HETEROGENEOUS DISCRETE KINETIC EQUATION OF STRATONOVICH TYPE

HO-YOUN KIM, YONG-JUNG KIM, AND HYUN-JIN LIM

Abstract. A heterogeneous discrete kinetic system is introduced in a form of Stratonovich type. We show that the parabolic scale singular limit exists and satisfies the heterogeneous diffusion law which was formally derived by Kim and Seo [11]. An energy functional is introduced which is monotone in time and provides uniform estimates for the convergence proof. The Div-Curl lemma is used in the proof.

Contents

1. Introduction

There have been a lot of discussions about the correct diffusion equation in a heterogeneous environment when the diffusivity $D = D(\mathbf{x})$ is a function of the space variable. The purpose of the paper is to introduce a discrete kinetic system with spatial heterogeneity and show that its diffusion limit (or parabolic scale limit) exists and satisfies a heterogeneous diffusion equation,

$$u_t = \nabla \cdot \Big(\sqrt{\mu^{-1}D}\ \nabla(\sqrt{\mu D}\, u)\Big), \tag{1.1}$$

where $\mu = \mu(\mathbf{x})$ is the turning frequency.

Kinetic theory provides a molecular level explanation of classical thermodynamics and is the foundation of the statistical thermodynamics. Discrete velocity kinetic equations are introduced to provide a methodology capable of mathematical proof of such Boltzmann dynamics (see Platkowski and Illner [18] for a discussion). Broadwell and Carleman models [1, 2] are famous

Date: January 7, 2020.

examples. In one space dimension, the Carleman model is written by

$$
\begin{aligned}
u_t^+ + v u_x^+ &= \mu(u^- - u^+),\\
u_t^- - v u_x^- &= \mu(u^+ - u^-),
\end{aligned}
\tag{1.2}
$$

where particles may take one of two velocities $\pm v$ with a constant speed $v > 0$. The turning frequency μ is proportional to the total population, say $\mu = u^+ + u^-$, in the Carleman model. On the other hand, Brownian particles collide with background molecules and the turning frequency is independent of the density of Brownian particles. Taylor [25], Goldstein [6], and Kac [10] took constant turning frequency and derived telegrapher's equation. More recently, Othmer *et al.* [15] developed the kinetic equation with constant μ as a velocity jump process with a continuum velocity. Hillen and Othemer [8, 16] formally derived diffusion equations from continuum velocity kinetic equation using a parabolic scaling limit.

The one dimensional discrete kinetic equations (1.2) have been intensively studied and used to derive various diffusion equation via parabolic scaling limit. A generalized Carleman model is often considered which takes $\mu = (u^+ + u^-)^\alpha$ with a general exponent α, where $\alpha = 1$ is the Carleman model and $\alpha = 0$ is the Goldstein-Taylor model. Pulvirenti and Toscani [19] considered the parabolic limit for $0 \le \alpha < 1$ and showed convergence to fast diffusion. Lions and Toscani [13] extended it to the case of all $\alpha < 1$, which now include the slow diffusion. See [20, 21] for subsequent results. In particular, Salvarani and Vazquez [22] obtained the diffusion limit using the Div-Curl lemma for the case with $|\alpha| \le 1$.

The generalized Carleman model successfully provided nonlinear diffusion equations. However, it is about homogeneous diffusion and does not provide any clue for a heterogeneous diffusion when the diffusivity $D = D(\mathbf{x})$ is not constant. It seems that there is no discrete kinetic equation with spatial heterogeneity in the literature. Three diffusion laws,

$$u_t = \nabla \cdot \big(D\nabla u\big), \tag{1.3}$$

$$u_t = \nabla \cdot \big(\sqrt{D}(\nabla \sqrt{D}u)\big), \tag{1.4}$$

$$u_t = \Delta(Du), \tag{1.5}$$

are often taken when the diffusivity is heterogeneous. The three laws are called Fick [5], Wereide [26], and Chapman [3], respectively. See Milligen *et al.* [14] for a comparison of the diffusion laws with experimental data, where none of them gives a satisfying result. The three diffusion laws are based on a hypothesis that the diffusion phenomenon is decided by the diffusivity only even in a heterogeneous environment. However, the diffusion law (1.1) claims that diffusivity D alone is not enough and an extra information such as the turning frequency μ is needed.

The heterogeneous discrete velocity kinetic equations introduced in this paper are

$$u_t^{k+} + \frac{1}{\epsilon}(v(\mathbf{x})u^{k+})_{x_k} = \frac{\mu(\mathbf{x})}{2n\epsilon^2}\sum_{j=1\pm}^{n\pm}(u^j - u^{k+}), \tag{1.6}$$

$$u_t^{k-} - \frac{1}{\epsilon}(v(\mathbf{x})u^{k-})_{x_k} = \frac{\mu(\mathbf{x})}{2n\epsilon^2}\sum_{j=1\pm}^{n\pm}(u^j - u^{k-}), \tag{1.7}$$

where $(\mathbf{x},t) \in \Omega_\infty$ and $k = 1,\cdots,n$. We denote $\mathbf{x} = (x_1,\cdots,x_n) \in \mathbb{R}^n$, $\Omega := [-1,1]^n \subset \mathbb{R}^n$, $\Omega_T := \Omega \times (0,T)$, and $\Omega_\infty := \Omega \times (0,\infty)$. The solution consists of $2n$ functions, u^i for $i = 1\pm,\cdots,n\pm$, which are the population densities of individuals that move with one of $2n$ velocities, $\pm v\mathbf{e}_k$, where $\mathbf{e}_k$ denotes the unit vector of the rectangular coordinate system. We consider the problem with initial value,

$$u^i(\mathbf{x},0) = u_0^i(\mathbf{x}), \quad i = 1+, 1-, ,\cdots, n+, n-, \tag{1.8}$$

and the periodic boundary condition,

$$u^{k\pm}(\mathbf{x},t) = u^{k\pm}(\mathbf{y},t) \quad \text{if} \quad \bmod\,(\mathbf{x}-\mathbf{y},2) = 0. \tag{1.9}$$

The periodic boundary condition (1.9) implies that the space domain is actually an n dimensional torus which has no boundary. This means that the boundary condition will be forgotten for simplicity in this paper.

Find that the spatial heterogeneity is introduced to the coefficients $v = v(\mathbf{x})$ and $\mu = \mu(\mathbf{x})$, which are now scalar valued periodic functions. Since the diffusivity of the discrete kinetic equation is given by

$$D = \frac{v^2}{n\mu} \tag{1.10}$$

and n is the space dimension, v and μ are only possible places to include the spatial heterogeneity. Note that, if v is not constant, it should be placed inside the derivative as in the equation to obtain the conservation law correctly. We denote the population density of the whole species by

$$u(\mathbf{x},t) = \sum_{j=1\pm}^{n\pm} u^j(\mathbf{x},t). \tag{1.11}$$

The parameter $\epsilon > 0$ appears after a change of time and space variables in a parabolic scaling. The solution of the system depends on the small parameter and we will denote the solution by $u = u^\epsilon$ when the dependency on ϵ is needed explicitly.

The diffusion law (1.1) is an isotropic version of the general anisotropic diffusion law which has been formally obtained by Kim and Seo [11, Eq. (1.6)] from a modified kinetic equation of Stratonovich type. The system of $2n$ equations in (1.6)–(1.7) is an example of such a system. There is no directional heterogeneity in the system and the system ends up with the desired isotropic diffusion law. It is believed that a large class of discrete kinetic equations of Stratonovich type will ends up with the same diffusion law (1.1) and the proof of this paper depend on the specific formation of the system (1.6)–(1.7). The purpose of the paper is to prove the following theorem.

