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Asymptotic behavior of solutions to scalar conservation laws and optimal convergence orders to N-waves☆

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Abstract

The goal of this article is to develop a new technique to obtain better asymptotic estimates for scalar conservation laws. General convex flux, $f''(u)\geqslant 0$, is considered with an assumption $\lim_{u\to 0} uf'(u)/f(u)=\gamma>1$. We show that, under suitable conditions on the initial value, its solution converges to an N-wave in L^1 norm with the optimal convergence order of $O(1/t)$. The technique we use in this article is to enclose the solution with two rarefaction waves. We also show a uniform convergence order in the sense of graphs. A numerical example of this phenomenon is included.

Keywords: Asymptotics; Characteristics; Convergence order; N-wave; Similarity

1. Introduction

In this article we obtain the optimal convergence order of sign-changing solutions to the Cauchy problem of a scalar conservation law in one-space dimension

$$u_t+f(u)_x=0,\qquad u(x,0)=u_0(x),\quad x\in \boldsymbol{R},\ \ t>0, \tag{1}$$

where the initial value u_0 is integrable and has a compact support,

$$u_0\in L^1(\boldsymbol{R}),\quad \mathrm{supp}(u_0)\subset[-L,L],\quad L\in\boldsymbol{R}. \tag{2}$$

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We assume that the flux $f(u)$ is convex,

$$f(0) = f'(0) = 0, \quad f''(u) \geqslant 0, \tag{3}$$

and has an algebraic growth of order $\gamma > 1$ near the zero state $u = 0$ in the sense that

$$\lim_{u \to 0} uf'(u)/f(u) = \gamma, \quad \gamma > 1. \tag{4}$$

Note that we may assume $f(0) = f'(0) = 0$ without loss of generality.

It is well known that the nonlinearity in the flux $f(u)$ may generate a singularity and, hence, a smooth solution does not exist globally in time. In this article we consider weak solutions satisfying the entropy condition

$$u(x-, t) \geqslant u(x+, t), \quad x \in \boldsymbol{R}, \ t \geqslant 0. \tag{5}$$

Since the convexity of the flux $f(u)$ is not strict, the flux can be linear in an interval and the Cauchy problem may accept a discontinuity which violates the entropy condition. This kind of discontinuities can be avoided by simply assuming that the initial value does not include any of them, i.e., by assuming $u_0(x-) \geqslant u_0(x+)$, $x \in \boldsymbol{R}$.

Under the convexity hypothesis (3), $f'(u)$ is an increasing function and the *similarity profile* $u = g(x)$ is uniquely defined by the relation

$$g(0) = 0, \qquad f'(g(x)) = x, \quad x \in \boldsymbol{R}. \tag{6}$$

We can easily check that $g(x)$ is also an increasing function and rarefaction waves have this profile, i.e., $u(x,t) = g((x - x_0)/(t + t_0))$ for some constants $x_0 \in \boldsymbol{R}$, $t_0 \geqslant 0$.

It is well known that the asymptotic structure of the solution $u(x,t)$ is a member of two parameters family of N-waves,

$$N_{p,q}(x,t) = \begin{cases} g(x/t), & -a_p(t) < x < b_q(t), \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

where p, q are the invariants of the Cauchy problem (1), i.e.,

$$p = -\inf_x \int_{-\infty}^{x} u_0(y)\, dy, \quad q = \sup_x \int_x^{\infty} u_0(y)\, dy,$$

$$q - p = M = \int u_0(y)\, dy, \tag{8}$$

and $a_p(t), b_q(t) \geqslant 0$ satisfy

$$p = -\int_{-a_p(t)}^{0} g(y/t)\, dy, \quad q = \int_0^{b_q(t)} g(y/t)\, dy. \tag{9}$$

The convergence of the solution $u(x,t)$ to the N-wave has been studied in various contexts. Liu and Pierre [12] show

$$\lim_{t\to\infty} t^{(r-1)/\gamma r}||u(\cdot,t)-N_{p,q}(\cdot,t)||_{L^r}=0 \tag{10}$$

for the general L^1 initial value u_0 under the power law,

$$f(u)=\frac{1}{\gamma}|u|^\gamma,\quad \gamma>1. \tag{11}$$

It is clear that the long-time behavior of the solution mostly depends on the structure of the flux $f(u)$ near the zero state $u=0$ since $u(x,t)\to 0$ as $t\to\infty$. So assumption (4) is a natural generalization of the power law (11). If L^1 norm is considered ($r=1$ in (10)), the estimate gives the convergence to the N-wave, but it does not give any convergence order.

Lax [10] shows the asymptotic structure for the strictly convex case $f''(u)>0$, which is the N-wave, and the technique is employed to show $O(1/\sqrt{t})$ convergence rate in L^1 norm by DiPerna [4] (also see [15, Chapter 16]). A different approach based on generalized characteristics has been used in [3, Chapter XI] obtaining similar results. Basically, the strictly convex flux represents the Burgers equation $\gamma=2$ and these techniques have been extended to genuinely nonlinear hyperbolic systems (see [4,10,11,14]).

It is well known that the $\mu\to 0$ limit of solutions $u^\mu(x,t)$ to a regularized problem,

$$u_t^\mu+f(u^\mu)_x=\mu u_{xx}^\mu,\quad u^\mu(x,0)=u_0(x), \tag{12}$$

is the entropy solution of the inviscid problem. The asymptotic behavior of this type of equation has been studied in [1,5,6] under the power laws. For the viscous Burgers equation, diffusive N-waves are suggested for its asymptotics in [8,9,16]. The convergence order to a diffusive N-wave in L^1 norm is given in [8], which is $O(1/t)$. Since $\lim_{\mu\to 0}\lim_{t\to\infty}u^\mu(x,t)\neq\lim_{t\to\infty}\lim_{\mu\to 0}u^\mu(x,t)$ in general, we cannot expect the same convergence order for the inviscid problem.

The main goal of this article is to develop a new technique to obtain better asymptotic L^1 estimates for general scalar conservation laws with the convexity. For the case without the convexity, we refer to [2,13,17,18]. In our method we monitor the evolution of the solution more closely. The main idea is to enclose the solution $u(x,t)$ with two rarefaction waves $g(x/t)$ and $g(x/(t+\alpha))$, or $g((x+L)/t)$ and $g((x-L)/t)$, where constants α, L are decided from the initial value. In [7] a piecewise self-similar solution has been employed to approximate a general solution. That study shows the effectiveness of estimates based on rarefaction waves. The main result in this article is:

Theorem 1. *Let the flux $f(u)$ be convex* (3) *and have an algebraic growth rate $\gamma>1$ near the zero state* (4). *Let the bounded initial value u_0 have a compact support*

$\subset [-L, L]$ *and invariant constants* p, q (8). *Then, the solution* $u(x,t)$ *to problem* (1) *satisfies*

$$||u(x,t) - N_{p,q}(x,t)||_1 \leqslant \text{const.} \max_x |N_{p,q}(x,t)| \quad \text{as } t \to \infty. \tag{13}$$

Furthermore, if a point $x = \beta$ *satisfying*

$$-p \equiv \inf_x \int_{-\infty}^{x} u_0(y)\,dy = \int_{-\infty}^{\beta} u_0(y)\,dy, \quad q \equiv \sup_x \int_x^{\infty} u_0(y)\,dy = \int_{\beta}^{\infty} u_0(y)\,dy \tag{14}$$

is unique and there exist constants $\alpha, \delta > 0$ *satisfying*

$$|u_0(x+\beta)| \geqslant |g(x/\alpha)|, \quad |x| \leqslant \delta, \tag{15}$$

then

$$||u(x+\beta,t) - N_{p,q}(x,t)||_1 \leqslant \text{const.}\, t^{-1} \quad \text{as } t \to \infty. \tag{16}$$

The asymptotic estimate in (13) shows that the solution converges to an N-wave in L^1 norm with the same order of the height of the N-wave. If the flux is given by the power law, $f(u) = |u|^\gamma/\gamma$, $\gamma > 1$, we can easily check that the order is $O(t^{-\frac{1}{\gamma}})$. The coefficients in the asymptotic estimates (13) and (16) can be estimated by the limits in Lemmas 2 and 3 (see Remark 10). For the strictly convex case, $f'' > 0$, we can easily verify that $\max_x |N_{p,q}(x,t)|$ is of order $O(t^{-\frac{1}{2}})$ (see Remark 11).

