Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xxxxxxx

pp. **X–XX**

BIOLOGICAL INVASION WITH A STARVATION-DRIVEN DIFFUSION IN A HETEROGENEOUS SPACE

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(Communicated by the associate editor name)

Abstract. The propagation speed of biological invasion varies in a spatially heterogeneous environment. Taking a singular limit in a hyperbolic scale provides a way to specify the propagation speed at a specific place since hyperbolic scaling does not change the wave speed. In this paper, we study the effect of starvation-driven diffusion to wave speed in a spatially heterogeneous environment. The model equation is

$$U_t = \epsilon\Delta(\gamma(u)U) + \frac{1}{\epsilon}U\Big(1 - \frac{U}{m(x)}\Big),$$

where $m(x)$ is the carrying capacity at position $x \in \Omega$, $u = \frac{U}{m}$ is the starvation measure, and the motility (or departing rate) γ is an increasing function of the starvation measure. We show that the propagation speed is constant under such a starvation-driven diffusion even if m is nonconstant.

1. Introduction.

Temporal and spatial changes in the environment affect the life of living things, and biological invasion is becoming an increasingly important issue in relation to these changes. The biological invasion speed has been intensively studied mathematically using reaction-diffusion equations. These mathematical studies provide fundamental insights on the dynamics of biological invasion. The paper's primary interest is the effect of a heterogeneous environment on the invasion speed when a starvation-driven diffusion is taken. Since the seminal papers by Fisher [9],

2020 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Dimension theory, Poincaré recurrences, multifractal analysis, discrete-time model, singular Hopf bifurcation.

The first author is supported by NSF grant xx-xxxx.

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Kolmogorov, Petrovsky, and Piskunov [17], there have been many mathematical studies on this issue. In the context of population genetics, Fisher proposed a reaction-diffusion equation,

$$U_t = \Delta U + U(1-U), \tag{1.1}$$

to describe the process of spatial spreading of a mutant phenotype. The unknown solution U is the population density of the mutant phenotype and all coefficients are normalized by one. Skellam [22] introduced the intrinsic growth rate $r > 0$ and the carrying capacity $m > 0$ into the system and considered

$$U_t = \Delta U + rU\Big(1 - \frac{U}{m}\Big) \tag{1.2}$$

to explain spatial patterns of biological individuals. The two model equations, (1.1) and (1.2), are homogeneous models and consist of the population growth and the random migration, which are the two main components of the biological invasion.

Shigesada *et al.* introduced biological invasion models in spatially periodic environments (see [19, 21, 20]). They segmented habitats spatially into favorable and unfavorable regions which appear periodically and analyzed how the pattern and scale of spatial fragmentation affect the dispersal size. Their reaction-diffusion equations are written as

$$U_t = \nabla \cdot (A(x)\nabla U) + f(x,U),$$

where the spatial heterogeneity of the reaction function $f(x,U)$ represented by step functions which take two different values periodically. Berestycki *et al.* [2, 3, 4] extended the work of Shigesada *et al.* in a general setting with rather general smooth periodic coefficients.

1.1. **Random diffusion in a heterogeneous environment.** The importance of having a biologically meaningful diffusion model has been stressed by many researchers (see Skellam [23, 24] and Okubo & Levin [18, Chapter 5]). To obtain a biologically meaningful dispersal phenomenon, a diffusion model should include the effect of the interaction among individual organisms and the response to the environmental variations. However, diffusion models which are not appropriate in heterogeneous environments may lead to wrong conclusions. We start by briefly discussing the meaning of diffusion models (see [7, Section 2] for more detail).

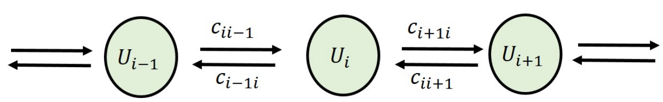

FIGURE 1. A linearly connected patch system is viewed as a general diffusion in $\mathbf{R}^1$.

Consider a linearly connected patch system. The population at patch i is denoted by U_i and the migration rate from patch j to patch i by c_{ij}, i.e., $c_{ij} = c_{i\leftarrow j}$(see Figure 1). We assume $c_{ij} = 0$ for $j \neq i \pm 1$. The corresponding dispersal model is

$$\dot{U}_i = c_{ii+1}U_{i+1} + c_{ii-1}U_{i-1} - c_{i-1i}U_i - c_{i+1i}U_i, \tag{1.3}$$

where $U_i = U_i(t)$ and $\dot{U}_i$ is the ordinary differentiation in the time variable $t \geq 0$. Such a dispersal model is called **symmetric** if $c_{ij} = c_{ji}$. The heterogeneity in the two adjacent patches is ignored in a symmetric dispersal and the physical meaning of it is unclear. Since $c_{i+1i} = c_{ii+1}$, we may treat the dispersal rate as if it is decided at the middle point between the two patches and denote $\gamma_{i+1/2} := c_{i+1i} = c_{ii+1}$. Then, (1.3) is written as

$$\dot{U}_i = \gamma_{i+1/2}(U_{i+1} - U_i) - \gamma_{i-1/2}(U_i - U_{i-1}).$$

One can easily see that this is a discretization of Fick's law,

$$U_t = \nabla \cdot (\gamma \nabla U),$$

in one the one space dimension. In other words, Fick's law models a symmetric dispersal, and a constant steady-state of Fick's law is a result of symmetry, not of randomness.