Theorem 1.1 (Singular limit of a Stratonovich type discrete kinetic equations)**.** *Let* $u_0^i \in L^4(\Omega)$ *for* $i = 1\pm, \cdots, n\pm$, $v(\mathbf{x})$ *and* $\mu(\mathbf{x})$ *be bounded and bounded away from zero, and* ∇v *be bounded (see* (3.1)–(3.3)*). Let* $u^{i,\epsilon}$ *be the solution of* (1.6)–(1.9) *and* u^ϵ *the corresponding total population. Then,* u^ϵ *converges in* $L^2(\Omega_T)$ *and its limit* u *is the weak solution of* (1.1)*.*

The key step of the proof is in the construction of the energy functional given in Definition 4.1 and applicability of the Div-Curl lemma.

2. Revertible velocity jump process of Stratonovich types

In this section we introduce the concept of revertibility and a continuum velocity kinetic equation, which are the background of the discrete kinetic equations in (1.6)–(1.7). In a homogeneous case, a kinetic equation with a constant turning frequency,

$$p_t + \frac{1}{\epsilon}\mathbf{v}\cdot\nabla p = \frac{\mu}{\epsilon^2}\int_V \Big(q(\mathbf{v})p(\mathbf{v}',\mathbf{x},t) - q(\mathbf{v}')p(\mathbf{v},\mathbf{x},t)\Big)d\mathbf{v}', \tag{2.1}$$

is often taken as a velocity jump process and its singular limit is investigated to obtain a diffusion model (see Hillen and Othmer [8]). In the equation, $p = p(\mathbf{v},\mathbf{x},t)$ is the density (or probability) of particles with velocity $\mathbf{v} \in V$ at $(\mathbf{x},t) \in Q_\infty$, $V \subset \mathbb{R}^n$ is the set of all possible velocities that a particle may take, and $q(\mathbf{v})$ is the probability for a particle to take the velocity $\mathbf{v}$ after a collision. The total population density in (1.11) is now given by

$$u(\mathbf{x},t) := \int_V p(\mathbf{v},\mathbf{x},t)d\mathbf{v}.$$

The discrete kinetic equation for this homogeneous case is obtained by taking

$$V = \{\pm\mathbf{e}_k : k = 1,\cdots,n\},\ q(\pm\mathbf{e}_k) = \frac{1}{2n},\ \text{and } \mu = \text{constant}.$$

Then, one obtains the same equations as (1.6)–(1.7) after replacing the non-constant speed $v(\mathbf{x})$ by a constant one $v = 1$. Find that there is no directional dependency in V nor q. The convergence of the discrete kinetic equation to the diffusion equation,

$$u_t = D\Delta u, \qquad D = \frac{1}{n\mu},$$

has been obtained.

The spatial heterogeneity can be included in μ, $q(\mathbf{v})$, and V. Then, the kinetic equation becomes

$$p_t + \frac{1}{\epsilon}\mathbf{v}\cdot\nabla p = \frac{\mu(\mathbf{x})}{\epsilon^2}\int_V \Big(q(\mathbf{v},\mathbf{x})p(\mathbf{v}',\mathbf{x},t) - q(\mathbf{v}',\mathbf{x})p(\mathbf{v},\mathbf{x},t)\Big)d\mathbf{v}'. \tag{2.2}$$

It has been formally derived that the diffusion equation obtained by the heterogeneous kinetic equation is

$$u_t = \nabla \cdot \frac{1}{\mu} \nabla(\mu \mathbb{D} u), \quad \mathbb{D} := \frac{1}{\mu} \int_V (\mathbf{v} \otimes \mathbf{v}) q(\mathbf{v}, \mathbf{x}) d\mathbf{v}.$$

See Hillen and Painter [9] for a derivation with constant μ and Kim and Seo [11] with nonconstant μ. However, a rigorous convergence proof for the heterogeneous case has not been obtained. The difficulty is in the construction of a heterogeneous discrete kinetic equation with which a rigorous mathematical convergence proof is possible.

The main trouble of the heterogeneous kinetic equation is that it is not revertible. Let X_ℓ be the position after ℓ number of walks (or collisions). The expectation is $E(X_\ell) = X_0$ if the random walk system is spatially homogeneous, and $E(X_\ell) \neq X_0$ otherwise. We call a random walk system revertible if $E(X_2) = X_0$ whenever the second walk is in the opposite direction of the first one. The velocity jump process given by the heterogeneous kinetic equation (2.2) is not revertible. To make the kinetic equation revertible, Kim and Seo [11] introduced an idea taking a vector field after a collision instead of taking a velocity. The corresponding kinetic equation is written by

$$p_t + \frac{1}{\epsilon} \nabla \cdot (\mathbf{v}_\alpha p) = \frac{\mu(\mathbf{x})}{\epsilon^2} \int_A \Big(q(\alpha, \mathbf{x}) p(\alpha', \mathbf{x}, t) - q(\alpha', \mathbf{x}) p(\alpha, \mathbf{x}, t) \Big) d\alpha', \tag{2.3}$$

where A is the index set of velocity vector fields, i.e., $V = \{\mathbf{v}_\alpha; \alpha \in A\}$, and $q(\alpha, \mathbf{x})$ is the probability to take the velocity vector field $\mathbf{v}_\alpha$ after a collision. The correspnding diffusion equation is

$$u_t = \nabla \cdot \frac{1}{\mu} \Big(\nabla \cdot (\mu \mathbb{D} u) \Big) - \nabla \cdot \Big(\frac{1}{\mu} \mathbb{N} u \Big), \quad \mathbb{N} = \int_A (D \mathbf{v}_\alpha) \mathbf{v}_\alpha q(\alpha, \mathbf{x}) d\alpha,$$

where the extra correction term $\mathbb{N}$ appears since $\mathbf{v}_\alpha \cdot \nabla p \neq \nabla \cdot (\mathbf{v}_\alpha p)$. In the isotropic diffusion case, this diffusion equation turns into (1.1).

Discrete kinetic equations corresponding to the revertible velocity jump process is naturally constructed, which is another key difference in comparison with a non-revertible one. For example, we may take

$$V = \{\mathbf{v}_{k\pm}(\mathbf{x}) := \pm v(\mathbf{x}) \mathbf{e}_k : k = 1, \cdots, n\}, \; q(\mathbf{v}_{k\pm}, \mathbf{x}) = \frac{1}{2n}, \text{ and } \mu = \mu(\mathbf{x}),$$

where there are $2n$ discrete vector fields $\mathbf{v}_i$, $i = 1\pm, \cdots, n\pm$, with an identical speed $v(x) > 0$. If u^i is the particle density that moves along the vector field $\mathbf{v}_i$, the revertible kinetic equation (2.3) is written by

$$u^i_t + \frac{1}{\epsilon} \nabla \cdot (u^i \mathbf{v}_i) = \frac{\mu(\mathbf{x})}{2n\epsilon^2} \sum_{j=1\pm}^{n\pm} (u^j - u^i), \quad i = 1\pm, \cdots, n\pm.$$

These are the discrete kinetic equations of the paper in (1.6)–(1.7).