The convergence order in (16) is optimal in the sense that we may construct a solution which has the convergence order of $O(t^{-1})$, but not $o(t^{-1})$. We can also easily check that the uniqueness of the point $x = \beta$ in (14) is necessary and that assumption (15) which restrict the profile of the initial value near the point $x = \beta$ is also needed. Without these assumptions the convergence order in (13) is optimal which is already mentioned in [4] for the strictly convex case.

Our approach is as follows. In Section 2, we consider the evolution of N-waves under the power law (11). In this case we can explicitly evaluate the areas enclosed by the N-wave $N_{p,q}(x,t)$ and rarefaction waves $g(x/(t+\alpha))$ or $g((x \pm L)/t)$ (see Figs. 1 and 2). In Section 3, we obtain convergence orders of these areas for a general flux satisfying (4). In Section 4, we show that the solution $u(x,t)$ stays inside of the area for sufficiently large $t > 0$ (Proposition 9) and prove Theorem 1. These estimates also provide uniform convergence orders (Theorem 13) in the sense of graphs in Section 5. Finally, in Section 6, we provide a numerical simulation which shows how the solution evolves and be placed inside of the area.

2. Evolution of N-waves under the power law

In this section, roughly speaking, we consider the convergence order of the thin areas enclosed by two N-waves, Fig. 1 or 2. These areas converge to zero as $t \to \infty$,

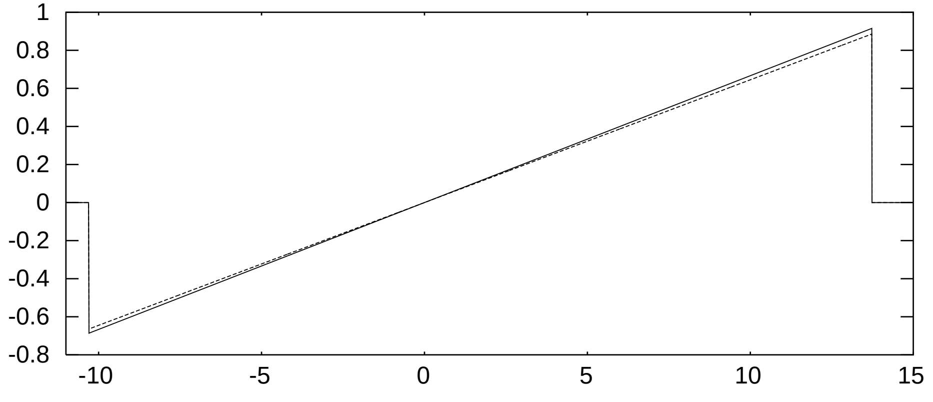


Fig. 1. The area of the thin layer enclosed with an N-wave $N_{p,q}(x,t)$ (solid lines) and a rarefaction wave $g(x/(t+\alpha))$ (dashed line) is of order $O(1/t)$ as $t\to\infty$. In the figure, $p=8\pi/9,\ \ q=2\pi,\ \ t=15$ and $\alpha=0.5$.

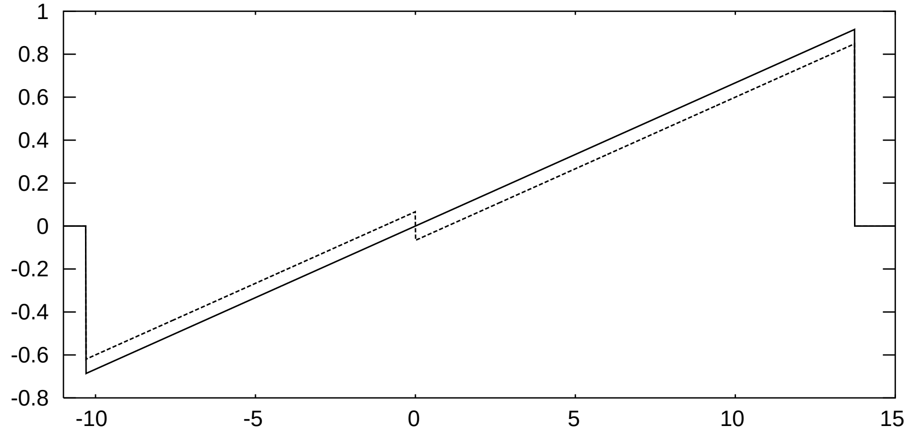


Fig. 2. The area of the thin layer enclosed with an N-wave $N_{p,q}(x,t)$ (solid lines) and a rarefaction wave $g((x\pm L)/t)$ (dashed lines) is of order $O(1/\sqrt[\gamma]{t})$ as $t\to\infty$. In this figure, $p=9\pi/8,\ \ q=2\pi,\ \ t=15$ and $L=1$.

and the solution $u(x,t)$ converges to the N-wave $N_{p,q}(x,t)$ with the same order in L^1 norm. Consider N-waves under the power law

$$f(u)=\frac{1}{\gamma}|u|^{\gamma},\quad \gamma>1. \tag{11}$$

Then, the similarity profile $g(x)$ is given by

$$g(x)=\operatorname{sign}(x)\sqrt[\gamma-1]{|x|},$$

and N-waves are by

$$N_{p,q}(x,t)=\begin{cases} g(x/t), & -a_p(t)<x<b_q(t),\\ 0 & \text{otherwise}, \end{cases}$$

where

$$a_p(t)=\left(\frac{\gamma p}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma}}\sqrt[\gamma]{t},\quad b_q(t)=\left(\frac{\gamma q}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma}}\sqrt[\gamma]{t}. \tag{17}$$

These N-waves are bounded by $-A_p(t)\leqslant N_{p,q}(x,t)\leqslant B_q(t)$, where

$$\begin{aligned} A_p(t)&=-g\left(\frac{-a_p(t)}{t}\right)=\left(\frac{\gamma p}{\gamma-1}\right)^{\frac{1}{\gamma}}t^{-\frac{1}{\gamma}}=O(t^{-\frac{1}{\gamma}}),\\ B_q(t)&=g\left(\frac{b_q(t)}{t}\right)=\left(\frac{\gamma q}{\gamma-1}\right)^{\frac{1}{\gamma}}t^{-\frac{1}{\gamma}}=O(t^{-\frac{1}{\gamma}}) \end{aligned} \tag{18}$$

measure the height of the N-wave.

Lemma 2. *For any* $\alpha>0$,

$$\lim_{t\to\infty} t\int_{-a_p(t)}^{b_q(t)}\left|N_{p,q}(x,t)-g\left(\frac{x}{t+\alpha}\right)\right|dx=\frac{\alpha(p+q)}{\gamma-1}. \tag{19}$$

Proof. Since $g(\frac{x}{t}\frac{t}{t+\alpha})=g(\frac{x}{t})(\frac{t}{t+\alpha})^{\frac{1}{\gamma-1}}$, we have

$$\int_0^{b_q(t)} g\left(\frac{x}{t+\alpha}\right)dx=q\left(\frac{t}{t+\alpha}\right)^{\frac{1}{\gamma-1}}=q\left(1-\frac{\alpha}{t+\alpha}\right)^{\frac{1}{\gamma-1}}.$$

Clearly, $g(x/t)>g(x/(t+\alpha))$ for $x>0$, and the Taylor expansion gives

$$\begin{aligned} &\int_0^{b_q(t)} N_{p,q}(x,t)-g\left(\frac{x}{t+\alpha}\right)dx\\ &\quad=q-q\left(1-\frac{\alpha}{t+\alpha}\right)^{\frac{1}{\gamma-1}}=\frac{\alpha q}{\gamma-1}\frac{1}{t+\alpha}-\frac{(2-\gamma)q\alpha^2}{2(\gamma-1)^2}\frac{1}{(t+\alpha)^2}y^{\frac{3-2\gamma}{\gamma-1}}, \end{aligned}$$

where $1-\frac{\alpha}{t+\alpha}\leqslant y\leqslant 1$. So we have

$$\lim_{t\to\infty} t\int_0^{b_q(t)}\left|N_{p,q}(x,t)-g\left(\frac{x}{t+\alpha}\right)\right|dx=\frac{\alpha q}{\gamma-1}.$$

Similarly we may show that $\lim_{t\to\infty} t\int_{-a_p(t)}^{0} |N_{p,q}(x,t) - g(\frac{x}{t+\alpha})|\,dx = \frac{\alpha p}{\gamma-1}$, and obtain (19). □

For example, consider the flux of the Burgers equation $f(u) = \frac{1}{2}u^2$. In the case the similarity profile is simply $g(x) = x$. In Fig. 1, N-wave $N_{8\pi/9,2\pi}(x,t)$ has been displayed together with a rarefaction wave $g(x/(t+\alpha))$. Lemma 2 implies that the thin area enclosed by these two waves converges to zero with order $O(t^{-1})$ as $t\to\infty$. Furthermore, we have the coefficient part of the convergence rate, which is $\alpha(p+q)/(\gamma-1)$.