We call the dispersal model (1.3) **random** if $c_{i+1i} = c_{i-1i}$. This is the case when a species migrates to one of the two adjacent patches with equal probability and the physical meaning of this case is clear. Let u_i denote the environmental harshness at the i-th patch and assume the migration rates $c_{i\pm1i}$ to depart the patch are decided by the harshness, i.e., $c_{i\pm1i} = \gamma(u_i)$ for a function γ. Then, (1.3) is written as

$$\dot{U}_i = \gamma(u_{i+1})U_{i+1} + \gamma(u_{i-1})U_{i-1} - 2\gamma(u_i)U_i.$$

This is a finite difference scheme corresponding to Chapman's (or Ito's) diffusion law,

$$U_t = \Delta(\gamma(u)U). \tag{1.4}$$

One of the simplest harshness measure is $u_i = \frac{U_i}{m_i}$, where U_i is the population at the i-th patch and m_i is the carrying capacity at it. This is an excellent indicator when the population dynamics is given by the logistic function as in (1.2). The indicator gives the number of populations that share a unit amount of carrying capacity. It is called a starvation measure and, if γ is an increasing function, we call the diffusion in (1.4) a starvation-driven diffusion. Notice that a constant state is not a steady-state anymore. A steady state is given by '$\gamma(u)U$ = constant' or 'u = constant'. In other words, the randomness produces nonconstant steady-state if the environment is spatially nonconstant.

The dispersal model (1.3) is both **symmetric** and **random** only when $c_{i\pm1i} = d_0$ for a constant and the corresponding diffusion is the linear diffusion

$$U_t = d_0\Delta U.$$

It is known well that an initial disturbance becomes trivialized eventually and the solution distribution converges to a constant steady-state. However, that is not because the linear diffusion is random, but because it is symmetric.

There is a difference between biological diffusion and Brownian particle diffusion. The migration rate γ in the biological diffusion model (1.4) allows us to introduce biological mechanisms into a dispersal model. However, for the dispersal of Brownian particles, the departing rate γ is taken as a constant or simply as one since non-organic particles do not have the choice to stay and everyone departs as soon as it arrives. In such a case, particle velocity, turning frequency, walk length, and jumping time characterize the diffusion phenomenon. For example, a heterogeneous diffusion model,

$$U_t = \nabla\cdot\big(\sqrt{\mu D}\,\nabla(\sqrt{\mu^{-1}D}U)\big),$$

has been introduced using a discrete kinetic equations (see [14, 16]), where $D = D(x)$ is the diffusivity and $\mu = \mu(x)$ is the turning frequency. If the two components are combined, we obtain

$$U_t = \nabla\cdot\big(\sqrt{\mu D}\,\nabla(\sqrt{\mu^{-1}D}\gamma(u)U)\big).$$

Mixing the two components does not make the point clear, so in this article we use diffusion model in (1.4). If there is no enough food or resource, biological organisms start to migrate in search of food even if they do not know where foods are. Starvation-driven diffusion (1.4) has been introduced to model such a random dispersal (see [5, 6]). This diffusion models is being used in chemotaxis papers (see [8, 13, 26, 27, 28]). If the starvation-measure is simply the population size, $u = U$, then it return to porous medium equation type diffusion (see [10, 25]).

1.2. **Hyperbolic scale singular limit.** The question which we wish to answer is how does the invasion speed change in a heterogeneous environment. However, there are several subtle issues to discuss nonconstant invasion speed. For example, consider a solution of a reaction-diffusion equation,

$$U_t = (\gamma(u)U)_{xx} + U\Big(1 - \frac{U}{m(x)}\Big), \quad U(x,t) = U_0(x).$$

One may say that the invasion speed is the speed at which the solution support expands. However, if the problem is uniformly parabolic, the solution support becomes the whole real line at any time $t > 0$ even if the initial value is compactly support. In other words, the support size does not provide the information of invasion speed. The most favorite way to obtained the invasion speed is to find a traveling wave solution. However, if the problem is heterogeneous, there does not exist such an ideal solution.

In the paper, we take a hyperbolic singular limit which gives a clear way to answer the question. After change of variables,

$$x \to \varepsilon x, \qquad t \to \varepsilon t,$$

the reaction-diffusion equation is written as

$$(U^\varepsilon)_t = \varepsilon(\gamma(u)U^\varepsilon)_{xx} + \frac{1}{\varepsilon}U^\varepsilon\Big(1 - \frac{U^\varepsilon}{m(x)}\Big).$$

Let U^0 be the singular limit of U^ε as $\varepsilon \to 0$. Since the wave speed is not changed under the hyperbolic scaling, the limit U^0 keeps the information of the wave speed of the original problem. If $\sup(U_0) = (-\infty, 0]$, the limit U^0 turns out to be a function with an interface $\xi(t)$ and satisfies

$$U^0(x,t) = \begin{cases} m(x), & x < \xi(t), \\ 0, & x > \xi(t). \end{cases}$$

In other words, we obtain a sharp interface after taking a hyperbolic singular limit. The interface gives a heterogeneous invasion speed, where $\xi'(t)$ is the wave speed at the position $x = \xi(t)$.

In the paper, we consider a case when the starvation measure is given by $u = \frac{U}{m(x)}$ and show that the wave speed is constant, i.e., $\xi'(t) = c_k$. In other words, the invasion speed is independent of the heterogeneity in the carrying capacity $m(x)$. This is a special property of the starvation-driven diffusion. If the motility (or departing rate) is a function of population, i.e., $\gamma = \gamma(U)$, we obtain a porous medium equation type diffusion and the invasion speed is nonconstant (see [15]).

2. Mathematical model and the main results.

Consider a reaction-diffusion system,

$$\begin{cases} (U^\varepsilon)_t = \varepsilon\Delta\left(\gamma\Big(\dfrac{U^\varepsilon}{m}\Big)U^\varepsilon\right) + \dfrac{1}{\varepsilon}U^\varepsilon\Big(1 - \dfrac{U^\varepsilon}{m}\Big) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial U^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ U^\varepsilon(x,0) = U_0(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 1)$, ν is the outward unit normal vector to the boundary $\partial\Omega$, $\varepsilon > 0$ is a small parameter, and γ is a smooth and increasing function of a starvation measure

$$u^\varepsilon := U^\varepsilon/m.$$

We take a power function as the motility function,

$$\gamma(u) = u^{\tilde{k}}, \qquad \tilde{k} \geq 1,$$

and the carrying capacity $m(x) > 0$ is positive and depends on the space variable x. The unknown solution U^ε is the population density of a species whose evolution is governed by the logistic growth and the random migration.