In the kinetic theory, the velocity of a particle is not changed between two consecutive collisions. However, in the modified kinetic equations (2.3), a particle of the revertible velocity jump process moves according to a vector field $\mathbf{v}_\alpha$. In other words, a particle moves along an integral curve of v_α and changes its speed and direction according to it until the next collision. This variation converges to zero after taking a singular limit as $\epsilon \to 0$. On the other hand, the velocity jump process given by the classical kinetic equation (2.1) is not revertible and this property is not trivialized after taking a singular limit. The strange behavior observed in [11, Section 3] is due to this behavior.

The spatial heterogeneity in $q(\mathbf{v}, \mathbf{x})$ or $q(\alpha, \mathbf{x})$ is activated at the moment of collision. A velocity jump process based on the kinetic equation (2.1) can be called an Ito type since the spatial heterogeneity is involved at the moment of collision only. The corresponding diffusion equation is Chapman's law (1.5) which is satisfied by the probability density function of a stochastic process if the Ito integral is used. If a velocity jump process follows the modified kinetic equation (2.3), the other spatial heterogeneity in $\mathbf{v}_\alpha$ is involved continuously along the path of a particle. This behavior gives a similar behavior of Stratonovich integral and makes the process revertible. Indeed, the corresponding diffusion equation is written by (1.1) which is identical to Wereide's law (1.4) if μ is constant and is satisfied by the probability density function if the Stratonovich integral is used.

3. Notation and existence

We introduce notations for the modified kinetic equations used in the paper. The special feature of the model is in the spatial heterogeneity in the particle speed $v = v(\mathbf{x})$ and the turning frequency $\mu = \mu(\mathbf{x})$. For the consistency of the problem, the two coefficients are assumed to satisfy the periodic boundary condition,

$$v(\mathbf{x}) = v(\mathbf{y}),\ \mu(\mathbf{x}) = \mu(\mathbf{y}) \quad \text{if} \quad \bmod\,(\mathbf{x} - \mathbf{y}, 2) = 0. \tag{3.1}$$

We also assume that $v(\mathbf{x})$ and $\mu(\mathbf{x})$ are bounded and bounded away from zero. For notational convenience, we assume that there exists $M > 0$ such that

$$M^{-1} < v(\mathbf{x}) < M, \quad M^{-1} < \mu(\mathbf{x}) < M. \tag{3.2}$$

In addition, we assume that $\frac{\partial v(\mathbf{x})}{\partial x_k}$ is bounded,

$$\Big|\frac{\partial v(\mathbf{x})}{\partial x_k}\Big| < M, \quad k = 1, \cdots, n. \tag{3.3}$$

We are interested in the singular limit of solutions of (1.6)–(1.8) as $\epsilon \to 0$. In these limiting process, the spatial heterogeneity in v and μ are treated as macroscopic-scale distributions. We take following notations;

$$\begin{aligned}
\mathbf{u} &= (u^{1+}, u^{1-}, \cdots, u^{n+}, u^{n-}) \in \mathbb{R}^{2n},\\
J_{i,j} &= \frac{v(\mathbf{x})}{\epsilon}(u^i - u^j), \quad i,j = 1\pm, \cdots, n\pm,\\
u^k &= u^{k+} + u^{k-}, \qquad k = 1, \cdots, n,\\
J_k &= J_{k+,k-}, \qquad k = 1, \cdots, n,\\
\mathbf{J} &= (J_1, \cdots, J_n) \in \mathbb{R}^n.
\end{aligned}$$

Solutions of (1.6)–(1.8) depend on the parameter ϵ. If needed, we explicitly denote the dependency on ϵ, i.e.,

$$u^\epsilon,\ u^{j,\epsilon},\ J^\epsilon_{i,j},\ J^\epsilon_k, \text{ and } \mathbf{J}^\epsilon.$$

However, for simplicity, we denote them without ϵ when the parameter ϵ is fixed. If needed, we also denote the dependency of the flux in u, i.e.,

$$\mathbf{J}^{u,\epsilon},\ J^{u,\epsilon}_{i,j}, \text{ and } J^{u,\epsilon}_k.$$

We seperate the use of indexes by denoting $i, j \in \{1+, 1-, \cdots, n+, n-\}$ and $k \in \{1, \cdots, n\}$.

Remark 3.1. *If one wants to know the dependency of upper and lower bounds of v and μ, one may take $0 < m_1 < v(\mathbf{x}) < M_1$ and $0 < m_2 < \mu(\mathbf{x}) < M_2$. In this paper we look for simpler expressions using only on parameter $M > 0$.*

The existence and uniqueness of the weak solution of (1.6)–(1.8) come from classical semigroup theory (see [4, Section 3], [7, Section 3], [17, Sections 1, 3, and 4]). For example, we may write (1.6)–(1.7) in an operator form,

$$\frac{\partial}{\partial t}U = GU + BU,$$

where

$$U := \begin{pmatrix} u^{1+}\\ u^{1-}\\ \vdots\\ u^{n+}\\ u^{n-}\end{pmatrix}, \qquad GU := \frac{v(\mathbf{x})}{\epsilon}\begin{pmatrix} -\partial_{x_1}u^{1+}\\ \partial_{x_1}u^{1-}\\ \vdots\\ -\partial_{x_n}u^{n+}\\ \partial_{x_n}u^{n-}\end{pmatrix},$$

$$\text{and} \quad BU := \frac{1}{\epsilon}\begin{pmatrix} -u^{1+}\partial_{x_1}v(\mathbf{x})\\ u^{1-}\partial_{x_1}v(\mathbf{x})\\ \vdots\\ -u^{n+}\partial_{x_n}v(\mathbf{x})\\ u^{n-}\partial_{x_n}v(\mathbf{x})\end{pmatrix} + \frac{\mu(\mathbf{x})}{2n\epsilon^2}\begin{pmatrix} \sum_{j=1\pm}^{n\pm}(u^j - u^{1+})\\ \sum_{j=1\pm}^{n\pm}(u^j - u^{1-})\\ \vdots\\ \sum_{j=1\pm}^{n\pm}(u^j - u^{n+})\\ \sum_{j=1\pm}^{n\pm}(u^j - u^{n-})\end{pmatrix}.$$

Since $v(\mathbf{x})$ is bounded and the above differential operator G is a contraction (see [7, Section 3]), we can verify that $G : D(G) \to [L^p(\Omega_T)]^{2n}$ is a continuous semigroup on U and we are interested for the case with $p \geq 2$. The other operator B is bounded and linear which is considered as a perturbation [17, Section 3]. The domain of the linear operator G is

$$D(G) = \left\{ (u^{1+}, \cdots, u^{n-}) \in [L^p(\Omega)]^{2n} \mid \partial_k u^{k\pm} \in L^p(\Omega) \text{ and periodic} \right\}.$$

Now, we may apply Theorem 1.3 in [17, Section 4] and obtain a unique solution $\mathbf{u}^\epsilon(\mathbf{x}, t) \in C([0,T], [L^p(\Omega)]^{2n})$ of (1.6)–(1.7) for an initial value $\mathbf{u}_0^\epsilon \in [L^p(\Omega)]^{2n}$.

4. Energy functional and its monotonicity

In this section, we obtain uniform L^2-estimates of u^ϵ and $\mathbf{J}^{u,\epsilon}$ which depend only on the initial value. First, we introduce an energy functional.

