(\left(\frac{u(\xi)}{m(\xi)}\right)^p\right)' = \left(\frac{p+q-1}{p} F(m(\xi), m_+)\right)^{\frac{p}{p+q-1}-1} \partial_1 F(m(\xi), m_+) m'(\xi). \tag{4.2}$$

From the definition of F given by (3.2),

$$\partial_1 F(m(\xi), m_+) = -c^2 \frac{m^{2-q}(\xi)}{\alpha(m(\xi))}. \tag{4.3}$$

From (3.6), (4.2) and (4.3), it follows that

$$\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)' = \Big(\frac{p+q-1}{p}F(m(\xi),m_+)\Big)^{\frac{p}{p+q-1}-1}\Big(-c^2\frac{m^{2-q}(\xi)}{\alpha(m(\xi))}\Big)$$
$$\times\frac{1}{c}\Big(\frac{p+q-1}{p}\Big)^{\frac{q}{p+q-1}}\alpha(m)m(\xi)^q\Big(F(m(\xi),m_+)\Big)^{\frac{q}{p+q-1}}.$$

After a simplification of the right side, we obtain

$$\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)' = -c\Big(\frac{p+q-1}{p}F(m(\xi),m_+)\Big)^{\frac{1}{p+q-1}}m^2(\xi).$$

By the definition of u, we have

$$-cu(\xi) = \frac{1}{m(\xi)}\Big(\Big(\frac{u(\xi)}{m(\xi)}\Big)^p\Big)',$$

which gives the first equation in (2.1) after differentiation. Therefore, we have shown that (u,m) is a classical solution in the domain (ξ_-,ξ_+).

If $\xi_\pm = \pm\infty$, the (u,m) is a classical solution on $\mathbb{R}$ and $\text{supp}(u) = \text{supp}(m) = \mathbb{R}$. Let $\xi_+ < \infty$. Then, (4) gives $u(\xi_+) = 0$ since $m(\xi_+) = m_+$, $F(m_+,m_+) = 0$, and $p+q-1>0$. Hence, u is continuous at ξ_+. A front interface appears at $\xi = \xi_+$ only for $p>1$. Then, since $\frac{p}{p+q-1} > 1$, $\big(\frac{u}{m}\big)^p$ given by (4.1) is differentiable at ξ_+. Next, consider the case $\xi_- > -\infty$. Similarly, $u(\xi_-) = 0$ since $m(\xi_-) = 0$. Therefore, we conclude $\text{supp}(u) = \text{supp}(m)$ and $u \in C(\mathbb{R})$ for any case. A tail interface appears at $\xi = \xi_-$ only when $\alpha(x)x^q$ is not Lipschitz continuous. Then,

$$F(x,m_+) = \int_x^{m_+}\frac{\eta^2}{\alpha(\eta)\eta^q}d\eta$$

is differentiable at $x=0$. Therefore, $\big(\frac{u}{m}\big)^p$ given by (4.1) is differentiable at ξ_-. Therefore, (u,m) is a weak solution of Definition 2.1 by Lemma 2.1

Now we present the boundary conditions. The cases with finite $\xi_\pm$ are actually already verified by checking the continuity of u and v on $\mathbb{R}$. We consider the case where $\xi_\pm$ are infinite. Then, (u,m) are classical solutions on $\mathbb{R}$. The boundary conditions for m have been obtained in Theorem 3.1 and we check the boundary conditions for u. Since $p+q-1>0$, we have $\lim_{\xi\to\infty}u(\xi) = u_+ = 0$. Now we show the other boundary value u_- given by (2.4). Since $m(\xi)\to 0$ as $\xi\to-\infty$, we have

$$\lim_{\xi\to-\infty}u(\xi) = \lim_{m\to 0}\Big(c^2\frac{p+q-1}{p}m^{p+q-1}\int_m^{m_+}\frac{\eta^2}{\alpha(\eta)\eta^q}d\eta\Big)^{\frac{1}{p+q-1}}.$$