Lemma 3. *For any given* $L>0$,

$$\lim_{t\to\infty} \sqrt[\gamma]{t}\int_0^{b_q(t)} N_{p,q}(x,t) - g\left(\frac{x-L}{t}\right)dx = \left(\frac{\gamma q}{\gamma-1}\right)^{\frac{1}{\gamma}} L \tag{20}$$

and

$$\lim_{t\to\infty} \sqrt[\gamma]{t}\int_{-a_p(t)}^{0} g\left(\frac{x+L}{t}\right) - N_{p,q}(x,t)\,dx = \left(\frac{\gamma p}{\gamma-1}\right)^{\frac{1}{\gamma}} L. \tag{21}$$

Proof. After a simple translation, we can easily check that

$$\sqrt[\gamma]{t}\int_0^{b_q(t)} g\left(\frac{x}{t}\right) - g\left(\frac{x-L}{t}\right)dx = \sqrt[\gamma]{t}\int_0^{L} g\left(\frac{x}{t}\right)dx + \sqrt[\gamma]{t}\int_{b_q(t)-L}^{b_q(t)} g\left(\frac{x}{t}\right)dx.$$

The first term is bounded by

$$0\leqslant \sqrt[\gamma]{t}\int_0^{L} g\left(\frac{x}{t}\right)dx \leqslant \sqrt[\gamma]{t}g\left(\frac{L}{t}\right)L = \sqrt[\gamma]{t}\left(\frac{L}{t}\right)^{\frac{1}{\gamma-1}} L\to 0$$

as $t\to\infty$ since $\gamma>1$. The second term is bounded by

$$\sqrt[\gamma]{t}g\left(\frac{b_q(t)-L}{t}\right)L\leqslant \sqrt[\gamma]{t}\int_{b_q(t)-L}^{b_q(t)} g\left(\frac{x}{t}\right)dx\leqslant \sqrt[\gamma]{t}\,g\left(\frac{b_q(t)}{t}\right)L.$$

Since both of the lower and the upper bounds converge to $(\frac{\gamma q}{\gamma-1})^{\frac{1}{\gamma}}L$ as $t\to\infty$, the asymptotic estimate in (20) holds. The other estimate (21) is obtained in the same way. □

In Fig. 2, N-wave $N_{9\pi/8,2\pi}(x,t)$ has been displayed together with $g((x\pm L)/t)$ for $L=1, t=15$. Lemma 3 implies that the area enclosed with these waves has order $O(t^{-\frac{1}{\gamma}})$ as $t\to\infty$ with a coefficient $(\sqrt[\gamma]{p}+\sqrt[\gamma]{q})(\frac{\gamma}{\gamma-1})^{\frac{1}{\gamma}}L$.

In the proof of Theorem 1 we show that these rarefaction waves form thin layers and the solution $u(x,t)$ of (1) lies inside of them over the interval $[-a_p(t), b_q(t)]$. These observations immediately give the convergence orders in Theorem 1.

3. Evolution of general N-waves

In this section we obtain estimates corresponding to Lemmas 2 and 3 with a general flux under the condition

$$\lim_{u\to 0} uf'(u)/f(u)=\gamma, \quad \gamma>1. \tag{4}$$

First, we observe that the similarity profile $g(x)$ also has the similar property near $x=0$.

Lemma 4. *Let $g(x)$ be the similarity profile, i.e., $f'(g(x))=x$. Then*

$$\lim_{x\to 0} xg'(x)/g(x)=(\gamma-1)^{-1}. \tag{22}$$

Proof. Differentiating the basic relation $f'(g(x))=x$ with respect to x, we obtain $f''(g(x))g'(x)=1$, i.e., $f''(g(x))=1/g'(x)$. Apply the l'Hopital's rule to the limit in (4) and get

$$\gamma=1+\lim_{u\to 0} uf''(u)/f'(u). \tag{23}$$

Setting $u=g(x)$ in (23), we obtain

$$\lim_{x\to 0} g(x)f''(g(x))/f'(g(x))=\lim_{x\to 0} g(x)/xg'(x)=\gamma-1. \qquad \square$$

It is natural to expect that the relation $f'(g(x))=x$ between the similarity profile $g(x)$ and the wave speed $f'(u)$ will give basic estimates in the evolution of solutions. We start with a trivial lemma.

Lemma 5. *Let a function $G(x)$ be increasing on $[0,x_0]$ and have values $G(0)=0$, $G(x_0)=u_0$. Then*

$$\int_0^{x_0} G(x)\,dx+\int_0^{u_0} F(u)\,du=x_0u_0, \tag{24}$$

where $F(u)$ is the unique function satisfying $F(G(x))=x$ for $x\in(0,x_0)$.

Proof. The integration by parts and the change of variable $u = G(x)$ give

$$\int_0^{u_0} F(u)\,du = \int_0^{x_0} xG'(x)\,dx = [xG(x)]_0^{x_0} - \int_0^{x_0} G(x)\,dx,$$

that implies (24). $\square$

In the following lemma we observe how the height and the width of an N-wave evolve asymptotically. Remember that the support $[-a_p(t), b_q(t)]$ measures the width of the N-wave $N_{p,q}(x,t)$ and the values at the end points,

$$A_p(t) \equiv -g\left(\frac{-a_p(t)}{t}\right), \quad B_q(t) \equiv g\left(\frac{b_q(t)}{t}\right), \tag{25}$$

measure the height.

Lemma 6. *Let $[-a_p(t), b_q(t)]$ be the support of an N-wave $N_{p,q}(x,t)$ and $A_p(t) = -g(-a_p(t)/t), B_q(t) = g(b_q(t)/t)$. Then*

$$\lim_{t\to\infty} tf(-A_p(t)) = \frac{p}{\gamma - 1}, \quad \lim_{t\to\infty} a_p(t)A_p(t) = \frac{\gamma p}{\gamma - 1}, \tag{26}$$

$$\lim_{t\to\infty} tf(B_q(t)) = \frac{q}{\gamma - 1}, \quad \lim_{t\to\infty} b_q(t)B_q(t) = \frac{\gamma q}{\gamma - 1}. \tag{27}$$

Proof. Setting $G(x) = g(x/t)$ and $F(u) = tf'(u)$, we have $F(G(x)) = x$ for $x > 0$, $G(0) = 0$ and $G(b_q(t)) = B_q(t)$. So Lemma 5 implies that

$$\int_0^{b_q(t)} g(x/t)\,dx + \int_0^{B_q(t)} tf'(u)\,du = b_q(t)B_q(t).$$

Since $\int_0^{b_q(t)} g(x/t)\,dx = q, f(0) = 0$ and $b_q(t)/t = f'(B_q(t))$, we have

$$q + tf(B_q(t)) = tf'(B_q(t))B_q(t).$$

Take $t\to\infty$ limit after dividing the both side by $tf(B_q(t))$ and obtain

$$\frac{q}{\lim_{t\to\infty} tf(B_q(t))} + 1 = \gamma,$$

which implies the first part of (27). Since $b_q(t)B_q(t) = q + tf(B_q(t))$, the second part of (27) is clear from the first part. Estimate (26) is obtained similarly. $\square$

Now we consider the property corresponding to Lemma 2.