Definition 4.1 (Energy functional)**.** *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function with $\psi(0) = 0$. For a given nonconstant speed $v : \Omega \to \mathbb{R}_+$, the energy of a population distribution $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ at time $t > 0$ is defined by*

$$\mathcal{E}_v(u)(t) := \int_\Omega \Phi_v(u(\mathbf{x}, t), \mathbf{x}) d\mathbf{x} \quad \text{with} \quad \Phi_v(u, \mathbf{x}) = \int_0^u \psi(v(\mathbf{x})\tau) d\tau.$$

For a given fractional population distribution $\mathbf{u} = (u^{1+}, \cdots, u^{n-}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+^{2n}$, the total energy at time $t > 0$ is defined by

$$\mathcal{E}_v(\mathbf{u})(t) = \sum_{j=1\pm}^{n\pm} \mathcal{E}_v(u^j)(t).$$

Find that Φ_v is not an antiderivative of ψ. Let Ψ be the antiderivative of ψ with $\Psi(0) = 0$. Then, Φ_v is given by

$$\Phi_v(u, \mathbf{x}) = \frac{1}{v(\mathbf{x})} \Psi(v(\mathbf{x})u) \quad \text{or} \quad \Psi(v(\mathbf{x})u) = v(\mathbf{x}) \Phi_v(u, \mathbf{x}).$$

We take $\psi(s) = s^\kappa$ with $\kappa \geq 1$ in this paper, which is a convex function.

Theorem 4.2. *Let v and μ satisfy (3.1)–(3.3) and u^i, $i = 1\pm, \cdots, n\pm$, be the solutions of (1.6)–(1.8). Then,*

(1) *The total energy $\mathcal{E}_v(\mathbf{u})(t)$ is decreasing in time and hence*

$$\mathcal{E}_v(\mathbf{u})(t) \leq \mathcal{E}_v(\mathbf{u}_0(\mathbf{x})), \quad t > 0.$$

(2) *For all $T > 0$,*

$$\sum_{i,j=1\pm}^{n\pm} \|J_{i,j}^{u,\epsilon}\|_{L^2(\Omega_T)}^2 \leq 2nM^3 \sum_{j=1\pm}^{n\pm} \|u_0^j\|_{L^2(\Omega)}^2.$$

Proof. Multiply $\psi(v(\mathbf{x})u^{k\pm})$ to (1.6) and (1.7) and obtain

(4.1)
$$\psi(v(\mathbf{x})u^{k\pm}) u_t^{k\pm} \pm \frac{\psi(v(\mathbf{x})u^{k\pm})}{\epsilon} (v(\mathbf{x})u^{k\pm})_{x_k} = \frac{\psi(v(\mathbf{x})u^{k\pm})\mu(\mathbf{x})}{2n\epsilon^2} \sum_{j=1\pm}^{n\pm} (u^j - u^{k\pm}).$$

Integrate (4.1) over Ω and add them for $i = 1\pm, \cdots, n\pm$. Then, the sum of time derivative terms becomes

$$\sum_{i=1\pm}^{n\pm} \int_\Omega \psi(v(\mathbf{x})u^i)u^i_t d\mathbf{x} = \sum_{i=1\pm}^{n\pm} \int_\Omega \Big(\frac{d}{dt}\int_0^{u^i} \psi(v(\mathbf{x})\tau)d\tau\Big)d\mathbf{x} = \frac{d}{dt}\mathcal{E}_v(\mathbf{u})$$

The periodic boundary condition gives that

$$\int_\Omega \psi(v(\mathbf{x})u^{k\pm})(v(\mathbf{x})u^{k\pm})_{x_k} d\mathbf{x} = \int_\Omega \frac{\partial}{\partial x_k}\Psi(v(\mathbf{x})u^{k\pm})d\mathbf{x} = 0.$$

Therefore, the sum of the second flux terms is zero. Now we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_v(\mathbf{u}) &\le \frac{1}{2n\epsilon^2}\int_\Omega \mu(\mathbf{x})\sum_{i,j=1\pm}^{n\pm} \psi(v(\mathbf{x})u^i)(u^j - u^i)d\mathbf{x} \\
&= -\frac{1}{4n\epsilon^2}\int_\Omega \mu(\mathbf{x})\sum_{i,j=1\pm}^{n\pm} (\psi(v(\mathbf{x})u^i) - \psi(v(\mathbf{x})u^j))(u^i - u^j)d\mathbf{x} \\
&= -\frac{1}{4n\epsilon^2}\int_\Omega \frac{\mu(\mathbf{x})}{v(\mathbf{x})}\sum_{i,j=1\pm}^{n\pm} (\psi(v(\mathbf{x})u^i) - \psi(v(\mathbf{x})u^j))(v(\mathbf{x})u^i - v(\mathbf{x})u^j)d\mathbf{x} \\
&\le -\frac{1}{4n\epsilon^2 M^2}\int_\Omega \sum_{i,j=1\pm}^{n\pm} (\psi(v(\mathbf{x})u^i) - \psi(v(\mathbf{x})u^j))(v(\mathbf{x})u^i - v(\mathbf{x})u^j)d\mathbf{x} \le 0,
\end{aligned}$$

where the last inequality is from the estimate $\frac{\mu(\mathbf{x})}{v(\mathbf{x})} \ge M^{-2}$ by (3.2) and the convexity of ψ with minimum $\psi(0) = 0$. Therefore,

$$\frac{d}{dt}\mathcal{E}_v(\mathbf{u}) \le 0,$$

and the first assertion of the theorem is completed.

To show the second assertion we take $\psi(s) = s$. Then, the above inequality is written by

$$\frac{\partial}{\partial t}\mathcal{E}_v(\mathbf{u}) \quad \le \quad -\frac{1}{4nM^2}\int_\Omega \sum_{i,j=1\pm}^{n\pm} |J^{u,\epsilon}_{i,j}|^2 d\mathbf{x}. \tag{4.2}$$

The integration of (4.2) over $(0, T)$ gives

$$\frac{1}{4nM^2}\int_0^T \int_\Omega \sum_{i,j=1\pm}^{n\pm} |J^{u,\epsilon}_{i,j}|^2 d\mathbf{x}dt \le \mathcal{E}_v(\mathbf{u}_0) - \mathcal{E}_v(\mathbf{u}(\mathbf{x}, T)) \le \mathcal{E}_v(\mathbf{u}_0).$$

Therefore, the L^2-norm of $J^{u,\epsilon}_{i,j}$ is uniformly bounded with respect to ϵ and $T > 0$ by

$$\sum_{i,j=1\pm}^{n\pm} \|J^{u,\epsilon}_{i,j}\|^2_{L^2(\Omega_T)} \le 4nM^2\mathcal{E}_v(\mathbf{u}_0).$$

For $\psi(s)=s$,

$$\mathcal{E}_v(\mathbf{u}_0)=\int_\Omega \frac{v(\mathbf{x})}{2}\sum_{j=1\pm}^{n\pm}(u_0^j)^2 d\mathbf{x}\le \frac{M}{2}\sum_{j=1\pm}^{n\pm}\|u_0^j\|_{L^2(\Omega)}^2.$$

Therefore, $2nM^3\sum_{j=1\pm}^{n\pm}\|u_0^j\|_{L^2(\Omega)}^2$ is an upper bound. □

Find that we don't need the smoothness of the solution in Theorem 4.2. A weak solution in $C([0,T],[L^p(\Omega)]^{2n})$ is enough since we need only integral calculations. We discuss a weak solution case in Corollary 4.4. The case of our main interest is with $\psi(s)=s^{p-1}$ for $p=2$ or 4.