If $\lim_{m\to 0}\big(\int_m^{m_+}\frac{\eta^2}{\alpha(\eta)\eta^q}d\eta\big)$ is finite, then u_- is zero. If it is not finite, we may apply L'Hospital's rule to compute the limit. Then, we obtain

$$\lim_{m\to 0}m^{p+q-1}\int_m^{m_+}\frac{\eta^2}{\alpha(\eta)\eta^q}d\eta = \frac{1}{p+q-1}\lim_{m\to 0}\frac{m^{p+2}}{\alpha(m)}.$$

Substitute this to the above and obtain

$$\lim_{\xi\to-\infty}u(\xi) = \lim_{m\to 0}\Big(\frac{c^2}{p}\frac{m^{p+2}}{\alpha(m)}\Big)^{\frac{1}{p+q-1}} = u_-.$$

Since $m(\xi) \to m_\pm$ as $\xi \to \pm\infty$, we have

$$m'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty.$$

Note that

$$\begin{aligned}(u^p)'(\xi) &= \frac{d}{d\xi}\left(m^p(\xi)\left(\frac{p+q-1}{p}F(m(\xi),m_+)\right)^{\frac{p}{p+q-1}}\right)\\ &= pm^{p-1}(\xi)m'(\xi)\left(\frac{p+q-1}{p}F(m(\xi),m_+)\right)^{\frac{p}{p+q-1}}\\ &\quad -c^2m^p(\xi)\left(\frac{p+q-1}{p}F(m(\xi),m_+)\right)^{\frac{p}{p+q-1}-1}\frac{m^{2-q}(\xi)}{\alpha(m(\xi))}m'(\xi).\end{aligned}$$

From (3.6), it follows that

$$\begin{aligned}(u^p)'(\xi) &= \frac{p}{c}\alpha(m(\xi))m^{p+q-1}(\xi)\left(\frac{p+q-1}{p}F(m(\xi),m_+)\right)^{\frac{p+q}{p+q-1}}\\ &\quad -cm^{p+2}(\xi)\left(\frac{p+q-1}{p}F(m(\xi),m_+)\right)^{\frac{1}{p+q-1}}.\end{aligned}$$

Since $p+q-1>0$ and $m(\xi) \to m_\pm$ as $\xi \to \pm\infty$,

$$(u^p)'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty.$$

Therefore, all boundary values are satisfied and the proof is completed. □

Now we prove Theorem 2.1, which contains the of main result of this paper. Most parts of the theorem are already proved in previous propositions and theorem. We prove the other parts below.

Proof of Theorem 2.1. The first part of the theorem has been shown in Proposition 4.1 and Theorem 3.1 except for the regularity of $u \in C^1(\mathbb{R})$ when $p \le 2$. Differentiate (3.4) and write the derivative of u by

$$u' = C_1m'\big(F(m,m_+)\big)^{\frac{1}{p+q-1}} + C_2mm'\big(F(m,m_+)\big)^{\frac{1}{p+q-1}-1}\frac{m^2}{\alpha(m)m^q},$$

where C_1 and C_2 are constants. Replace m' by the right side of (3.5) and obtain

$$u' = C_3\alpha(m)m^q\big(F(m,m_+)\big)^{\frac{q+1}{p+q-1}} + C_4m^3\big(F(m,m_+)\big)^{\frac{q+1}{p+q-1}-1},$$

where C_3 and C_4 are constants. Clearly, $u' \to 0$ as $m \to 0$. Therefore, u is differentiable at ξ_-. The concern is at the interface ξ_+. If $p \le 1$, there is no interface and hence $u \in C^2(\mathbb{R})$. The first part of the right side converges to zero as $m \to m_+$ since $\frac{q+1}{p+q-1} > 0$. If $p < 2$, then $\frac{q+1}{p+q-1} - 1 > 0$ and hence the second part also converges to zero as $m \to m_+$. Therefore we have $u \in C^1(\mathbb{R})$. We already proved that $m \in C^1(\mathbb{R})$ in Theorem 3.1.

In the rest of the proof we show the five items in the list of the theorem.