Lemma 7. *For any* $\alpha > 0$,

$$\lim_{t\to\infty} t\int_{-a_p(t)}^{b_q(t)} \left| N_{p,q}(x,t) - g\left(\frac{x}{t+\alpha}\right)\right| dx = \frac{\alpha(p+q)}{\gamma-1}. \tag{28}$$

Proof. Let $h \equiv x/t - x/(t+\alpha) = \alpha x/(t+\alpha)t$. Since $g(x/t) > g(x/(t+\alpha))$ on the interval $(0, b_q(t))$, we have

$$\int_0^{b_q(t)} \left| g\left(\frac{x}{t}\right) - g\left(\frac{x}{t+\alpha}\right)\right| dx = \frac{\alpha}{t+\alpha}\int_0^{b_q(t)} \frac{x}{t}\frac{g(\frac{x}{t}) - g(\frac{x}{t}-h)}{h}\, dx.$$

Since $b_q(t)/t \to 0$ as $t\to\infty$, estimate (22) implies that

$$\lim_{t\to\infty} t\int_0^{b_q(t)} \left| g\left(\frac{x}{t}\right) - g\left(\frac{x}{t+\alpha}\right)\right| dx = \lim_{t\to\infty}\frac{\alpha t}{t+\alpha}\int_0^{b_q(t)} \frac{1}{\gamma-1} g\left(\frac{x}{t}\right) dx = \frac{\alpha q}{\gamma - 1}.$$

Similarly, we may show that $\lim_{t\to\infty} t\int_{-a_p(t)}^0 |g(\frac{x}{t}) - g(\frac{x}{t+\alpha})|\, dx = \frac{\alpha p}{\gamma-1}$ and complete (28). $\square$

Now we consider the last lemma which corresponds to Lemma 3.

Lemma 8. *For any* $L > 0$,

$$\int_0^{b_q(t)} \left| g\left(\frac{x}{t}\right) - g\left(\frac{x-L}{t}\right)\right| dx + \int_{-a_p(t)}^0 \left| g\left(\frac{x}{t}\right) - g\left(\frac{x+L}{t}\right)\right| dx$$
$$= O(A_p(t) + B_q(t)). \tag{29}$$

Proof. We can easily check that

$$\int_0^{b_q(t)} g\left(\frac{x}{t}\right) - g\left(\frac{x-L}{t}\right) dx = -\int_{-L}^0 g\left(\frac{x}{t}\right) dx + \int_{b_q(t)-L}^{b_q(t)} g\left(\frac{x}{t}\right) dx.$$

The first term is bounded by

$$0 \leqslant -\int_{-L}^0 g\left(\frac{x}{t}\right) dx \leqslant -g\left(\frac{-L}{t}\right) L = O(A_p(t))$$

as $t\to\infty$. The second term is bounded by

$$0 \leqslant \int_{b_q(t)-L}^{b_q(t)} g\left(\frac{x}{t}\right) dx \leqslant g\left(\frac{b_q(t)}{t}\right) L = O(B_q(t)).$$

So we have obtained the half of (29), i.e., $\int_0^{b_q(t)} |g(\frac{x}{t}) - g(\frac{x-L}{t})|\,dx = O(A_p(t) + B_q(t))$. The other half is obtained similarly. □

4. Optimal convergence order in L^1 norm

In the followings we briefly review the theory of characteristics (see [3, Chapter XI] for a detailed introduction). A minimal backward characteristic $x = \xi_-(t)$ associated to the solution $u(x,t)$ that emanates from a given point (x_0, t_0) is a straight line in the half-plane $\boldsymbol{R} \times \boldsymbol{R}^+$ such that $\xi_-(t) = x_0 + (t - t_0)f'(u(x_0-, t_0))$, $0 < t < t_0$. A maximal one $x = \xi_+(t)$ is defined similarly by $\xi_+(t) = x_0 + (t - t_0)f'(u(x_0+, t_0))$. If the solution $u(x,t)$ is continuous at the given point (x_0, t_0), then they are identical, and we write $\xi(t) \equiv \xi_-(t) = \xi_+(t)$. The solution $u(x,t)$ is constant along a characteristic line, i.e., $u(\xi_\pm(t), t) = u(x_0\pm, t_0)$, $0 < t < t_0$. Setting $\bar{x}_\pm = x_0 - t_0 f'(u(x_0\pm, t_0))$, we may write it as $\xi_\pm(t) = \bar{x} + tf'(u_0(\bar{x}_\pm))$.

A characteristic line $x = \bar{x} + tf'(u_0(\bar{x}))$ is called a *divide* if there exists a sequence (x_m, t_m) such that $t_m \to \infty$ as $m \to \infty$ and $x_m + (t - t_m)f'(u(x_m, t_m)) \to \bar{x} + tf'(u_0(\bar{x}))$ uniformly on any closed interval $[0, T]$ as $m \to \infty$. In the case we may show that

$$\int_z^{\bar{x}} [u_0(y) - u_0(\bar{x})]\,dy \leqslant 0, \quad -\infty < z < \infty. \tag{30}$$

(See Theorem 11.4.1 in [3].) It is well known that, if $q = \int_\beta^\infty u_0(y)\,dy = \sup_x \int_x^\infty u_0(y)\,dy$, then, for all $t > 0$, $u(\beta, t) = 0$ and

$$\int_{-\infty}^{\beta} u(y,t)\,dy = -p, \quad \int_\beta^\infty u(y,t)\,dy = q. \tag{31}$$

(See Theorem 11.4.2 in [3].)

An N-wave $N_{p,q}(x,t)$ is a special solution of the conservation law (1). Let $v(x,t) = N_{p,q}(x - L, t)$ and $\xi(t)$ be a characteristic line that emanates from a point (x_0, t_0), $L - a_p(t_0) < x_0 < L + b_q(t_0)$. Then the slope of the characteristic line is $1/f'(g((x_0 - L)/t_0)) = t_0/(x_0 - L)$ and it always passes through the point $(L, 0)$.

Suppose two solutions u_1, u_2 are given. Since the flux is convex, $f'' \geqslant 0$, we may compare the solutions using the wave speed, i.e.,

$$f'(u_1(x_0,t_0)) \leqslant f'(u_2(x_0,t_0)) \;\Rightarrow\; u_1(x_0,t_0) \leqslant u_2(x_0,t_0). \tag{32}$$

The following proposition comes from these observations on characteristics and their speed of propagation.

Proposition 9. *Let the flux $f(u)$ be convex and have an algebraic growth rate near the zero state as in* (4). *Let the bounded initial value u_0 have a compact support $\subset [-L, L]$*

and invariant constants p, q *in* (8). *Then, for any point* $\beta \in \boldsymbol{R}$ *that satisfies*

$$-p \equiv \inf_x \int_{-\infty}^{x} u_0(y)\,dy = \int_{-\infty}^{\beta} u_0(y)\,dy, \quad q \equiv \sup_x \int_x^{\infty} u_0(y)\,dy = \int_{\beta}^{\infty} u_0(y)\,dy, \tag{14}$$

the solution $u(x,t)$ *to problem* (1) *satisfies*

$$|u(x+\beta,t)| \leqslant |g(x/t)|, \quad -\infty < x < \infty, \tag{33}$$

$$g((x-L)/t) \leqslant u(x,t) \leqslant g((x+L)/t), \quad -a(t) < x < b(t), \ t \geqslant 0, \tag{34}$$

where $b(t) = \max\{x \in \operatorname{supp}(u(\cdot,t))\}$ *and* $-a(t) = \min\{x \in \operatorname{supp}(u(\cdot,t))\}$ *are estimated by*

$$|a_p(t) - a(t)| \leqslant L, \quad |b_q(t) - b(t)| \leqslant L. \tag{35}$$

Furthermore, if such a point β *in* (14) *is unique and*

$$|g(x/\alpha)| \leqslant |u_0(x+\beta)|, \quad |x| \leqslant \delta, \tag{15}$$

for some constants $\alpha, \delta > 0$, *then there exists* $T > 0$ *such that*

$$|g(x/(t+\alpha))| \leqslant |u(x+\beta,t)|, \quad a(t) < x < b(t), \ t \geqslant T, \tag{36}$$

where the support $\operatorname{supp}(u(x,t)) = [-a(t), b(t)]$ *is estimated by*

$$a_p(t) \leqslant a(t) - \beta \leqslant a_p(t+\alpha), \quad b_q(t) \leqslant b(t) - \beta \leqslant b_q(t+\alpha) \tag{37}$$

for all $t > T$.