Corollary 4.3. *Let $p\ge 2$ and $\mathbf{u}_0\in[L^p(\Omega)]^{2n}$. Then, $\mathbf{u}\in C([0,T];[L^p(\Omega)]^{2n})$ and there exists a constant C such that, for all $t>0$,*

$$\|\mathbf{u}(t)\|_{[L^p(\Omega)]^{2n}}\le C\|\mathbf{u}_0\|_{[L^p(\Omega)]^{2n}}. \tag{4.3}$$

Proof. Let $\psi(s)=s^{p-1}$. Then, ψ is convex for $p\ge 2$ and

$$\Phi_v(u,\mathbf{x},t)=\int_0^u \psi(v(\mathbf{x})\tau)d\tau=\int_0^u v(\mathbf{x})^{p-1}\tau^{p-1}d\tau=v(\mathbf{x})^{p-1}\frac{u^p}{p}.$$

Since the speed $v(\mathbf{x})$ is bounded by (3.2), we obtain

$$\frac{1}{M^{p-1}}\frac{u^p}{p}\le \Phi_v(u,\mathbf{x})\le M^{p-1}\frac{u^p}{p},\quad u\ge 0.$$

Replace u with the solution u^j and integrate it over Ω. Then, for $t\ge 0$,

$$\frac{1}{pM^{p-1}}\|u^j(t)\|_{L^p(\Omega)}^p\le \mathcal{E}_v(u^j)(t)\le \frac{M^{p-1}}{p}\|u^j(t)\|_{L^p(\Omega)}^p.$$

The monotonicity of the total energy (Theorem 4.2) implies that

$$\begin{aligned}\|\mathbf{u}(t)\|_{[L^p(\Omega)]^{2n}}^p &= \sum_{j=1\pm}^{n\pm}\|u^j(t)\|_{L^p(\Omega)}^p\\ &\le pM^{p-1}\mathcal{E}_v(\mathbf{u}(\mathbf{x},t))\\ &\le pM^{p-1}\mathcal{E}_v(\mathbf{u}_0(\mathbf{x}))\\ &\le M^{2(p-1)}\sum_{j=1\pm}^{n\pm}\|u_0^j\|_{L^p(\Omega)}^p=M^{2(p-1)}\|\mathbf{u}_0\|_{[L^p(\Omega)]^{2n}}^p.\end{aligned}$$

Therefore, (4.3) holds with $C=M^{\frac{2(p-1)}{p}}$. □

Corollary 4.4. *Let $\mathbf{u}$ be the solution of* (1.6)–(1.8) *with an initial value $\mathbf{u}_0\in[L^2(\Omega)]^{2n}$. Then, Theorem 4.2(1) and Corollary 4.3 still hold.*

Proof. Let $\mathbf{u}=\{u^j\}_{j=1\pm}^{n\pm}$ be the weak solution with an initial value $\mathbf{u}_0=\left\{u_0^j\right\}_{j=1\pm}^{n\pm}\in[L^2(\Omega)]^{2n}$. Let $\mathbf{u}_0^\delta=\left\{u_0^{j,\delta}\right\}_{j=1\pm}^{n\pm}$ be a sequence of smooth functions which converge to $\left\{u_0^j\right\}_{j=1\pm}^{n\pm}$ as $\delta\to 0$ and $\mathbf{u}^\delta$ be a smooth solution with these smooth initial values. Since $\mathbf{u}_0 \to \mathbf{u}$ is a Lipschitz continuous mapping from $[L^p(\Omega)]^{2n}$ to $C([0,T];[L^p(\Omega)]^{2n})$ and the problem is linear, we have

$$\|\mathbf{u}^{\delta_1} - \mathbf{u}^{\delta_2}\|_{C([0,T],[L^p(\Omega)]^{2n})} \le C\|\mathbf{u}_0^{\delta_1} - \mathbf{u}_0^{\delta_2}\|_{[L^p(\Omega)]^{2n}}$$

for a constant $C > 0$. Therefore, the sequence of smooth solutions $\{\mathbf{u}^\delta\}$ converge to the weak solution $\mathbf{u}$. Since a smooth solution $\mathbf{u}^\delta$ satisfies Theorem 4.2 and Corollary 4.3, we can deduce that $\mathbf{u}$ satisfies Theorem 4.2(1) and Corollary 4.3 by the continuity of the norms. □

5. Convergence and Div-Curl lemma

The main theoretical part of the paper is in obtaining the singular limit as $\epsilon \to 0$. First, by adding the $2n$ equations in (1.6)–(1.7), we obtain a conservation law for the total population,

$$u_t + \nabla \cdot \mathbf{J} = 0. \tag{5.1}$$

By subtracting (1.7) from (1.6), we obtain n equations for each components of the flux,

$$\frac{\epsilon^2}{v(\mathbf{x})}\frac{\partial J_k}{\partial t} + (v(\mathbf{x})u^k)_{x_k} = -\frac{\mu(\mathbf{x})}{v(\mathbf{x})}J_k, \quad k = 1, \cdots, n. \tag{5.2}$$

To show the convergence of the singular limit as $\epsilon \to 0$, we need to show the convergence of the following two sequences,

$$(v(\mathbf{x})u^{k,\epsilon})_{x_k} \to (v(\mathbf{x})\frac{u}{n})_{x_k}, \quad \frac{\epsilon^2}{v}\frac{\partial J_k^\epsilon}{\partial t} \to 0.$$

If they are done, Eq. (5.2) implies

$$J_k^\epsilon \to -\frac{v(\mathbf{x})}{n\mu(\mathbf{x})}(v(\mathbf{x})u)_{x_k}.$$

After substituting them into (5.1), we obtain

$$u_t = \nabla \cdot \Big(\frac{v(\mathbf{x})}{n\mu(\mathbf{x})}\nabla(v(\mathbf{x})u)\Big) \quad \text{for} \quad (\mathbf{x}, t) \in \Omega_\infty. \tag{5.3}$$

If we denote the diffusivity D by (1.10), we may rewrite (5.3) in the form of our diffusion law (1.1) and completes the proof of Theorem 1.1. In this section, we show that the solutions of (1.6)–(1.8) converge to a weak solution of (5.3) as $\epsilon \to 0$.

Lemma 5.1. *Let $u^{j,\epsilon}$, $j = 1\pm, \cdots, n\pm$, be weak solutions of (1.6)–(1.8) with initial values $u_0^j \in L^2(\Omega)$. Then, for any given $T > 0$, there is a weakly convergent subsequence u^{j,ϵ_ℓ} such that, as $\epsilon_\ell \to 0$,*

$$u^{j,\epsilon_\ell} \rightharpoonup u^j \quad \textit{weakly in } L^2(\Omega_T),$$

$$J_{i,j}^{u,\epsilon_\ell} \rightharpoonup J_{i,j}^u \quad \textit{weakly in } L^2(\Omega_T).$$

Proof. We have already shown that $\mathbf{u}^{j,\epsilon}$ and $\mathbf{J}^{u,\epsilon}$ are uniformly bounded in $L^2(\Omega_T)$. Therefore, there exist weakly convergent subsequences $\{u^{j,\epsilon_\ell^j}\}$ and their limits $u^j \in L^2(\Omega_T)$ such that

$$u^{j,\epsilon_\ell^j} \rightharpoonup u^j \text{ weakly in } L^2(\Omega_T).$$

We take ϵ_ℓ^j as a subsequence of $\epsilon_\ell^{j'}$ for $j' < j$. By denoting $\epsilon_\ell := \epsilon_\ell^{n+}$, we have

$$u^{j,\epsilon_\ell} \rightharpoonup u^j \text{ weakly in } L^2(\Omega_T), \quad j = 1\pm, \cdots, n\pm .$$