Part (1): The second equation of (2.1) gives

$$u^q = c\frac{m'}{\alpha(m)}.$$

Hence,

$$\int_{\mathbb{R}} u^q d\xi = c\int_{\mathbb{R}} \frac{m'}{\alpha(m)} d\xi = c\int_0^{m_+} \frac{1}{\alpha(\eta)} d\eta. \tag{4.4}$$

Part (2): Since $q<1$ and u is bounded, we have

$$\int_{\mathbb{R}} u\, d\xi = \|u\|_\infty \int_{\mathbb{R}} \frac{u}{\|u\|_\infty} \le \|u\|_\infty \int_{\mathbb{R}} \frac{u^q}{\|u\|_\infty^q}\, d\xi < \infty$$

if $\int_0^{m_+} \frac{1}{\alpha(\eta)} d\eta < \infty$.

Part (3): The existence of a front interface $\xi_+ \in (0,\infty)$ of $m(\xi)$ has been shown in Proposition 3.3 when $p>1$. Since $u(\xi)$ is given by

$$u(\xi) = m(\xi)\Big(\frac{p+q-1}{p} F(m(\xi), m_+)\Big)^{\frac{1}{p+q-1}}, \tag{4.5}$$

it is clear that $u(\xi)=0$ for all $\xi>\xi_+$ and hence $u(\xi)$ also has a front interface at the same place $\xi=\xi_+$. It has been shown by Choi and Kim (2015) that if $p=q=1$, there is no such interface and $0<m(\xi)<m_+$ and $u(\xi)>0$ for all $\xi\in\mathbb{R}$. The same argument holds for the case $p\le 1$, which is omitted here.

Part (4): Suppose that $\alpha(m)m^q > m^{1-\epsilon}$ for some $0<\epsilon<1$ and for all small $m>0$. The existence of a tail interface $\xi_- \in \mathbb{R}_-$ of $m(\xi)$ has been shown in Proposition 3.2. Since $F(0,m_+)$ is finite, $u(\xi)=0$ for all $\xi\le\xi_-$ by the formula (4.5) and $u(\xi)>0$ if $\xi_-<\xi<\xi_+$. Therefore, u has a tail interface at the same place $\xi=\xi_-$.

Part (5). Since $p\le 1$, there is no front interface, as shown in (3). If $\alpha(m)m^q < C_1 m$ for small m, then $m=0$ is a steady state of the ODE (3.5) and the right side is Lipschitz continuous. Hence, by the Cauchy-Lipschitz theorem, $m(\xi)>0$ for all $\xi\in\mathbb{R}$. Since $0<m(\xi)<m_+$ and $u(\xi)>0$ for all $\xi\in\mathbb{R}$, u and m are $C^2(\mathbb{R})$ and classical solutions. □

5. Numerical simulations

In this section we compare the shapes of the traveling wave solutions, which depend on the parameters. The case with $p=q=1$ has been considered in Choi and Kim (2015) which shows the structure of traveling waves without an interface. In this section, we consider five cases with front interfaces by choosing $p>1$. We will fix the two parameters by

$$p = 1.5 \quad \text{and} \quad q = 0.5 \tag{5.1}$$

for the first four cases. In these cases, we have $\delta = 1-q = 0.5$. Since $p<2$, the solution is $C^1(\mathbb{R})$ in the four cases. In the fifth case, we take $p=2.5$ and the solution is only $C(\mathbb{R})$.

The traveling wave type of the population distribution, $u=u(\xi)$, is now decided by the consumption rate. Under the power law, $\alpha(m)=m^r$, the consumption rate is given by

$$\kappa(m,u) = m^r u^{-\delta}, \quad 0<r,\delta<1.$$

There are five cases in Corollary 2.1. We first consider four cases given in Table 1. In the simulation, we take the following parameters:

$$m_+ = 1, \quad m(0) = 0.5, \quad c = 2. \tag{5.2}$$

The boundary value u_- is decided by the relation (2.4). Using the parameters in (5.1) and (5.2), u_- turns into

$$u_- = \lim_{m\to 0}\Big(\frac{c^2}{p}\frac{m^{p+2}}{\alpha(m)}\Big)^{\frac{1}{p+q-1}} = \lim_{m\to 0}\frac{4}{1.5}m^{3.5-r}.$$