Proof. Let $v(x,t) = g(x/t)$. Then we can easily check that $f'(v)_x = 1/t$ and $v(0,t) = 0$ for all $t > 0$. The Oleinik estimate, $f'(u)_x \leqslant 1/t$, gives $f'(u)_x \leqslant f'(v)_x$. Since $u(\beta,t) = v(0,t) = 0$ for all $t > 0$, we obtain $f'(u(x+\beta,t)) \leqslant f'(v(x,t))$ for $x > 0$ and $f'(u(x+\beta,t)) \geqslant f'(v(x,t))$ for $x < 0$. The convexity of the flux implies (33).

Fix $t_0 > 0$ and $-a(t_0) < x_0 < b(t_0)$. Let $\xi_\pm(t)$ be the extremal backward characteristics associated to the solution $u(x,t)$ that emanates from the point (x_0, t_0). Since $y = -a(t), b(t)$ are (generalized) characteristics, the uniqueness of the forward characteristics implies that $-a(t) < \xi_\pm(t) < b(t)$ for all $0 < t < t_0$ and, hence, $-L \leqslant \xi_\pm(0) \leqslant L$. Since backward characteristics associated to special solutions $v_\pm(x,t) = g((x \pm L)/t)$ always pass through the points $(\pm L, 0)$ respectively, the speed of characteristics are ordered as

$$f'\left(g\left(\frac{x-L}{t}\right)\right) \leqslant f'(u(x\pm,t)) \leqslant f'\left(g\left(\frac{x+L}{t}\right)\right), \quad -a(t) < x < b(t), \ t \geqslant 0.$$

So we may conclude (34) using (32). Furthermore, since

$$q = \sup_x \int_x^\infty u(x,t)\,dx \geqslant \int_L^{b(t)} u(x,t)\,dx$$
$$\geqslant \int_L^{b(t)} g\left(\frac{x-L}{t}\right) dx = \int_0^{b(t)-L} g\left(\frac{x}{t}\right) dx,$$

we have $b(t)-L \leqslant b_q(t)$. On the other hand, since there exists $\beta > -L$ such that $\int_\beta^\infty u(x,t)\,dx = q$ (31),

$$q = \int_\beta^{b(t)} u(x,t)\,dx \leqslant \int_\beta^{b(t)} g\left(\frac{x+L}{t}\right) dx$$
$$\leqslant \int_{-L}^{b(t)} g\left(\frac{x+L}{t}\right) dx = \int_0^{b(t)+L} g\left(\frac{x}{t}\right) dx.$$

So $b(t)+L \geqslant b_q(t)$ and we may conclude that $|b_q(t)-b(t)| \leqslant L$. The other half of (35) can be shown similarly.

Now we show (36) and (37) assuming (15) and the uniqueness of the point $x=\beta$ that satisfies (14). Note that we may assume that $\beta = 0$ without loss of generality. The uniqueness of such a point implies that

$$\int_0^x u_0(y)\,dy > 0 \quad \text{for all } x \neq 0. \tag{38}$$

Let $y = \xi_-^t(s)$, $0 < s < t$ be the minimal backward characteristic that emanates from the point $(b(t),t)$ (see Fig. 3). Then, since $\xi_-^t(0)$ is decreasing as $t \to \infty$, there exists a point $\bar{x}$ such that $\xi_-^t(0) \to \bar{x}$ as $t \to \infty$. Since $|u(x,t)| \to 0$ as $t \to \infty$ and $u(x,t)$ is

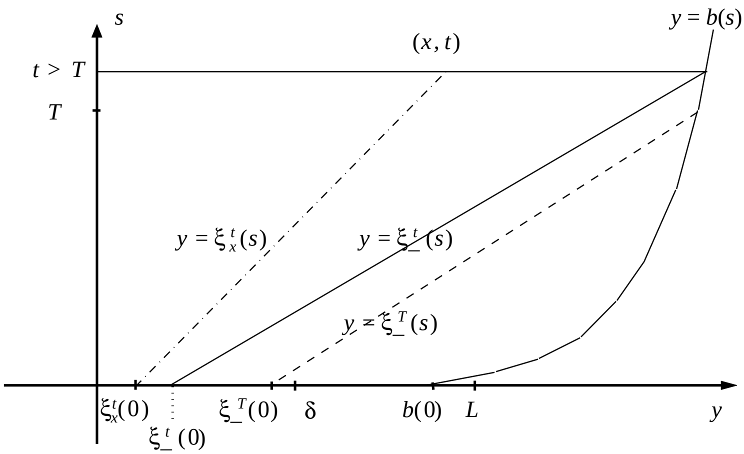


Fig. 3. Characteristics in the proof of Proposition 9.

constant along a characteristic line, $u(\bar{x},t)\to 0$ as $t\to 0+$ and the constant characteristic line $y=\bar{x}(\equiv \bar{x}+tf'(0))$ is a divide. Putting $z=0$ in (30) we obtain

$$\int_0^{\bar{x}} u_0(y)\,dy\leqslant 0.$$

Hence, (38) implies that $\bar{x}=0$. So there exists $T>0$ such that $0\leqslant \xi_-^t(0)\leqslant \delta$ for all $t>T$. To complete estimate (36) it is enough to consider continuity points of the solution since there is no isolated discontinuity (the Oleinik estimate). Clearly, for any $0<x<b(t), t>T$, the backward characteristic $\xi_x^t(\cdot)$ that emanates from the point (x,t) satisfy $0<\xi_x^t(0)<\delta$ and, hence,

$$u(x,t)=u_0(\xi_x^t(0))>g((\xi_x^t(0))/\alpha).$$

Let $u_0(\xi_x^t(0))=g(\xi_x^t(0)/\alpha_0$ for some $\alpha_0>0)$. Then, clearly, $\alpha_0<\alpha$ and the forward characteristic associated to $v(x,t)=g(x/(\alpha_0+t))$ that emanates from the point $(\xi_x^t(0),0)$ is identical to $\xi_x^t(s)$ for $0<s<t$. Hence,

$$u(x,t)=g\left(\frac{x}{\alpha_0+t}\right)>g\left(\frac{x}{\alpha+t}\right),\quad 0<x<b(t),\ t>T.$$

Similarly, we may show that $u(x,t)<g(x/(t+\alpha))$ for any $-a(t)<x<0,\ t>T$ and obtain (36).

Furthermore, since

$$q=\int_0^{b(t)} u(x,t)\,dx\leqslant \int_0^{b(t)} g\left(\frac{x}{t}\right)dx,$$
$$q=\int_0^{b(t)} u(x,t)\,dx\geqslant \int_0^{b(t)} g\left(\frac{x}{t+\alpha}\right)dx$$

for $t>T$, we have $b_q(t)\leqslant b(t)\leqslant b_q(t+\alpha)$. Similarly, $a_p(t)\leqslant a(t)\leqslant a_p(t+\alpha)$ and (37) is obtained. $\square$

The existence of β that satisfies (14) is obvious. First, since $\int_{-\infty}^x u_0(y)\,dy$ is continuous and $\mathrm{supp}(u_0)\subset[-L,L]$, there exists a point $\beta\in[-L,L]$ such that

$$\int_{-\infty}^{\beta} u_0(y)\,dy=\inf_x \int_{-\infty}^{x} u_0(y)\,dy.$$

Furthermore, since

$$\int_{\beta}^{\infty} u_0(y)\,dy-\int_x^{\infty} u_0(y)\,dy=\int_{\beta}^{x} u_0(y)\,dy=\int_{-\infty}^{x} u_0(y)\,dy-\int_{-\infty}^{\beta} u_0(y)\,dy\geqslant 0,$$

the point $\beta \in [-L, L]$ also satisfies

$$\int_\beta^\infty u_0(y)\,dy = \sup_x \int_x^\infty u_0(y)\,dy.$$

In the proposition we have seen that the solution essentially stays inside of an envelope enclosed with similarity profiles. This property gives the optimal order of convergence to the N-wave. In the following, we show our main result of this article:

Proof. (*Theorem* 1). First, we check that $u(x,t) \geqslant 0$ for all $x > L$. If not, there exists a continuity point $(x_0,\ t_0)$ such that $u(x_0,\ t_0) < 0,\ x_0 > L,\ t_0 > 0$. Let $\xi(s),\ 0 < s < t_0$, be the genuine backward characteristic that emanates from the point (x_0, t_0). Then, since the characteristic speed $f'(u(x_0, t_0))$ is negative, we have $\xi(0) > x_0 > L$, which contradicts to the assumption $\mathrm{supp}(u_0) \subset [-L, L]$. Similarly, we may show $u(x,t) \leqslant 0$ for all $x < -L$.