One of key ideas in this paper is to rewrite the equation in terms of starvation measure. If the environment is homogeneous, i.e., if the distribution of the carrying capacity, $m(x)$, is constant, the starvation measure can be considered as a population density normalized by the carrying capacity. In terms of the starvation measure, (2.1) is written as

$$(P^\varepsilon) \qquad \begin{cases} (u^\varepsilon)_t = \varepsilon\dfrac{1}{m(x)}\Delta(m(x)(u^\varepsilon)^k) + \dfrac{1}{\varepsilon}f(u^\varepsilon) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial (u^\varepsilon)^k}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $f(u) = u(1-u)$, $k = \tilde{k} + 1 \geq 2$, and $u_0 := U_0/m$. This is the main problem of the paper.

For technical reasons, we take a few assumptions on m and u_0. The carrying capacity m is assumed to satisfy

$$m > 0,\ m \in C^1(\overline{\Omega}) \quad \text{and} \quad \frac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.2}$$

The initial value is smooth $u_0 \in C^2(\bar{\Omega})$ and nonnegative $u_0 \geq 0$. These assumptions yield that there exists a positive upper bound $C_0 > 0$ such that

$$|u_0| + |\nabla u_0| + |\Delta u_0| \leq C_0.$$

The support of u_0, denoted by Ω_0, is compact in Ω and its boundary, denoted by Γ_0, is a smooth closed hyper-surface, i.e.,

$$\Omega_0 := \operatorname{supp}(u_0) \subset\subset \Omega,\ \Gamma_0 := \partial\Omega_0 \ \text{ is smooth closed hyper-surface.} \tag{2.3}$$

Finally, we assume that the initial value is not flat at the boundary, i.e.,

$$\frac{\partial u_0}{\partial \mathbf{n}_0}(y) < 0 \text{ for all } y \in \Gamma_0, \tag{2.4}$$

where $\mathbf{n}_0$ is the outward unit normal vector on the boundary Γ_0.

The behaviour of the solution u^ε for a small $\varepsilon > 0$ is divided into two stages. In the first one, we show that u^ε satisfies

$$u^\varepsilon(x,t) \approx Y\left(\frac{t}{\varepsilon}; u_0(x)\right)$$

in the time interval $[0, t^\varepsilon]$, where $t^\varepsilon = O(\varepsilon|\log\varepsilon|)$ and $Y(\tau,\xi)$ is the solution of

$$\begin{cases} Y_\tau = f(Y), \quad \tau \geq 0, \\ Y(0;\xi) = \xi. \end{cases}$$

This initial dynamics gives an interface generation phenomenon for a short time period $t^\varepsilon = O(\varepsilon|\ln\varepsilon|)$. When the first stage is finished, i.e., at $t = t^\varepsilon$, the solution $u^\varepsilon(x,t^\varepsilon)$ takes values close to either zero or one except steep and thin transition layers. In the second stage, i.e., during a time interval $[t^\varepsilon, T]$ with a macroscopic time scale $T > 0$, the solution keeps the thin and steep transition layers, where the position of a layer is approximated by a transition layer flow given by

$$(IP) \qquad \begin{cases} V_n = c_k \text{ on } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0. \end{cases}$$

In this flow, Γ_t is the interface at time $t > 0$, V_n is the speed of the moving interface in the outward normal direction, and c_k is the minimal traveling wave speed given in (5.1). We denote the maximal time interval for the existence of the smooth solution of (IP) by $[0, T_{max})$.

Next, we estimate the first stage of interface generation. To do that, we introduce the signed distance function $\overline{d}(x,t)$ to Γ_t given by

$$\overline{d}(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t), & x \notin \Omega_t, \\ -\operatorname{dist}(x,\Gamma_t), & x \in \Omega_t, \end{cases} \tag{2.5}$$

where Ω_t is the region enclosed by Γ_t.

Theorem 2.1 (Generation of interface)**.** *Let $k \geq 2$, $m(x) > 0$ satisfy (2.2), $t^\varepsilon = \varepsilon|\ln\varepsilon|$, and the initial value $u_0 \geq 0$ be smooth and satisfy (2.3)-(2.4) for some $c_0 > 0$. Let u^ε be the solution of (P^ε). Then, for any given $\eta_0 \in (0,1/2)$, there exist positive constants ε_0 and M_0 such that for all $\varepsilon \in (0,\varepsilon_0)$*

$$0 \leq u^\varepsilon(x,t^\varepsilon) \leq 1 + \eta_0 \qquad \textit{for} \qquad x \in \Omega, \tag{2.6}$$

$$u^\varepsilon(x,t^\varepsilon) \geq 1 - \eta_0 \qquad \textit{if} \qquad u_0(x) \geq M_0\varepsilon \tag{2.7}$$

$$u^\varepsilon(x,t^\varepsilon) = 0 \qquad \textit{if} \qquad \overline{d}(x,0) \geq M_0\varepsilon. \tag{2.8}$$

Thus we obtain two following theorems; the generation of the interface (Theorem 2.1) and a result combining the generation and the motion of the interface (Theorem 2.2).

Theorem 2.2 (Motion of interface)**.** *Let $k \geq 2$, $m(x) > 0$ satisfy (2.2), $t^\varepsilon = \varepsilon|\ln\varepsilon|$, and the initial value $u_0 \geq 0$ be smooth and satisfy (2.3)-(2.4) for some $c_0 > 0$. Let u^ε be the solution of (P^ε). Let Γ_t be the interface given by the interface problem (IP), and $\overline{d}(x,t)$ be the signed distance defined by (2.5). Then, there exists $T_{max} > 0$ such that, for any $T < T_{max}$ and $\eta_1 \in (0,1/2)$, there exist $\varepsilon_0 > 0$ and $M_1 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$ and for all $t \in (t^\varepsilon, T]$,*

$$0 \leq u^\varepsilon(x,t) \leq 1 + \eta_1 \qquad \textit{for} \qquad x \in \Omega, \tag{2.9}$$

$$u^\varepsilon(x,t) \geq 1 - \eta_1 \qquad \textit{if} \qquad \bar{d}(x,t) \leq -\varepsilon M_1, \tag{2.10}$$

$$u^\varepsilon(x,t) = 0 \qquad \textit{if} \qquad \bar{d}(x,t) \geq \varepsilon M_1. \tag{2.11}$$

We basically use the methods and results of Hilhorst *et al.* [11]. However, while the convexity of the initial interface Γ_0 was necessary for [11], a new choice of super-solution of the paper permits avoiding this hypothesis here.