Therefore, $u_- = \frac{8}{3}$ for the case $r = 3.5$ and $u_- = 0$ for $0 < r < 3.5$. The quantity $\int u^q d\xi$ is computed by (4.4),

$$\int_{\mathbb{R}} u^q d\xi = c\int_0^{m_+}\frac{1}{\alpha(\eta)}d\eta = 2\int_0^1 \eta^{-r}d\eta.$$

Therefore, $\int_{\mathbb{R}} u^q d\xi = \infty$ for all $r \geq 1$, $\int_{\mathbb{R}} u^q d\xi = \frac{20}{3}$ if $r = 0.7$, and $\int_{\mathbb{R}} u^q d\xi = \frac{20}{7}$ if $r = 0.3$. A tail interface exists only when $r < \delta$. Hence, the infimum of the support of u is $-\infty$ if $r \geq 0.5$ in the example. The values of these quantities are provided in Table 1.

Table 1. The parameters are $p = 1.5,\ q = \delta = 0.5,\ c = 2$, and $m_+ = 1$. Traveling waves have an interface in the front of their propagation since $p > 1$.

r	parameter regime	u_- (wave type)	$\int u^q d\xi$	$\inf\mathrm{supp}(u)$
3.5	$r = p + 2$	$\frac{8}{3}$ (front type)	∞	$-\infty$
1.5	$1 < r < p + 2$	0 (pulse type)	∞	$-\infty$
0.7	$\delta < r < 1$	0 (pulse type)	$\frac{20}{3}$	$-\infty$
0.3	$r < \delta$	0 (pulse type)	$\frac{20}{7}$	finite

Now we observe the structures of the traveling waves of the four cases numerically.

Case 1 ($r = 3.5 = p + 2$). We find m first by solving (3.6) numerically and then find u by using (3.4). Under the parameters of this case, F in (3.2) becomes

$$F(x, y) = c^2\int_x^y \frac{\eta^2}{\alpha(\eta)\eta^q}d\eta = 4\int_x^y \eta^{-2}d\eta = 4(x^{-1} - y^{-1}).$$

Therefore, (3.6) turns into

$$m' = (1.5)^{-0.5}m^4(m^{-1} - 1)^{0.5}, \quad m(0) = 0.5.$$

Note that $m = 0$ is an unstable steady state and $m = 1$ is a stable one. Furthermore, the right side of the differential equation is Lipschitz continuous at $m = 0$ and is not at $m = 1$. This is the reason why the solution has a front interface only.

The numerical solution $m(\xi)$ of this differential equation and the population distribution $u(\xi)$ obtained by (3.4) are displayed in Figure 1. We have presented the solution with a 150 times larger space scale for the tail part ($\xi < 0$) than for the front part ($\xi > 0$) to show the tail behavior in more detail. Only under such a large scale difference, we can clearly observe that the traveling wave has a font interface approximately at $\xi = 4.57$ and $u(\xi)$ converges to a value approximately $u_- \cong 2.67$ as $\xi \to -\infty$.

Case 2 ($r = 1.5$). In this case F in (3.2) becomes

$$F(x, y) = c^2\int_x^y \frac{\eta^2}{\alpha(\eta)\eta^q}d\eta = 4\int_x^y d\eta = 4(y - x).$$

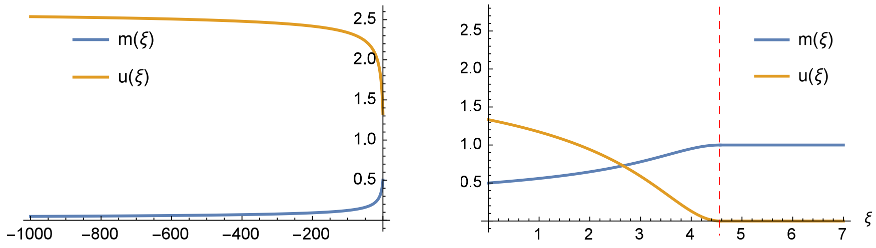


FIGURE 1. ($p = 1.5, q = 0.5, r = p+2, c = 2, m_+ = 1$). The traveling wave for u is of a front type. We can find that $u(\xi) \to u_- = 8/3$ as $\xi \to -\infty$. The front interface is at $\xi \cong 4.57$.