Let $q > 0$. Then, we may take $T > 0$ such that $b_q(T) > L$. Estimates (33) and (34) imply

$$g((y-(L-\beta))/t) \leqslant u(x+\beta, t) \leqslant g(x/t), \quad 0 \leqslant x \leqslant b_q(t).$$

Since $\int_\beta^{b_q(t)+\beta} g((y-\beta)/t)\,dy = \int_\beta^\infty u(y+\beta, t)\,dy = q$, we obtain

$$\begin{aligned}\int_\beta^\infty |N_{p,q}(y-\beta, t) - u(y,t)|\,dy &= 2\int_0^{b_q(t)} (N_{p,q}(y,t) - u(y+\beta, t))\,dy \\ &\leqslant 2\int_0^{b_q(t)} (g(y/t) - g((y-(L-\beta)/t))\,dy. \qquad (39)\end{aligned}$$

So Lemma 8 implies that

$$\int_\beta^\infty |N_{p,q}(y-\beta, t) - u(y,t)|\,dy = O(B_q(t)).$$

If $q = 0$, we may take $\beta = L$ and the estimate is trivial. Similarly, we may show $\int_{-\infty}^\beta |N_{p,q}(y-\beta, t) - u(y,t)|\,dy = O(A_p(t))$. Hence, (13) is obtained from

$$\begin{aligned}&||N_{p,q}(x,t) - u(x,t)||_1 \\ &\quad \leqslant ||N_{p,q}(x,t) - N_{p,q}(x-\beta, t)||_1 + ||N_{p,q}(x-\beta, t) - u(x,t)||_1 \\ &\quad = O(A_p(t) + B_q(t)) = O(\max_x |N_{p,q}(x,t)|).\end{aligned}$$

The convergence order in (16) is obtained from (33) and (36). Let $T>0$ be the constant in (36). Then, for $t>T$,

$$\int_0^\infty |u(x+\beta,t)-N_{p,q}(x,t)|\,dx = 2\int_0^{b_q(t)} |N_{p,q}(x,t)-u(x+\beta,t)|\,dx$$
$$\leqslant 2\int_0^{b_q(t)} \left| g\left(\frac{x}{t}\right) - g\left(\frac{x}{t+\alpha}\right)\right| dx. \tag{40}$$

So Lemma 7 implies that $\int_0^\infty |u(x+\beta,t)-N_{p,q}(x,t)|\,dx = O(t^{-1})$. Similarly we may show that $\int_{-\infty}^0 |u(x+\beta,t)-N_{p,q}(x,t)|\,dx = O(t^{-1})$ and, therefore, (16) is complete. $\square$

Remark 10. We can estimate the coefficients in estimates (13) and (16). Under the power law, $f(u)=|u|^\gamma/\gamma$, Lemma 3 and estimate (39) imply that

$$\lim_{t\to\infty} \sqrt[\gamma]{t}\,||u(x,t)-N_{p,q}(x,t)||_1 \leqslant 2\left(\left(\frac{\gamma q}{\gamma-1}\right)^{\frac{1}{\gamma}} + \left(\frac{\gamma p}{\gamma-1}\right)^{\frac{1}{\gamma}}\right)L.$$

Under conditions (14), (15), Lemma 7 and estimate (40) imply that

$$\lim_{t\to\infty} t||u(x+\beta,t)-N_{p,q}(x,t)||_1 \leqslant 2\,\frac{\alpha(p+q)}{\gamma-1}.$$

Remark 11. The convergence order $O(\max_x |N_{p,q}(x,t)|)$ in (13) depends on the flux $f(u)$. Since estimates in (27) imply that $f(B_q(t))=O(t^{-1})$, we can easily see that $B_q(t)=O(t^{-\frac{1}{\gamma}})$ under the power law $f(u)=c|u|^\gamma$. Suppose the flux is strictly convex, i.e., $f''(u)>0$, with $f''(0)=c>0$. Then

$$\gamma = \lim_{u\to 0}\frac{uf'(u)}{f(u)} = 1+\lim_{u\to 0}\frac{uf''(u)}{f'(u)}.$$

Substituting $u=B_q(t)=g(b_q(t)/t)$, we obtain

$$\lim_{t\to\infty}\frac{tB_q(t)c}{b_q(t)} = \gamma-1 = O(1).$$

Since $b_q(t)B_q(t)$ is of order $O(1)$, we may conclude that $b_q(t)=O(\sqrt{t})$ and $B_q(t)=O(1/\sqrt{t})$. So we have achieved the well-known convergence rate for the strictly convex case.

Remark 12. Let $u(x,t)$ be a solution given by an N-wave $u(x,t)=N_{p,q}(x-L,t)$, $L\neq 0$. Then Lemma 3 implies that $||u(x,t)-N_{p,q}(x,t)||_1$ is of order $O(t^{-\frac{1}{\gamma}})$, not

$o(t^{-\frac{1}{\gamma}})$. So it is clear that the solution or the N-wave should be placed at the correct place, say β, to get the optimal convergence rate $O(t^{-1})$ in (16) and the initial value should have some growth near the point in the sense of (15). Without these extra conditions on the initial value $u_0(x)$, convergence order of $O(t^{-\frac{1}{\gamma}})$ is optimal.

5. Uniform convergence order

Proposition 9 gives a uniform convergence order to the N-wave. In this section we study the uniform estimate of the solution under the power law $f(u) = |u|^\gamma/\gamma,\ \gamma > 1$. Let the bounded initial value u_0 have a compact support $\subset [-L, L]$. First consider the case $1 < \gamma \leqslant 2$. From the Taylor series, we have $g(\frac{x}{t}) - g(\frac{x-L}{t}) = \frac{L}{t} g'(\frac{y}{t})$ for $y \in [x - L, x]$. Since $g'(x)$ is an increasing function for $x > 0$, we have a uniform estimate

$$\left| g\left(\frac{x}{t}\right) - g\left(\frac{x-L}{t}\right) \right| \leqslant \frac{L}{t} g'\left(\frac{b_q(t)}{t}\right), \quad L < x < b_q(t).$$

Using Lemmas 4 and 6, the right-hand side is estimated by

$$\frac{L}{t} g'\left(\frac{b_q(t)}{t}\right) = \frac{L}{b_q(t)} \frac{b_q(t)}{t} g'\left(\frac{b_q(t)}{t}\right) = O\left(\frac{1}{b_q(t)} g\left(\frac{b_q(t)}{t}\right)\right) = O(t^{-\frac{2}{\gamma}}).$$

A similar estimate for $|g(\frac{x}{t}) - g(\frac{x+L}{t})|$ holds for $x \in (-a_p(t), -L)$, and Proposition 9 implies that

$$|N_{p,q}(x,t) - u(x,t)| = O(t^{-\frac{2}{\gamma}}), \quad -\min(a_p(t), a(t)) < x < \min(b_q(t), b(t)),$$

$$|N_{p,q}(x,t) - u(x,t)| = 0, \quad x < -\max(a_p(t), a(t)) \text{ or } x > \max(b_q(t), b(t)).$$

This estimate shows a uniform convergence order $O(t^{-\frac{2}{\gamma}})$ away from the discontinuity points $x = -a(t),\ x = b(t)$. The essential difficulty in the uniform estimate lies in estimating the shock location.