3. **Notions.** We define solution, sub-solution, and super-solution of the problem (P^ε) in a weak sense similarly to the ones in [11].

Definition 3.1. Let $T > 0$ and $Q_T = \Omega \times (0,T)$. A non-negative function $u^\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ is said to be a (weak) solution of (P^ε) if
(i) $u^\varepsilon \in L^2(Q_T)$ and $\nabla(u^\varepsilon)^k \in L^2(Q_T)$,

$$(ii) \qquad \int_\Omega u^\varepsilon(T)\varphi(T) = \int_\Omega u_0\varphi(0) + \int_0^T \int_\Omega \left(u^\varepsilon\varphi_t - \varepsilon\nabla(m(u^\varepsilon)^k)\nabla\frac{\varphi}{m} + \frac{1}{\varepsilon}f(u^\varepsilon)\varphi\right)$$

for all non-negative functions $\varphi \in C^1(\bar{Q}_T)$. If a function $\bar{u}^\varepsilon$ satisfies the conditions in (i) and

$$\int_\Omega \bar{u}^\varepsilon(T)\varphi(T) \geq \int_\Omega u_0\varphi(0) + \int_0^T \int_\Omega \left(\bar{u}^\varepsilon\varphi_t - \varepsilon\nabla(m(\bar{u}^\varepsilon)^k)\nabla\frac{\varphi}{m} + \frac{1}{\varepsilon}f(\bar{u}^\varepsilon)\varphi\right), \tag{3.1}$$

it is called a super-solution of (P^ε). If a function $\underline{u}^\varepsilon$ satisfies the conditions in (i) and (3.1) with reversed inequality, it is called a sub-solution of (P^ε).

The following lemma provides a sufficient conditions to be weak-, sub-, and super- solutions of Problem (P^ε). For a simpler writing, introduce a differential operator

$$\mathcal{L}(w) := w_t - \varepsilon\frac{1}{m(x)}\Delta(m(x)w^k) - \frac{1}{\varepsilon}f(w) \tag{3.2}$$

on the set where $\{w(x,t) > 0\}$.

Lemma 3.2 ([11, Lemma 2.2])**.** *Let u be a continuous function defined in $\overline{\Omega} \times [0,T]$. Let $\Omega_t^+ = \{(x \in \Omega : u(x,t) > 0\}$ and ν be the unit normal vector to Ω_t^+. Assume that*
(i) $\dfrac{\partial u^k}{\partial \nu} = 0$ *on* $\partial\Omega_t$,
(ii) $\mathcal{L}(u) = 0$ *in* $\{(x,t) \in \Omega \times [0,T] : u(x,t) > 0\}$,
(iii) ∇u^k *is continuous in* $\overline{\Omega} \times [0,T]$,
(iv) $\cup_{t\in[0,T]}\partial\Omega_t^+ \times \{t\}$ *is sufficiently smooth.*
Then, u is a weak solution to (P^ε). On the other hand, u is a sub-solution if the equalities in (i) and (ii) are replaced by "$\geq$", and u is a super-solution if the equalities are replaced by "$\leq$".

4. **Generation of interface.** In this section, we prove Theorem 2.1. The idea of the proof is based on the comparison principle, and hence we need to construct an appropriate pair of sub- and super-solutions. Since the reaction term plays an important role in the formation of interface, the form of sub-solutions and super-solutions will be based on the positive solutions of the equation without diffusion $u_\tau = f(u) + \delta$ with $\tau = t/\varepsilon^2$ and $\delta > 0$ small enough.

4.1. **Preliminary.** We will employ the key idea used for the bistable cases [1, 12]. To that purpose, we redefine f in $(-\infty, 0]$ so that $f(u) = -f(u), u \in (-\infty, 0]$. Let $\delta_0 > 0$ be small enough such that for all $\delta \in (-\delta_0, \delta_0)$, the function $f(u) + \delta$ has exactly three zeros denoted by $\alpha_-(\delta) < \alpha_0(\delta) < \alpha_+(\delta)$. Set $\mu(\delta) := f'(\alpha_0(\delta))$. Note that $1 = \mu(0) = f'(0)$ and that there exists a positive constant $C_1 > 0$ such that

$$|\mu(\delta) - 1| \leq C_1\delta \quad \text{for all} \quad \delta \in (-\delta_0, \delta_0).$$

Consider the solution $Y(\tau;\xi;\delta)$ of the ordinary differential equation:

$$\begin{cases} Y_\tau = f(Y) + \delta, & \tau \geq 0, \\ Y(0;\xi;\delta) = \xi. \end{cases}$$

The behaviour of Y is given in the following lemmas:

Lemma 4.1 ([1], Lemmas 4.2, 4.3, 4.4)**.** *There exist positive constants C_2, C_3 such that*
(i) $|Y| \leq C_2$,
(ii) $0 < Y_\xi \leq C_3 e^{\mu(\delta)\tau}$ *and* $\left|\dfrac{Y_{\xi\xi}}{Y_\xi}\right| \leq C_4(e^{\mu(\delta)\tau} - 1)$,
for all $(\tau, \xi, \delta) \in (0, \infty) \times (-2C_0, 2C_0) \times (-\delta_0, \delta_0)$.