Since $J_{i,j}^{u,\epsilon_\ell}$ is again a uniformly bounded sequence in $L^2(\Omega_T)$ for $i,j = 1\pm, \cdots, n\pm$, we may repeat the process and obtain a subsequence of ϵ_ℓ, which is denoted by ϵ_ℓ again, such that

$$J_{i,j}^{u,\epsilon_\ell} \rightharpoonup J_{i,j}^u \text{ weakly in } L^2(\Omega_T)$$

for all $i,j = 1\pm, \cdots, n\pm$. □

Next, we show that the obtained subsequence satisfies

$$(v(\mathbf{x})u^{k,\epsilon_\ell})_{x_k} \quad \to \quad (v(\mathbf{x})\frac{u}{n})_{x_k} \quad \text{in} \quad H^{-1}(\Omega_T), \tag{5.4}$$

$$\frac{\epsilon_\ell^2}{v}\frac{\partial J_k^{\epsilon_\ell}}{\partial t} \quad \to \quad 0 \quad \text{in} \quad H^{-1}(\Omega_T). \tag{5.5}$$

This is the part needed to make the formal derivation of the diffusion equation (5.3) rigorous. In the rest of this section we obtain the convergence and complete the convergence of the singular limit to the unique solution of (1.1).

Lemma 5.2. *Let u^{j,ϵ_ℓ} be the subsequence obtained in Lemma 5.1. Then, the convergence in* (5.4) *and* (5.5) *hold as $\ell \to \infty$.*

Proof. For $i,j = 1\pm, ..., n\pm$,

$$\|u^{i,\epsilon_\ell} - u^{j,\epsilon_\ell}\|_{L^2(\Omega_T)} = \|\frac{\epsilon_\ell}{v(\mathbf{x})}J_{i,j}^{\epsilon_\ell}\|_{L^2(\Omega_T)} \le 2\epsilon_\ell M\sqrt{U_0} \to 0 \ \text{ as } \ell \to \infty.$$

Equivalently, $u^{i,\epsilon_\ell} \to u^i = \frac{1}{2n}u$ weakly in $L^2(\Omega_T)$ and hence

$$(v(\mathbf{x})u^{k,\epsilon_\ell})_{x_k} = (v(\mathbf{x})(u^{k+,\epsilon_\ell} + u^{k-,\epsilon_\ell}))_{x_k} \to (v(\mathbf{x})\frac{u}{n})_{x_k} \ \text{ in } H^{-1}(\Omega_T).$$

Let $K := \{\phi \in H_0^1(\Omega_T) : \|\phi\|_{H_0^1(\Omega_T)} \le 1\}$. Then, since $J_{i,j}^{\epsilon_\ell}$ is bounded in $L^2(\Omega_T)$,

$$\begin{aligned} \Big\|\frac{\partial J_k^{\epsilon_\ell}}{\partial t}\Big\|_{H^{-1}(\Omega_T)} &= \sup_{\phi\in K}\langle \partial_t J_{k,-k}^{\epsilon_\ell}, \phi\rangle = \sup_{\phi\in K}(-\int_{\Omega_T} J_{k,-k}^{\epsilon_\ell}\phi_t d\mathbf{x}dt) \\ &\le \|J_{k,-k}^{\epsilon_\ell}\|_{L^2(\Omega_T)} \sup_{\phi\in K}\|\phi_t\|_{L^2(\Omega_T)} \le \|J_{k,-k}^{\epsilon_\ell}\|_{L^2(\Omega_T)} < \infty. \end{aligned}$$

Therefore, we have

$$\Big\|\frac{\epsilon_\ell^2}{v(\mathbf{x})}\frac{\partial J_k^{\epsilon_\ell}}{\partial t}\Big\|_{H^{-1}(\Omega_T)} \le M\epsilon_\ell^2\Big\|\frac{\partial J_k^{\epsilon_\ell}}{\partial t}\Big\|_{H^{-1}(\Omega_T)} \to 0 \quad \text{in } H^{-1}(\Omega_T).$$

The convergence in (5.4) and (5.5) are obtained. □

Now, we are going to prove strong convergence of the solution when the initial values u_0^j are placed in $L^4(\Omega)$. The key ingredient of the proof is the Div-Curl lemma.

Lemma 5.3 (Div-Curl Lemma)**.** *Suppose that $A \subset \mathbb{R}^{n+1}$ is open and $\mathbf{w}^\ell, \mathbf{z}^\ell : A \to \mathbb{R}^{n+1}$ are given for $\ell = 1, 2, \cdots$. Suppose further that*

$$\mathbf{w}^\ell \rightharpoonup \mathbf{w} \text{ weakly in } [L^2(A)]^{n+1}, \tag{5.6}$$

$$\mathbf{z}^\ell \rightharpoonup \mathbf{z} \text{ weakly in } [L^2(A)]^{n+1}, \tag{5.7}$$

$$\bigtriangledown \cdot \mathbf{w}^\ell \text{ is bounded in } L^2(A), \tag{5.8}$$

$$curl(\mathbf{z}^\ell) \text{ is bounded in } [L^2(A)]^{(n+1)^2}, \tag{5.9}$$

Then,

$$\langle \mathbf{w}^\ell, \mathbf{z}^\ell \rangle \longrightarrow \langle \mathbf{w}, \mathbf{z} \rangle \quad \text{in the distribution sense,}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$.

To apply the Div-Curl Lemma, we first arrange solutions and fluxes in the form of the lemma.

Lemma 5.4. *Let $u_0 \in L^2(\Omega)$, $\mathbf{w}^\ell = (\mathbf{J}^{\epsilon_\ell}, u^{\epsilon_\ell})$, and $\mathbf{z}^\ell = (\mathbf{0}, v(\mathbf{x})u^{\epsilon_\ell})$. Then, both sequences $\mathbf{w}^\ell$ and $\mathbf{z}^\ell$ are in $[L^2_{x,t}(\Omega_T)]^{n+1}$ and $\langle \mathbf{w}^\ell, \mathbf{z}^\ell \rangle \longrightarrow \langle \mathbf{w}, \mathbf{z} \rangle$ in the distribution sense.*

Proof. It is enough to show that $\mathbf{w}^\ell$ and $\mathbf{z}^\ell$ satisfy the four conditions in (5.6)–(5.9). By Lemma 5.1, $\mathbf{w}^\ell \rightharpoonup \mathbf{w}$ and $\mathbf{z}^\ell \rightharpoonup \mathbf{z}$ weakly in $[L^2(\Omega_T)]^{n+1}$ and hence the first two conditions (5.6) and (5.7) are satisfied. Eq. (5.1) implies (5.8), i.e.,

$$\bigtriangledown_{x,t} \cdot \mathbf{w}^\ell = u_t^{\epsilon_\ell} + \bigtriangledown \cdot \mathbf{J}^{\epsilon_\ell} = 0.$$