It is convenient to consider similarity variables

$$\xi = x/\sqrt[\gamma]{t}, \quad w(\xi, t) = \sqrt[\gamma]{t}\, u(x,t), \quad N_{p,q}(\xi, t) = \sqrt[\gamma]{t} N_{p,q}(x,t). \tag{41}$$

Then, the supports of solutions are also similarly transformed

$$\tilde{a}_p = \frac{a_p(t)}{\sqrt[\gamma]{t}} = \left(\frac{\gamma p}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}}, \quad \tilde{b}_q = \frac{b_q(t)}{\sqrt[\gamma]{t}} = \left(\frac{\gamma q}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}}, \quad \tilde{a}(t) = \frac{a(t)}{\sqrt[\gamma]{t}}, \quad \tilde{b}(t) = \frac{b(t)}{\sqrt[\gamma]{t}},$$

and the N-wave $N_{p,q}(x,t)$ is transformed to a time independent function

$$N_{p,q}(\xi) = \begin{cases} \operatorname{sign}(\xi)\ \sqrt[\gamma-1]{|\xi|}, & -\tilde{a}_p < \xi < \tilde{b}_q, \\ 0 & \text{otherwise}. \end{cases} \tag{42}$$

The supports $[-a_p(t), b_q(t)]$ of the N-wave $N_{p,q}(x,t)$ and $[-a(t), b(t)]$ of the similarity solution $w(\xi,t)$ are transformed to a fixed interval $[-\tilde{a}_p, \tilde{b}_q]$ and $[-\tilde{a}(t), \tilde{b}(t)]$, respectively. After the transformation we have a uniform estimate for $1 < \gamma \leqslant 2$,

$$|N_{p,q}(\xi) - w(\xi,t)| = O(t^{-\frac{1}{\gamma}}), \quad -\min(\tilde{a}_p, \tilde{a}(t)) < \xi < \min(\tilde{b}_q, \tilde{b}(t)),$$

$$|N_{p,q}(\xi) - w(\xi,t)| = 0, \quad \xi < -\max(\tilde{a}_p, \tilde{a}(t)) \text{ or } \xi > \max(\tilde{b}_q, \tilde{b}(t)).$$

Furthermore, since $|b_q(t) - b(t)|, |a_p(t) - a(t)| < L$, (35), we have

$$|\tilde{a}_p - \tilde{a}(t)| = O(1/\sqrt[\gamma]{t}), \quad |\tilde{b}_q - \tilde{b}(t)| = O(1/\sqrt[\gamma]{t}).$$

So we may conclude that the similarity solution $w(\xi,t)$, which is transformed by (41), converges to $N_{p,q}(\xi)$ with the uniform convergence order $O(1/\sqrt[\gamma]{t})$ in the sense of graphs.

If $\gamma > 2$, $g'(x)$ is a decreasing function for $x > 0$ and we have a uniform estimate

$$\left| g\left(\frac{x}{t}\right) - g\left(\frac{x \pm L}{t}\right) \right| \leqslant 2g\left(\frac{L}{t}\right) = O(t^{-\frac{1}{\gamma-1}}), \quad -\infty < x < \infty.$$

So, for $\gamma > 2$, the uniform convergence order in similarity variables is

$$|N_{p,q}(\xi) - w(\xi,t)| = O(t^{-\frac{1}{\gamma(\gamma-1)}}), \quad -\min(\tilde{a}_p, \tilde{a}(t)) < \xi < \min(\tilde{b}_q, \tilde{b}(t)),$$

$$|N_{p,q}(\xi) - w(\xi,t)| = 0, \quad \xi < -\max(\tilde{a}_p, \tilde{a}(t)) \text{ or } \xi > \max(\tilde{b}_q, \tilde{b}(t)).$$

Now consider the uniform convergence order under assumptions (14) and (15). Let $\beta = 0$ for the convenience. The Taylor expansion implies that

$$g\left(\frac{x}{t}\right) - g\left(\frac{x}{t+\alpha}\right) = \frac{\alpha x}{t(t+\alpha)} g'\left(\frac{y}{t+\alpha}\right),$$

where $y \in (x, x + \alpha x/t)$. The left-hand side is an increasing function for all $\gamma > 1$ and $x > 0$. So, for $x > 0$, we have

$$\left| g\left(\frac{x}{t}\right) - g\left(\frac{x}{t+\alpha}\right) \right| = O\left(\frac{\alpha}{t} \frac{b_q(t)}{t+\alpha} g'\left(\frac{b_q(t)}{t+\alpha}\right)\right) = O(t^{-\frac{\gamma+1}{\gamma}})$$

and, hence,

$$|N_{p,q}(x,t) - u(x,t)| = \begin{cases} O(t^{-\frac{\gamma+1}{\gamma}}), & -\tilde{a}_p < x < \tilde{b}_q, \\ 0, & x < -\tilde{a}(t) \text{ or } x > \tilde{b}(t). \end{cases}$$

The support $[-a(t), b(t)]$ of the solution $u(x,t)$ is estimated by (37), and (17) gives

$$|b(t) - b_q(t)| = O((t+\alpha)^{1/\gamma} - t^{1/\gamma}) = O(t^{\frac{1-\gamma}{\gamma}}).$$

If we transform these estimates into similarity variables, we obtain

$$|N_{p,q}(\xi) - w(\xi,t)| = O(1/t), \quad -\tilde{a}_p < \xi < \tilde{b}_q,$$

$$|N_{p,q}(\xi) - w(\xi,t)| = 0, \quad \xi < -\tilde{a}(t) \text{ or } \xi > \tilde{b}(t),$$

$$|\tilde{a}(t) - \tilde{a}_p| = O(1/t), \quad |\tilde{b}(t) - \tilde{b}_q| = O(1/t).$$

So we may conclude a uniform convergence to $N_{p,q}(\xi)$ of order $O(1/t)$ in the sense of graphs.

We summarize these uniform estimates in the following theorem:

Theorem 13. *Let $u(x,t)$ be the solution of the conservation law* (1) *with the power law $f(u) = |u|^\gamma/\gamma$, $\gamma > 1$. Let the bounded initial value u_0 have a compact support $\subset [-L, L]$ and invariance variables p, q* (8). *Then, the similarity solution $w(\xi,t)$ transformed by* (41) *uniformly converges to $N_{p,q}(\xi)$ in the sense of graphs and*

$$\sup_{\xi \in \boldsymbol{R}} \left\{ \inf_{\zeta \in \boldsymbol{R}} \{ |w(\xi,t) - N_{p,q}(\zeta)| + |\xi - \zeta| \} \right\} = \begin{cases} O(t^{-\frac{1}{\gamma(\gamma-1)}}), & \gamma \geqslant 2, \\ O(t^{-\frac{1}{\gamma}}), & 1 < \gamma \leqslant 2. \end{cases} \tag{43}$$

Furthermore, if a point $x = \beta$ such that

$$p = -\int_{-\infty}^{\beta} u_0(y)\,dy, \quad q = \int_{\beta}^{\infty} u_0(y)\,dy \tag{14}$$

is unique, and there exist constants $\alpha > t_0$, $\delta > 0$ such that

$$u_0(x+\beta) \leqslant g(x/\alpha), \quad -\delta \leqslant x \leqslant 0; \quad u_0(x+\beta) \geqslant g(x/\alpha), \ 0 \leqslant x \leqslant \delta, \tag{15}$$

then

$$\sup_{\xi \in \boldsymbol{R}} \left\{ \inf_{\zeta \in \boldsymbol{R}} \{ |w(\xi,t) - N_{p,q}(\zeta)| + |\xi - \zeta| \} \right\} = O(1/t). \tag{44}$$