Lemma 4.2 ([1], Lemma 4.7)**.** *Let $0 < \eta < 1/2$. There exist positive constants $\varepsilon_0 = \varepsilon_0(\eta)$ and C_4 such that, for all $\varepsilon \in (0, \varepsilon_0)$ and $\xi \in (-2C_0, 2C_0)$,*

(i) $$-1 - \eta \leq Y(|\ln\varepsilon|;\xi;\pm\varepsilon) \leq 1 + \eta,$$

(ii) $$\begin{aligned} &\text{if } \xi \geq C_4\varepsilon, \text{ then } Y(|\ln\varepsilon|;\xi;-\varepsilon) \geq 1 - \eta, \\ &\text{if } \xi \leq -C_4\varepsilon, \text{ then } Y(|\ln\varepsilon|;\xi;+\varepsilon) \leq -1 + \eta < 0. \end{aligned}$$

4.2. **Proof of Theorem 2.1.** To prove Theorem 2.1, we extend the initial condition u_0 smoothly to $\tilde{u}_0 \in C^2(\overline{\Omega})$. Such extension helps us to obtain sufficient conditions needed to use the comparison principle. For example, let

$$u_0(x) = \begin{cases} 1 - x^2 & \text{if } x \in (-1,1) \\ 0 & \text{if } x \in [-2,-1] \cup [1,2], \end{cases}$$

with $\Omega = [-2,2]$. Then, for any $a > 0$, we obtain

$$\partial_x (u_0(x) + a)^k = \begin{cases} -2kx(1 + a - x^2)^{k-1} & \text{if } x \in (-1,1) \\ le : sufficientforsub0 & \text{if } x \in [-2,-1] \cup [1,2], \end{cases}$$

which implies that $(u_0 + a)^k \notin C^1(\Omega)$. By a similar reasoning, if we use u_0 instead of $\tilde{u}_0$ for the construction of the super-solution in (4.2), then such a super-solution fails the condition (iii) of Lemma 3.2.

We let $\tilde{u}_0$ to be the extension of the initial condition u_0, which is possible by Section 6. Such $\tilde{u}_0$ belongs to $C^2(\overline{\Omega})$, and satisfies $\tilde{u}_0 = u_0$ in Ω_0 and

$$\begin{cases} \tilde{u}_0 < -\tilde{d}\, dist(x, \Omega_0) & \text{for } 0 < dist(x,\Omega_0) < \tilde{d} \\ \tilde{u}_0 = -1 & \text{for } \tilde{d} \le dist(x,\Omega_0) \end{cases} \tag{4.1}$$

for some positive constant $\tilde{d} < 1$. Next we construct a pair of sub- and super-solutions of problem (P^ε) in order to show (2.7).

Claim: There exists a constant $C_5 > 0$ such that the functions

$$w^{\varepsilon,\pm}(x,t) := \left[Y\left(\frac{t}{\varepsilon}; \tilde{u}_0(x) \pm C_5 \varepsilon^2 (e^{\mu(\pm\varepsilon)\frac{t}{\varepsilon}} - 1); \pm\varepsilon\right)\right]^+ \tag{4.2}$$

are a pair of sub- and super-solutions of problem (P^ε).

We only show $w^{\varepsilon,+}$ is a super-solution; the fact that $w^{\varepsilon,-}$ is a sub-solution can be proved in a similar way. We need to verify the sufficient conditions for sub-solution in Lemma 3.2. First note that on the set $\Omega_t^+[w^{\varepsilon,+}] := \{x \in \Omega : w^{\varepsilon,+}(x,t) > 0\}$ we have

$$\nabla (w^{\varepsilon,+})^k = (kY^{k-1}Y_\xi)\left(\frac{t}{\varepsilon}; \tilde{u}_0(x) + C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon}} - 1); \varepsilon\right)\nabla \tilde{u}_0.$$

Since $Y(\frac{t}{\varepsilon}; \tilde{u}_0(x) + C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon}} - 1)) = 0$ on $\partial\Omega_t^+[w^{\varepsilon,+}]$, we have that $\nabla(w^{\varepsilon,+})^k(x,t) \to 0$ as $(x,t) \to (x_0,t)$ with $x_0 \in \partial\Omega_t^+[w^{\varepsilon,+}]$. This verifies the condition (i) in Lemma 3.2.

Next, we show that $\mathcal{L}(w^{\varepsilon,+}) \ge 0$ in $\{(x,t) \in \Omega \times (0,t^\varepsilon) : w^{\varepsilon,+}(x,t) > 0\}$ where $\mathcal{L}$ is defined in (3.2). We have

$$\begin{aligned} w_t^{\varepsilon,+} &= \frac{Y_\tau}{\varepsilon} + C_5 \varepsilon \mu(\varepsilon) e^{\mu(\varepsilon)\frac{t}{\varepsilon}} Y_\xi, \\ \nabla w^{\varepsilon,+} &= Y_\xi \nabla \tilde{u}_0, \\ \Delta w^{\varepsilon,+} &= Y_{\xi\xi} |\nabla \tilde{u}_0|^2 + Y_\xi \Delta \tilde{u}_0. \end{aligned}$$

Also note that

$$\begin{aligned} \Delta(m(w^{\varepsilon,+})^k) =& (w^{\varepsilon,+})^k \Delta m + 2k(w^{\varepsilon,+})^{k-1}\nabla m \nabla w^{\varepsilon,+} + m\Delta(w^{\varepsilon,+})^k \\ =& (w^{\varepsilon,+})^k \Delta m + 2k(w^{\varepsilon,+})^{k-1}\nabla m \nabla w^{\varepsilon,+} \\ &+ m(k(w^{\varepsilon,+})^{k-1}\Delta w^{\varepsilon,+} + k(k-1)(w^{\varepsilon,+})^{k-2}|\nabla w^{\varepsilon,+}|^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\Delta(m(w^{\varepsilon,+})^k)}{m} = Y^k \frac{\Delta m}{m} &+ 2kY^{k-1}Y_\xi \frac{\nabla m}{m} \nabla \tilde{u}_0 + kY^{k-1}(Y_{\xi\xi}|\nabla \tilde{u}_0|^2 \\ &+ Y_\xi \Delta \tilde{u}_0) + k(k-1)Y^{k-2}Y_\xi^2 |\nabla \tilde{u}_0|^2. \end{aligned}$$