Therefore, (3.6) turns into

$$m' = (1.5)^{-0.5} m^2 (1-m)^{0.5}, \quad m(0) = 0.5.$$

Note that the Lipschitz continuity of the right side at the two steady states is same as the first case. Hence, there is no tail interface in this case either.

Numerical solutions are given in Figure 2. We can see that both u and m converge to zero as $\xi \to -\infty$. The scale of the tail is still a lot larger than the one of the front. However, $m \to 0$ faster than the first case. The traveling wave has a font interface approximately at $\xi = 2.81$, which is closer to the origin than the previous one.

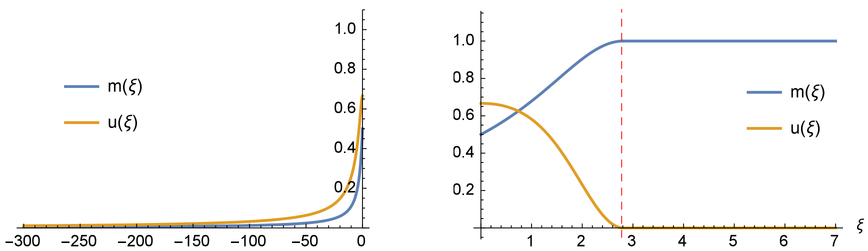


FIGURE 2. ($p = 1.5, q = 0.5, r = 1.5, c = 2, m_+ = 1$). The traveling wave for u is of pulse type. The front interface is at $\xi \cong 2.81$. Both m and u converge to zero as $\xi \to -\infty$.

Case 3 ($r = 0.7$)**.** Note that F in (3.2) becomes

$$F(x,y) = c^2 \int_x^y \frac{\eta^2}{\alpha(\eta)\eta^q} d\eta = 4 \int_x^y \eta^{0.8} d\eta = \frac{4}{1.8}(y^{1.8} - x^{1.8}),$$

and m satisfies

$$m' = \big(1.8 \times 1.5\big)^{-0.5} m^{1.2} (1 - m^{1.8})^{0.5}, \quad m(0) = 0.5.$$

Notice that the Lipschitz continuity at the two steady states is not changed.

Numerical solutions are given in Figure 3. We can see that both u and m converge to zero as $\xi \to -\infty$. The scale of the tail is now quite close to the one of the front. However, there is no tail interface. The traveling wave has a font interface approximately at $\xi = 2.37$.

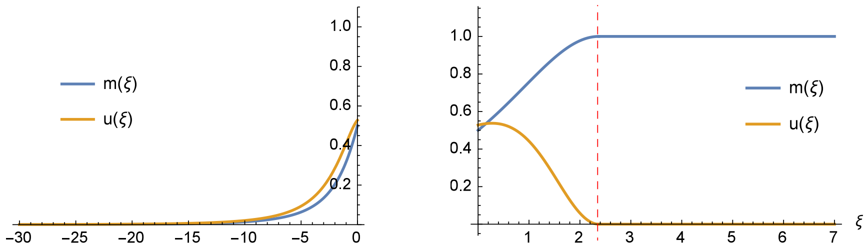


FIGURE 3. ($p = 1.5, q = 0.5, r = 0.7, c = 2, m_+ = 1$). The traveling wave for u is of a pulse type. The front interface is at $\xi \cong 2.37$. Both m and u converge to zero as $\xi \to -\infty$.

Case 4 (**$r = 0.3$**)**.** In this case, we have

$$F(x, y) = c^2 \int_x^y \frac{\eta^2}{\alpha(\eta)\eta^q} d\eta = 4 \int_x^y \eta^{1.2} d\eta = \frac{4}{2.2}(y^{2.2} - x^{2.2}),$$

and

$$m' = \big(2.2 \times 1.5\big)^{-0.5} m^{0.8}(1 - m^{2.2})^{0.5}, \quad m(0) = 0.5.$$

The right side of the differential equation is not Lipschitz continuous at both $m = 0$ and $m = 1$. This is the reason why the solution m has both front and tail interfaces.