6. A numerical example

Consider the solution to the Burgers equation

$$u_t + (\tfrac{1}{2}u^2)_x = 0, \quad t>0, \ x\in \boldsymbol{R}, \tag{45}$$

with its initial value

$$u(x,0) = u_0(x) = \begin{cases} -\sqrt{-x(3+x)}, & -3<x<0, \\ \sqrt{x(4-x)}, & 0<x<4, \\ 0 & \text{otherwise}. \end{cases} \tag{46}$$

In this example we can easily check that the center of the initial value is $\beta = 0$, the similarity profile is $g(x) = x$, and that invariants are

$$p = -\int_{-\infty}^{0} u_0(y)\,dy = 9\pi/8, \quad q = \int_0^{\infty} u_0(y)\,dy = 2\pi.$$

So the asymptotic behavior of the solution $u(x,t)$ is given by $N_{p,q}(x,t)$ with $p = 9\pi/8, q = 2\pi$. Near the center, $\beta = 0$, the initial value is bounded by

$$u_0(x) \leqslant g(x/0.5), \ -0.5 \leqslant x \leqslant 0 \quad u_0(x) \geqslant g(x/0.5), \ 0 \leqslant x \leqslant 0.5.$$

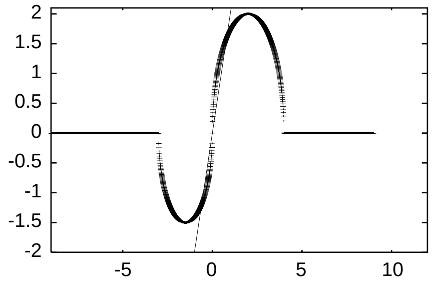
Let $x = \xi(t)$ be the characteristic line associated to the solution $u(x,t)$ that emanates from a point $(x_0, 15), -a_p(15) < x_0 < b_q(15)$. Then, since $|N_{p,q}(x,15)| < 1$ and $|u_0(\pm 0.5)| > 1$, we have $-0.5 < \xi(0) < 0.5$. Hence, $u(x,15)$ should be bounded by

$$\begin{aligned} & g(x/15) \leqslant u(x,15) \leqslant g(x/(15+0.5)), \quad -a_p(15) < x < 0, \\ & g(x/(15+0.5)) \leqslant u(x,15) \leqslant g(x/15), \quad 0 < x < b_q(15). \end{aligned} \tag{47}$$

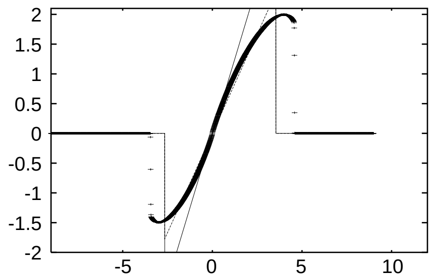
In Fig. 1, the N-wave $N_{p,q}(x,t)$ has been displayed together with $g(x/(t+\alpha))$ for $t = 15, \alpha = 0.5$. Estimate (47) implies that the solution $u(x,t)$ lies inside of the small and thin area enclosed with two similarity profiles $g(x/15)$ and $g(x/15.5)$ for $-a_p(15) < x < b_q(15)$. This observation together with Lemma 7 is the essence of the proof of Theorem 1.

Now we present a computational simulation to observe the phenomenon numerically which has been explained above. We briefly introduce our numerical scheme. We consider a uniform space $x_{j+1/2} = (j+1/2)\Delta x$ and time $t_n = n\Delta t$ mesh, where $j \in \boldsymbol{R}, \ n \in \boldsymbol{R}^+$. The cell-average of the solution is approximated by the solution of finite difference equation,

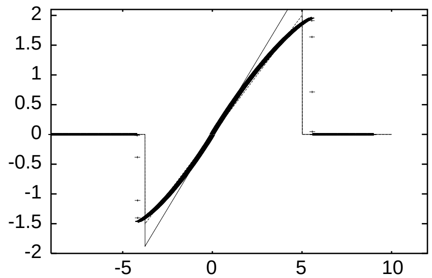
$$U_j^n \sim \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx,$$



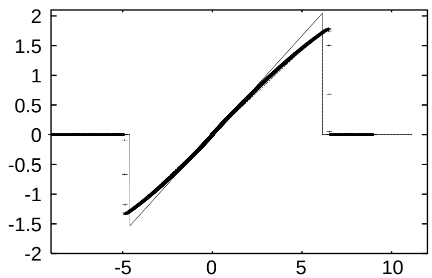
(a) Initial value (6.45) and $g(x/0.5)$



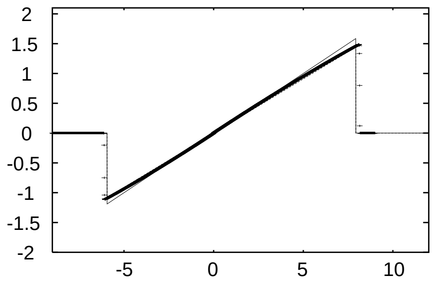
(b) $u(x, 1)$ and the layer at $t = 1$



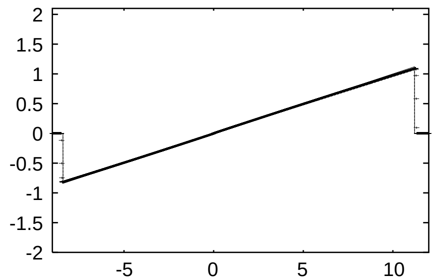
(c) $u(x, 2)$ and the layer at $t = 2$



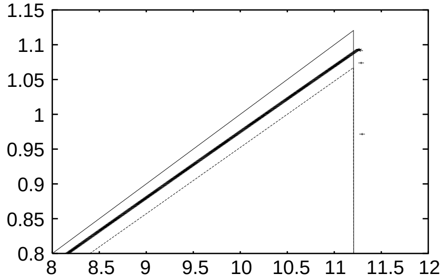
(d) $u(x, 3)$ and the layer at $t = 3$



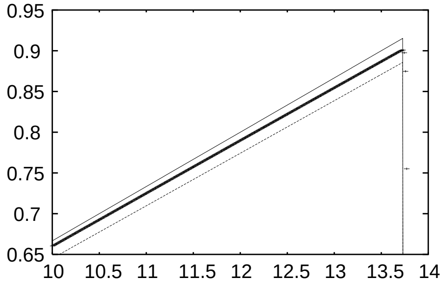
(e) $u(x, 5)$ and the layer at $t = 5$



(f) $u(x, 10)$ and the layer at $t = 10$



(g) $u(x, 10)$, near the shock



(h) $u(x, 15)$, near the shock

where U_j^n is given by an explicit method,

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)).$$

In our examples we employ the numerical flux of the Godunov method,

$$F(U_j^n, U_{j+1}^n) = \begin{cases} U_{j+1}^n & \text{if } U_j^n + U_{j+1}^n \leqslant 0, U_{j+1}^n \leqslant 0, \\ U_j^n & \text{if } U_j^n + U_{j+1}^n > 0, U_j^n > 0, \\ 0 & \text{if } U_j^n < 0, U_{j+1}^n > 0. \end{cases}$$

In Fig. 4, we set $\Delta x = 0.01$ and $\Delta t = 0.0025$. The numerical approximations of the solution $u(x,t)$ (dots) are displayed together with similarity profiles $g(x/t)$ and $g(x/(t+\alpha))$ (lines), which make thin layers. We can clearly observe that the numerical solution lies inside of them for $t \geqslant 10$.

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Fig. 4. Numerical simulations for solutions to (45)–(46) (dots: Godunov scheme with $\Delta x = 0.01, \Delta t = 0.0025$): We may observe that the solution $u(x,t)$ of the inviscid problem (1) lies inside of the thin layer eventually which consists of similarity profiles $g(x/t)$ and $g(x/(t+\alpha))$. (a) Initial value (6.45) and $g(x/0.5)$; (b) $u(x,1)$ and the layer at $t=1$; (c) $u(x,2)$ and the layer at $t=2$; (d) $u(x,3)$ and the layer at $t=3$; (e) $u(x,5)$ and the layer at $t=5$; (f) $u(x,10)$ and the layer at $t=10$; (g) $u(x,10)$ near the shock; (h) $u(x,15)$ near the shock.

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