Thus, in view of Lemma 4.1

$$\begin{aligned}\mathcal{L}(w^{\varepsilon,+}) = &\left[\frac{Y_\tau}{\varepsilon} - \frac{f(Y)}{\varepsilon} - \varepsilon Y^k \frac{\Delta m}{m}\right] \\ &+ \varepsilon Y_\xi \Big[C_5 \mu(\varepsilon) e^{\mu(\varepsilon)\frac{t}{\varepsilon}} - 2kY^{k-1} \frac{\nabla m}{m} \nabla \tilde{u}_0 - kY^{k-1} (\frac{Y_{\xi\xi}}{Y_\xi} |\nabla \tilde{u}_0|^2 + \Delta \tilde{u}_0) \\ &\qquad - k(k-1) Y^{k-2} Y_\xi |\nabla \tilde{u}_0|^2 \Big] \\ &= [1 + O(\varepsilon)] + \varepsilon Y_\xi \left[C_5 \mu(\varepsilon) e^{\mu(\varepsilon)\frac{t}{\varepsilon}} + O(1) + O(1) e^{\mu(\varepsilon)\frac{t}{\varepsilon}} \right],\end{aligned}$$

By choosing C_5 large enough and ε_0 small enough we obtain $\mathcal{L}(w^{\varepsilon,+}) \geq 0$. Thus $w^{\varepsilon,+}$ is a super-solution and similarly $w^{\varepsilon,-}$ is a sub-solution.

By Lemma 4.2 (i) and the fact that $w^{\varepsilon,\pm}$ are a sub- and super-solutions we can deduce that

$$0 \leq w^{\varepsilon,-} \leq u^\varepsilon \leq w^{\varepsilon,+} \leq 1 + \eta_0$$

which implies (2.6). We now show (2.7) and (2.8). Let $M_0 \geq \tilde{d}^{-1}(C_4 + 2C_5) > C_4 + 2C_5$ where the last inequality holds since $0 < \tilde{d} < 1$. And we choose ε_0 small enough such that $(C_4 + 2C_5)\varepsilon_0 \leq 1$. Also, since we only consider the case $0 < t < t^\varepsilon$, by choosing ε_0 small enough we obtain

$$e^{(\mu(\pm\varepsilon)-1)\frac{t}{\varepsilon}} \leq e^{C_1 t^\varepsilon} \leq 2. \tag{4.3}$$

Then, for any point $x \in \Omega$ satisfying $u_0(x) \geq M_0\varepsilon$ which is equivalent to $\tilde{u}_0(x) \geq M_0\varepsilon$ we have

$$\begin{aligned}\tilde{u}_0(x) - C_5\varepsilon^2(e^{\mu(-\varepsilon)\frac{t}{\varepsilon}} - 1) &\geq M_0\varepsilon - C_5\varepsilon^2(e^{\frac{t}{\varepsilon}} e^{(\mu(-\varepsilon)-1)\frac{t}{\varepsilon}} - 1) \\ &\geq M_0\varepsilon - C_5\varepsilon^2(2e^{\frac{t^\varepsilon}{\varepsilon}} - 1) \\ &\geq M_0\varepsilon - C_5\varepsilon(2 - \varepsilon) \geq C_4\varepsilon,\end{aligned}$$

where the first inequality holds by (4.3) and the last inequality holds since $M_0 \geq C_4 + 2C_5$. Thus, if $u_0(x) \geq M_0\varepsilon$ we deduce frrom Lemma 4.2 (ii) that

$$\begin{aligned}u^\varepsilon(x, t^\varepsilon) &\geq w^{\varepsilon,-}(x, t^\varepsilon) \\ &= \left[Y\left(|\ln\varepsilon|; \tilde{u}_0(x) - C_5\varepsilon^2(e^{\mu(-\varepsilon)\frac{t}{\varepsilon}} - 1); -\varepsilon \right) \right]^+ \geq 1 - \eta_0,\end{aligned}$$

which completes the proof of (2.7). And for any $x \in \Omega$ satisfying $dist(x, \Omega_0) \geq M_0\varepsilon$ we deduce from (4.1) and (4.3) that

$$\begin{aligned}\tilde{u}_0(x) + C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon}} - 1) &\leq \min(-\tilde{d}\, dist(x, \Omega_0), -1) + C_5\varepsilon^2(e^{\frac{t}{\varepsilon}} e^{(\mu(-\varepsilon)-1)\frac{t}{\varepsilon}} - 1) \\ &\leq \min(-\tilde{d}\, dist(x, \Omega_0), -1) + C_5\varepsilon(2 - \varepsilon) \\ &\leq \min(-\tilde{d} M_0\varepsilon, -1) + C_5\varepsilon(2 - \varepsilon) \leq -C_4\varepsilon.\end{aligned} \tag{4.4}$$

Thus again by Lemma 4.2 (ii) we deduce from (4.4) that

$$\begin{aligned}u^\varepsilon(x, t) &\leq w^{\varepsilon,+}(x, t) \\ &= \left[Y\left(\frac{t}{\varepsilon}; \tilde{u}_0(x) + C_5\varepsilon^2(e^{\mu(\varepsilon)\frac{t}{\varepsilon}} - 1); \varepsilon \right) \right]^+ = 0,\end{aligned}$$

if $dist(x, \Omega_0) \geq M_0\varepsilon$, where the last equality holds by Lemma 4.2 which implies (2.8).

5. **Motion of interface.** In this section, we prove Theorem 2.2. Let U be the travelling wave solution with the minimum speed c_k,

$$\begin{cases} (U^k)''(z) + c_k U'(z) + f(U) = 0 & \text{for all } z \in \mathbf{R}, \\ U(-\infty) = 1, & \\ U(z) > 0 & \text{for all } z < 0, \\ U(z) = 0 & \text{for all } z \geq 0. \end{cases} \tag{5.1}$$