It is left to show the compactness condition (5.9) on $\text{curl}(\mathbf{z}^\ell)$ in $[H^{-1}(\Omega_T)]^{(n+1)^2}$. Since $u^{\epsilon_\ell} = \sum_{k=1}^n u^{k,\epsilon_\ell}$,

$$u^{\epsilon_\ell} = n u^{k,\epsilon_\ell} + \sum_{k'=1}^n (u^{k',\epsilon_\ell} - u^{k,\epsilon_\ell}),$$

and

$$\partial_{x_k}(v(\mathbf{x})u^{\epsilon_\ell}) = n\partial_{x_k}(v(\mathbf{x})u^{k,\epsilon_\ell}) + \sum_{k'=1}^n \partial_{x_k}[v(\mathbf{x})(u^{k',\epsilon_\ell} - u^{k,\epsilon_\ell})]. \tag{5.10}$$

By (5.2), we have

$$\partial_{x_k}(v(\mathbf{x})u^{k,\epsilon_\ell}) = -\frac{\mu(\mathbf{x})}{v(\mathbf{x})} J_k^{\epsilon_\ell} - \frac{{\epsilon_\ell}^2}{v(\mathbf{x})} \frac{\partial J_k^{\epsilon_\ell}}{\partial t}.$$

The compactness of $\frac{{\epsilon_\ell}^2}{v(\mathbf{x})} \frac{\partial J_k^{\epsilon_\ell}}{\partial t}$ has been obtained in Lemma 5.2. In addition, $J_k^{\epsilon_\ell} = \frac{1}{\epsilon_\ell} v(\mathbf{x})(u^{k+,\epsilon_\ell} - u^{k-,\epsilon_\ell})$ is bounded in $L^2(\Omega_T)$ by Theorem 4.2 and is a compact operator in $H^{-1}(\Omega_T)$. Thus, we can conclude that $(v(\mathbf{x})u^{k,\epsilon_\ell})_{x_k}$ is compact in $H^{-1}(\Omega_T)$. We also have

$$\|v(\mathbf{x})(u^{k',\epsilon_\ell} - u^{k,\epsilon_\ell})\|_{L^2(\Omega_T)} = \|\epsilon_\ell(J^{\epsilon_\ell}_{k'+,k+} - J^{\epsilon_\ell}_{k'-,k-})\|_{L^2} \le 2\epsilon_\ell\sqrt{U_0},$$

which converges to 0 as $\ell \to 0$. Thus, $\partial_{x_k}[v(\mathbf{x})(u^{k',\epsilon_\ell} - u^{k,\epsilon_\ell})]$ are compact in $H^{-1}(\Omega_T)$. By (5.10), $\partial_{x_k}(v(\mathbf{x})u^{\epsilon_\ell})$ is also compact in $H^{-1}(\Omega_T)$ for all k and it completes the proof. □

Lemma 5.5. *If $u_0^i \in L^4(\Omega)$ for all $i = 1\pm, \cdots, n\pm$ (or $\mathbf{u}_0 \in [L^4(\Omega)]^{2n}$), there is a sequence $\epsilon_\ell \to 0$ as $\ell \to \infty$ such that, for all $T > 0$,*

$$u^{\epsilon_\ell} \to u \quad \textit{strongly in } L^2(\Omega_T).$$

Proof. Lemma 5.1 implies that, for all $i = 1\pm, \cdots, n\pm$,

$$u^{i,\epsilon_\ell} \rightharpoonup u^i, \quad u^{\epsilon_\ell} \rightharpoonup u \quad \text{weakly in } L^2(\Omega_T).$$

From Lemma 5.2, we have $u^i = \frac{1}{2n}u$ for some $u \in L^2(\Omega_T)$. We denote $u^{k,\epsilon_\ell} := u^{k+,\epsilon_\ell} + u^{k-,\epsilon_\ell}$. Then, the Div-Curl lemma implies that

$$v(\mathbf{x})(u^{k,\epsilon_\ell})^2 \to v(\mathbf{x})(u^k)^2 = \frac{v(\mathbf{x})}{n^2}u^2$$

in the distribution sense. The uniform boundedness of $(u^{k,\epsilon_\ell})^2$ and the fact that $(\frac{1}{n}u)^2$ is in L^2 comes from Corollary 4.3 with $p = 4$. In addition, $v(\mathbf{x})$ is well-defined and bounded and bounded away from zero. Therefore,

$$(u^{k,\epsilon_\ell})^2 \rightharpoonup \frac{1}{n^2}u^2 \quad \text{weakly in } L^2(\Omega_T).$$

Since $u^{\epsilon_\ell} = \sum_{k=1}^n u^{k,\epsilon_\ell}$, we obtain the weak convergence of $(u^{\epsilon_\ell})^2 \rightharpoonup u^2$. The strong L^2 convergence comes from [22, Lemma 7], which states that $u^{\epsilon_\ell} \to u$ strongly in $L^2(\Omega_T)$ if $|\Omega_T| < \infty$, $u^{\epsilon_\ell} \rightharpoonup u$, and $(u^{\epsilon_\ell})^2 \rightharpoonup u^2$ weakly in $L^2(\Omega_T)$. □

Now we finish the proof of the main theorem.

Proof of Theorem 1.1. Note that $u^{\epsilon_\ell} := \sum_{j=1\pm}^{n\pm} u^{j,\epsilon_\ell} \to u$ strongly in $L^2(\Omega_T)$. In addition, for $i, j = 1\pm, \cdots, n\pm$,

$$\|u^{i,\epsilon_\ell} - u^{j,\epsilon_\ell}\|_{L^2(\Omega_T)} = \|\frac{\epsilon_\ell}{v(\mathbf{x})}J^{\epsilon_\ell}_{i,j}\|_{L^2(\Omega_T)} \le \epsilon_\ell M\sqrt{U_0} \to 0 \quad \text{as } \ell \to \infty,$$

and

$$u^{i,\epsilon_\ell} \to \frac{1}{2n}u \quad \text{strongly in } L^2(\Omega_T).$$

Therefore, the solution of the system (5.1)–(5.2),

$$\begin{aligned} u_t^{\epsilon_\ell} + \nabla\cdot\mathbf{J}^{\epsilon_\ell} &= 0, \\ \frac{\epsilon_\ell^2}{v}\frac{\partial J_k^{\epsilon_\ell}}{\partial t} + (v(\mathbf{x})u^{i,\epsilon_\ell})_{x_k} &= -\frac{\mu(\mathbf{x})}{v(\mathbf{x})}J_k^{\epsilon_\ell}, \end{aligned}$$

converges to a solution of

$$u_t + \bigtriangledown \cdot \mathbf{J} = 0, \tag{5.11}$$

$$\Big(\frac{v(\mathbf{x})}{n} u\Big)_{x_k} = -\frac{\mu(\mathbf{x})}{v(\mathbf{x})} J_k \tag{5.12}$$

in the distribution sense. After the substitution of (5.12) into (5.11), we can see that the limit u is the weak solution of the diffusion equation

$$u_t = \bigtriangledown \cdot (\frac{v(\mathbf{x})}{n\mu(\mathbf{x})} \bigtriangledown (v(\mathbf{x})u)) \tag{5.13}$$

with the periodic boundary condition and the initial condition. It is classical that the weak solution of (5.13) is unique. This implies that the subsequential convergence as $\epsilon_\ell \to 0$ is actually the convergence as $\epsilon \to 0$. Since the diffusivity D is given by (1.10), (5.13) is written as (1.1) and the proof is completed. $\square$