Numerical solutions are given in Figure 4. We can see that both u and m have tail interfaces at approximately $\xi = -7.98$. The tail and the front parts of the traveling wave are now displayed with the same scale. The traveling wave has a font interface approximately at $\xi = 2.19$.

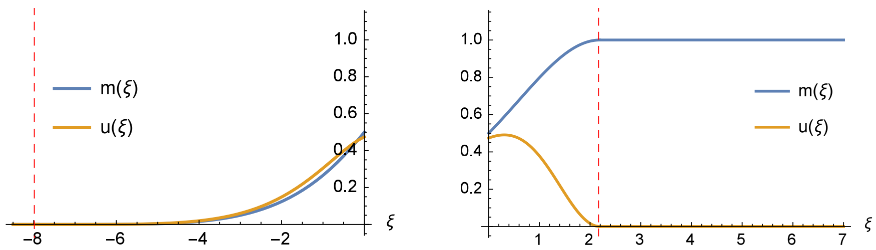


FIGURE 4. ($p = 1.5, q = 0.5, r = 0.3, c = 2, m_+ = 1$). The traveling wave for u is of a pulse type. The front interface is at $\xi \cong 2.19$ and the tail one is at $\xi \cong -7.98$.

Case 5 (**$p = 2.5$, $q = 0.5$, $r = 0.3$**)**.** In the previous four cases, we took $p = 1.5$ and hence the population distribution $u(\xi)$ is $C^1(\mathbb{R})$ by Theorem 2.1. Now we take

$$p = 2.5.$$

The other parameters are the same as the ones of Case 4, i.e.,

$$q = 0.5, \quad m_+ = 1, \quad m(0) = 0.5, \quad c = 2.$$

Then, the function $F(x, y)$ is identical to the one of Case 4. Numerical solutions are given in Figure 5. We can see that both u and m have a tail interface at approximately $\xi = -7.96$

and the traveling wave has a font interface approximately at $\xi = 1.53$. We emphasize that u is not smooth at the front interface $\xi \cong 1.53$. In fact, it is not even Lipschitz continuous and $u'(\xi)$ blows up as $\xi \nearrow 1.53$, which is the case of solutions of porous medium equations with $p > 2$.

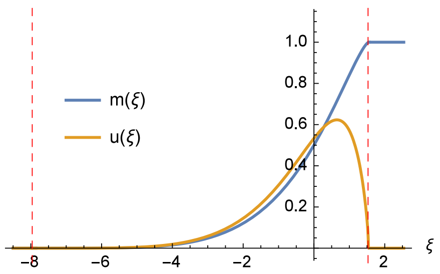


FIGURE 5. ($p = 2.5, q = 0.5, r = 0.3, c = 2, m_+ = 1$). The traveling wave for u is of a pulse type. The front interface is at $\xi \cong 1.53$ and the tail one is at $\xi \cong -7.96$.

6. DISCUSSION AND CONCLUSION

The chemotaxis model of this paper is based on the way how to measure the distance. If food is the reason for migration, for example, the migration distance should be related to food. Let x be the Euclidean space variable and $m(x)$ the distribution of food. Consider a new variable,

$$y(x) = \int_0^x m(z)dz,$$

which measures the amount of food between 0 and x. If migration distance is decided by the amount of food, having model equations in the variable y will provide a new insight into the migration phenomenon.