Lemma 5.1 ([11]). *For all $z \in (-\infty, 0)$, we have $U'(z) < 0$. The travelling wave U is smooth outside 0. Moreover, there exist $C > 0$ and $\beta > 0$ such that the following properties hold:*

$$|(U^k)'(z)| \le CU(z) \quad \text{for all } z \in \mathbf{R}, \tag{5.2}$$

$$0 < 1 - U(z) \le Ce^{-\beta|z|} \quad \text{for all } z < 0, \tag{5.3}$$

$$|zU'(z)| \le CU(z) \quad \text{for all } z < -1. \tag{5.4}$$

The cut-off signed distance function: Let $\overline{d}(x,t)$ be the signed distance function to Γ_t. Choose $d_0 > 0$ small enough such that $\overline{d}(x,t)$ is smooth in the tubular neighborhood $\{(x,t) \in \Omega \times [0,T] : \overline{d}(x,t) < 3d_0\}$ of $\Gamma_{[0,T]}$ and satisfies $|\nabla \overline{d}| = 1$. Let $\zeta(s)$ be a smooth increasing function on $\mathbf{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \le d_0, \\ -2d_0 & \text{if } s \le -2d_0, \\ 2d_0 & \text{if } s \ge 2d_0. \end{cases}$$

We define the modified signed distance d by

$$d(x,t) = \zeta(\overline{d}(x,t)).$$

Note that

$$\{(x,t) \in \Omega \times [0,T] : |\overline{d}(x,t)| < d_0\} = \{(x,t) \in \Omega \times [0,T] : |d(x,t)| < d_0\},$$

and that d coincides with $\bar{d}$ in that region. We have the following properties

$$|\nabla d(x,t)|^2 + |\Delta d(x,t)| \le D_1, \tag{5.5}$$

$$|d_t + c_k |\nabla d|^2| \le D_2 |d(x,t)|, \tag{5.6}$$

for all $(x,t) \in Q_T$.

5.1. **Proof of Theorem 2.2.** We construct a pair of functions $u^{\varepsilon,\pm}$

$$u^{\varepsilon,\pm}(x,t) := (1 \pm q(t))U\left(\frac{d(x,t) \mp \varepsilon p(t)}{\varepsilon}\right) = aU\left(\frac{d(x,t) \mp \varepsilon p(t)}{\varepsilon}\right),$$

where

$$p(t) = -e^{-\frac{t}{\varepsilon}} + e^{Lt} + K,$$
$$q(t) = \sigma(e^{-\frac{t}{\varepsilon}} + \varepsilon L e^{Lt}),$$
$$a := 1 \pm q(t).$$

We first prove that $u^{\varepsilon,\pm}$ are in fact a sub- and super-solution.

Lemma 5.2. *Fix $K \ge 1$. For any fixed $\sigma \in (0, \sigma_0)$ for some σ_0 small enough, there exist $L = L(\sigma, K) > 0$ large enough and $\varepsilon_0 = \varepsilon_0(K, L)$ small enough such that for every $\varepsilon \in (0, \varepsilon_0)$,*
(i) $\mathcal{L}(u^{\varepsilon,+}) \ge 0$ in $\{(x,t) \in \Omega \times [0,T] : u^{\varepsilon,+}(x,t) > 0\}$,
(ii) $\mathcal{L}(u^{\varepsilon,-}) \le 0$ in $\{(x,t) \in \Omega \times [0,T] : u^{\varepsilon,-}(x,t) > 0\}$.

Proof. We only prove (i) and provide a proof under the hypotheses that $\varepsilon_0 L e^{LT} < 1$ and $\sigma < 1/4$ which implies that $0 < q(t) < 1/2$. Also Since $u^{\varepsilon,+}(x,t) = 0$ if $d(x,t) \ge \varepsilon p(t)$, we only need to consider the region $\{(x,t) \in Q_T, d(x,t) < \varepsilon p(t)\}$.

Direct computation gives

$$u_t^{\varepsilon,+} = aU'\left(\frac{d_t}{\varepsilon} - p'(t)\right) + q'(t)U,$$
$$\nabla((u^{\varepsilon,+})^k) = a^k (U^k)' \frac{\nabla d}{\varepsilon}$$
$$\Delta((u^{\varepsilon,+})^k) = a^k \left[(U^k)'' \frac{|\nabla d|^2}{\varepsilon^2} + (U^k)' \frac{\Delta d}{\varepsilon}\right]$$

Thus, we have

$$\begin{aligned} \varepsilon^2 \frac{\Delta(m(u^{\varepsilon,+})^k)}{m} &= a^k \left[(U^k)''|\nabla d|^2 + \varepsilon (U^k)' \Delta d\right] + 2\varepsilon \frac{\nabla m}{m} a^k (U^k)' \nabla d + \varepsilon^2 \frac{\Delta m}{m} a^k U^k \\ &= a^k \left[-|\nabla d|^2 (c_k U' + f(U)) + \varepsilon (U^k)' \Delta d\right] \\ &\quad + 2\varepsilon \frac{\nabla m}{m} a^k (U^k)' \nabla d + \varepsilon^2 \frac{\Delta m}{m} a^k U^k. \end{aligned}$$

By grouping the terms containing U' (as well U) together and using (5.1), we obtain

$$\begin{aligned}\varepsilon L(u^{\varepsilon,+}) &= \varepsilon u_t^{\varepsilon,+} - \varepsilon^2\frac{\Delta(m(x)(u^{\varepsilon,+})^k)}{m(x)} - f(u^{\varepsilon,+})\\ &= aU'\left(d_t - \varepsilon p'(t)\right) + a^k c_k|\nabla d|^2U'\\ &\quad + \varepsilon q'(t)U + a^k|\nabla d|^2 f(U) - f(aU)\\ &\quad - \varepsilon a^k(U^k)'\Delta d - 2\varepsilon\frac{\nabla m}{m}a^k(U^k)'\nabla d - \varepsilon^2\frac{\Delta m}{m}a^kU^k\\ &= T_1 + T_2 + T_3\end{aligned}$$

Step 1: Using (2.2), (5.2) and (5.5), we deduce that

$$|T_3| \le C_7\varepsilon a^k U \tag{5.7}$$

Step 2: Since $f(u) = u(1-u)$ we have

$$T_2 = U\Big[\varepsilon q'(t) + a^k(|\nabla d|^2 - 1)(1-U) + a^k(1-U) - a(1-aU)\Big].$$