Remark 5.6. *In one space dimension, $n = 1$, there is a simpler way to obtain the diffusion equation* (5.13)*. It is by a change of the space variable. Consider a new space variable given by*

$$y = \int_{x_0}^{x} \frac{1}{v(s)} ds.$$

This change of variable stretches or shrinks the space according to the given speed $v(x)$ and makes the nonconstant speed in x variable a constant one $\tilde{v}(y) = 1$ in y variable. The particle density in a new space variable becomes $w(y,t) = v(x)u(x,t)$. Since $\frac{dy}{dx} = \frac{1}{v(x)}$, we have

$$w_t = vu_t = v\Big(\frac{v(x)}{\mu(x)}(vu)_x\Big)_x = v\frac{dy}{dx}\Big(\frac{v(x)}{\mu(x)}\frac{dy}{dx}w_y\Big)_y = \Big(\frac{1}{\tilde{\mu}(y)}w_y\Big)_y,$$

which is Fick's law (1.3) *with $D(y) = \frac{1}{\tilde{\mu}(y)}$. The new turning frequency $\tilde{\mu}(y)$ is the same frequency μ, but in the new variable y. It is classical that the singular limit of a homogeneous discrete kinetic system*

$$w_t^+ + \frac{1}{\epsilon} w_y^+ = \frac{\tilde{\mu}(y)}{\epsilon^2}(w^- - w^+),$$

$$w_t^- - \frac{1}{\epsilon} w_y^- = \frac{\tilde{\mu}(y)}{\epsilon^2}(w^+ - w^-),$$

converges to Fick's law. Therefore, after changing back the space variable y to the original one x, we obtain the diffusion equation (5.13) *for $n = 1$. However, this technique works for one space dimension only.*

6. Discussion for possible applications

Diffusion plays a key role in various phenomenon and a wrong choice of diffusion law may end up with a wrong conclusion. The new diffusion law (1.1) may improve many diffusion related problems in a heterogeneous environment and we provide an example in this discussion section.

Shigesada *et al.* [24] took Fick's diffusion law and proposed a biological invasion model in a periodic environment with two kinds of patches:

$$u_t = \nabla \cdot (D\nabla u) + (r(x) - u)u, \tag{6.1}$$

where diffusivity D and growth rate r are periodic and piecewise constant,

$$(D(x), r(x)) = \begin{cases} (1, 1), & mL < x \le mL + L_a, \\ (D_b, -r_b), & mL + L_a < x \le mL + L, \end{cases} \tag{6.2}$$

where $L > L_a > 0$ is the period and $-r_b < 0$ is the negative growth rate in undesirable patches. One of the properties of the model is that the invasion is more successful (in terms of parameter regime size) if $D_b < 1$ and less successful if $D_b > 1$. In other words, the biological species should reduce migration rate in bad patches to be successful in invasion. This is physically incorrect conclusion and Fick's law does not fit to the situation.

One may replace Fick's law with the diffusion law (1.1) and obtain

$$u_t = \nabla \cdot \big(\sqrt{\mu^{-1}D}\,\nabla(\sqrt{\mu D}\,u)\big) + (r(x) - u)u. \tag{6.3}$$

If we simply let μ be constant due to lack of information, the diffusion becomes Wereide's law (1.4). If $D_b = 1$, the two models are identical. Seo and Kim [23] proposed (6.3) as a replacement of (6.1) and showed that the invasion model (6.3) has physically correct behavior. For example, the invasion is more successful if the species increases its dispersal rate in bad patches, i.e., if $D_b > 1$ and less successful if the species reduces its dispersal rate in bad patches, i.e., if $D_b < 1$.

References

[1] J. E. Broadwell, Shock structure in a simple discrete velocity gas, Phys. of Fluids 7 (1964) 1243–1247.

[2] T. Carleman, Problemes mathématiques dans la théorie cinétique de gaz. 2. Almqvist & Wiksell (1957).

[3] S. Chapman, On the Brownian displacements and thermal diffusion of grains suspended in a non-uniform fluid, Proc. Roy. Soc. Lond. A 119 (1928) 34–54.

[4] B. Choi and Y.-J. Kim, Diffusion of Biological Organisms: Fickian and Fokker–Planck Type Diffusions, SIAM J. Appl. Math., 79 (2019) 1501–1527.

[5] A. Fick, On liquid diffusion, Phil. Mag. 10 (1855) 30–39.

[6] S. Goldstein, On diffusion by discontinuous movements and on the telegraph equation, The Quarterly Journal of Mechanics and Applied Mathematics 4.2 (1951): 129-156.

[7] T. Hillen, Existence theory for correlated random walks on bounded domains, Can. Appl. Math. Q. 18 (2010) 1–40.

[8] T. Hillen and H. Othmer, The diffusion limit of transport equations derived from velocity-jump processes, SIAM J. Appl. Math. 61 (2000) 751–775.

[9] T. Hillen and K. Painter, Transport and anisotropic diffusion models for movement in oriendted habitats, Dispersal, individual movement and spatial ecology, Springer, Berlin, Heidelberg, (2013) 177–222.

[10] M. Kac, A stochastic model related to the telegrapher's equation, The Rocky Mountain Journal of Mathematics 4 (1974) 497–509.

[11] Y.-J. Kim and H. Seo, Model for heterogeneous diffusion, SIAM Appl. Math. (2020) submitted.

[12] N. Kinezaki, K. Kawasaki, F. Takasu, and N. Shigesada, Modeling biological invasions into periodically fragmented environments, Theoret. Popul. Biol., **64(3)** (2003), 291–302.

[13] P.-L. Lions and T. Giuseppe, Diffusive limit for finite velocity Boltzmann kinetic models, Revista Matematica Iberoamericana 13.3 (1997) 473-513.

[14] B. van Milligen, P. Bons, B. Carreras, and R. Sánchez, On the applicability of Fick's law to diffusion in inhomogeneous systems, European Journal of Physics 26 (2005) 913.

[15] H.G. Othmer, S.R. Dunbar, and W. Alt. Models of dispersal in biological systems, Journal of mathematical biology 26 (1988) 263–298.

[16] H.G. Othmer and T. Hillen, The diffusion limit of transport equations II: Chemotaxis equations, SIAM Journal on Applied Mathematics 62 (2002) 1222–1250.

[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences 44 (1983) Springer-Verlag, Newyork.

[18] T. Platkowski and R. Illner, Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory, SIAM review 30 (1988) 213–255.

[19] A. Pulvirenti and G. Toscani,. Fast diffusion as a limit of a two-velocity kinetic model, Rend. Circ. Mat. Palermo Suppl 45 (1996) 521–528.

[20] F. Salvarani, Diffusion limits for the initial-boundary value problem of the Goldstein-Taylor model, Rend. Sem. Mat. Univ. Politec. Torino 57 (1999) 209–220.

[21] F. Salvarani and T. Giuseppe, The diffusive limit of Carleman-type models in the range of very fast diffusion equations, Journal of Evolution Equations 9 (2009) 67–80.

[22] F. Salvarani and J. L. Vazquez, The diffusive limit for Carleman type kinetic models, Nonlinearity 18 (2005) 1223–1248.

[23] H. Seo and Y.-J. Kim, Biological invasion in a periodic environment, J. Math. Biol., submitted (2020) (http://amath.kaist.ac.kr/papers/Kim/58.pdf)

[24] N. Shigesada, K. Kawasaki, and E. Teramoto, Traveling periodic waves in heterogeneous environment, Theoret. Popul. Biol., **30** (1986), 143–160.

[25] G.I. Taylor, Diffusion by continuous movements, Proceedings of the london mathematical society 2.1 (1922) 196–212.

[26] M. Wereide, La diffusion dúne solution dont la concentration et la temperature sont variables, Ann. Physique 2 (1914) 67–83.