Let $u(x,t)$ and $v(y,t)$ be population density functions in the variables x and y, respectively. Then, we obviously have

$$\int_x^{x+h} u(z,t)dz = \int_{y(x)}^{y(x+h)} v(z,t)dz$$

since both sides give the total population between x and $x+h$. Therefore, if u and v are continuous,

$$\begin{aligned} u(x,t) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} u(z,t)dz \\ &= \lim_{h\to 0} \frac{y(x+h)-y(x)}{h} \frac{1}{y(x+h)-y(x)} \int_{y(x)}^{y(x+h)} v(z,t)dz \\ &= m(x)v(y,t). \end{aligned} \tag{6.1}$$

Recently, both experimental and theoretical results emphasize the importance of porous medium diffusion in bacterial dispersal. Indeed, we have shown in this paper that porous medium diffusion is essential to obtaining a compactly supported chemotactic traveling wave. We used the porous medium equation in the food metric space,

$$v_t = (v^p)_{yy}. \tag{6.2}$$

If we return to the original variables x and $u(x,t)$ by using (6.1), we obtain

$$u_t = \Big(\frac{1}{m}\Big(\Big(\frac{u}{m}\Big)^p\Big)_x\Big)_x = \Big(\frac{p}{m^2}\Big(\frac{u}{m}\Big)^{p-1}\Big(u_x - \frac{u}{m}m_x\Big)\Big)_x. \tag{6.3}$$

This is the equation we took for our chemotaxis model. The diffusion effect is understood well when the migration distance is given by the Euclidean metric. Equation (6.3) shows the diffusion effect under the food metric. Note that a realistic case will be a mixture of both and the results of this paper are an extreme case where dispersal is completely decided by the food metric.

A classical chemotaxis model consists of diffusion and advection terms, for example,

$$u_t = (\mu(m,u)u_x - \chi(m,u)m_x)_x. \tag{6.4}$$

The diffusivity μ is from the randomness assumption of migration and the chemosensitivity χ is from the gradient sensing assumption of the bacteria. These two terms are usually modeled and measured separately. For example, if we choose $\mu(m,u) = pu^{p-1}$ and $\chi(m,u) = \chi_0 u/m$, (6.4) turns into

$$u_t = \Big((u^p)_x - \chi_0\frac{u}{m}m_x\Big)_x. \tag{6.5}$$

We expect similar phenomena to the ones of this paper if the first equation of (1.6) is replaced by (6.5). However, even if (6.5) looks simpler than (6.3), its analysis seems more challenging and requires more sophisticated analysis techniques since the two terms should be treated separately. Notice that the surprisingly detailed results of this paper were obtained due to the specific form of equation (6.3), which actually turns into the classical diffusion equation (6.2) after a change of variables.

The interface in front of the wave propagation is called a front interface in this paper, and only the diffusion is involved in its appearance, where $p > 1$ is a necessary and sufficient condition. The interface behind the wave propagation is called a tail interface and only the consumption rate is involved in its appearance. For example, we may take

$$m_t = -\kappa(m,u)u, \quad \kappa(m,u) = m^r u^{-\delta}. \tag{6.6}$$

Then, a tail interface appears when $r < \delta$. The degeneracy of nonlinear diffusion is not involved at all in the appearance of the tail interface. In most chemotaxis models, the consumption rate κ is taken as a constant or as a function of m only. However, a tail interface is not obtained in such cases. The competition effect in the consumption, i.e., the population dependency of the consumption rate given in (6.6), is key in the appearance of a tail interface. On the other hand, the boundary value,

$$u_- = \lim_{m\to 0}(c^2 m^{p+2-r}/p)^{\frac{1}{p-\delta}},$$

shows that diffusion is involved in the parameter regime for pulse type traveling waves, which satisfy $r < p+2$. Front type traveling waves are obtained when $r = p+2$ and all parameters are involved in deciding u_-, except for m_+. However, m_+ is involved in deciding the population size, as given in (4.4).

Dispersal theories without the traditional assumption of sensing the chemical gradient have also been considered in ecology problem contexts by the second author. Fokker-Planck type diffusion equations are considered, which include advection phenomena (see

Cho and Kim (2013); Kim et al. (2013, 2014); Kim and Kwon (2016)). Advection terms that appear in such models often turn out to be advantageous. Choi and Kim (2017) studied a microscopic scale model related to the food metric diffusion model.

Acknowledgements. S.-H. Choi and Y.-J. Kim were supported by National Research Foundation of Korea under contract numbers 2017R1E1A1A03070692 and 2017R1A2B2010398, respectively.

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