First we show that

$$\Big|(|\nabla d|^2 - 1)|(1-U)\Big| \le C_8\varepsilon. \tag{5.8}$$

Indeed, if $-d_0 < d(x,t) < \varepsilon p(t)$, then for $\varepsilon > 0$ small enough, $|\nabla d(x,t)| = 1$, hence (5.8) is trivial. Next, if $d(x,t) \le -d_0$, (5.3) and (5.5) implies that

$$\begin{aligned}\Big|(|\nabla d|^2 - 1)(1-U)\Big| &= \Big|(|\nabla d|^2 - 1)\Big|\left(1 - U\left(\frac{d(x,t) - \varepsilon p(t)}{\varepsilon}\right)\right)\\ &\le \Big|(|\nabla d|^2 - 1)\Big|(1 - U(-d_0/\varepsilon))\\ &\le (D_1^2 + 1)Ce^{-\beta\frac{d_0}{\varepsilon}}\end{aligned}$$

where the first inequality holds since U is a decreasing function. Hence (5.8) follows from the fact that $xe^{-x} \to 0$ as $x \to \infty$, which means that $e^{-x} \le x^{-1}$ for x large enough. Therefore, using (5.7) and (5.8), we conclude that

$$\begin{aligned}T_2 + T_3 &\ge U[\varepsilon q'(t) + a^k(1-U) - a(1-aU) - \varepsilon(C_7 + C_8)a^k]\\ &= U[\varepsilon q'(t) + (a^k - a) - (a^k - a^2)U - \varepsilon(C_7 + C_8)a^k].\end{aligned}$$

Since $k \ge 2$, $a = 1 + q > 1$ and $0 < U < 1$ we have

$$\begin{aligned}T_2 + T_3 &\ge U[\varepsilon q'(t) + a(a-1) - \varepsilon(C_7 + C_8)a^k]\\ &\ge U[\varepsilon q'(t) + q(t) - \varepsilon(C_7 + C_8)a^k].\end{aligned} \tag{5.9}$$

Step 3: We will show that there exists a constant $C_9 > 0$ such that

$$T_1 \ge -\varepsilon C_9 aU. \tag{5.10}$$

First note that

$$T_1 = aU'\left(d_t - \varepsilon p'(t)\right) + a^k c_k|\nabla d|^2U' = aU'[d_t - \varepsilon p'(t) + a^{k-1}c_k|\nabla d|^2] = aU'T_1^*.$$

We estimate T_1^*,

$$\begin{aligned}T_1^* &= d_t - \varepsilon p'(t) + a^{k-1}c_k|\nabla d|^2\\ &= d_t + c_k|\nabla d|^2 - \varepsilon p'(t) + (a^{k-1} - 1)c_k|\nabla d|^2\\ &\le D_2|d(x,t)| - \varepsilon p'(t) + (a^{k-1} - 1)c_k|\nabla d|^2 \quad \text{by (5.6)}\\ &\le D_2|d(x,t) - \varepsilon p(t)| + \varepsilon D_2 p(t) - \varepsilon p'(t) + (a^{k-1} - 1)c_k|\nabla d|^2\\ &= D_2|d(x,t) - \varepsilon p(t)| + \varepsilon[D_2 p(t) - p'(t)] + (a^{k-1} - 1)c_k|\nabla d|^2.\end{aligned}$$

Note that by the mean value theorem and the fact that $0 < q(t) < 1$, we have the following inequality

$$0 < a^{k-1} - 1 = (1 + q(t))^{k-1} - 1 \le C^*(k-1)q(t).$$

This together with (5.5) yields that

$$\begin{aligned}T_1^* &\le D_2|d(x,t) - \varepsilon p(t)| + \varepsilon[D_2 p(t) - p'(t)] + C^*(k-1)D_1c_kq(t)\\ &=: D_2|d(x,t) - \varepsilon p(t)| + J.\end{aligned} \tag{5.11}$$

Claim: We will show that $J \leq -\varepsilon D_2 p(t) < 0$. Indeed, using the identity

$$p'(t) = \frac{e^{-t/\varepsilon}}{\varepsilon} + Le^{Lt},$$

and choosing $\sigma_0 > 0$ small enough such that $C^* D_1 (k-1) c_k \sigma_0 < 1/2$ we have

$$\begin{aligned} J + \varepsilon D_2 p(t) &\leq \varepsilon D_2 [-e^{-\frac{t}{\varepsilon}} + e^{Lt} + K] - \varepsilon [\frac{e^{-t/\varepsilon}}{\varepsilon} + Le^{Lt}] + C^* D_1 (k-1) c_k \sigma [e^{-\frac{t}{\varepsilon}} + \varepsilon L e^{Lt}] \\ &\leq \varepsilon D_2 [-e^{-\frac{t}{\varepsilon}} + e^{Lt} + K] - \varepsilon [\frac{e^{-t/\varepsilon}}{\varepsilon} + Le^{Lt}] + \frac{1}{2} [e^{-\frac{t}{\varepsilon}} + \varepsilon L e^{Lt}] \\ &\leq -\varepsilon D_2 e^{-t/\varepsilon} + \varepsilon \Big[e^{Lt} (D_2 - L/2) + D_2 K \Big] \\ &\leq 0, \end{aligned}$$

provided that L is large enough.

Next we consider two cases. In the first case, if $0 \leq d(x,t) \leq \varepsilon p(t)$, it follows (5.11) that $T_1^* \leq \varepsilon D_2 p(t) + J$ and hence $T_1^* \leq 0$ in view of the claim above. As a consequence, $T_1 = aU' T_1^* \geq 0$ which implies (5.10). In the second case where $d(x,t) \leq 0$, first note that

$$\frac{d - \varepsilon p(t)}{\varepsilon} \leq -K \leq -1. \tag{5.12}$$

Since by the claim above, $J \leq 0$ which implies by (5.11) that $T_1^* \leq D_2 |d(x,t) - \varepsilon p(t)|$ and hence $T_1 \geq a D_2 U' |d(x,t) - \varepsilon p(t)|$. Therefore (5.4) and (5.12) imply that $T_1 \geq -\varepsilon C_9 a U$. This completes the proof of (5.10).

Choose $\bar{C} > 2^k (C_8 + C_7) + 2C_9$. Combining (5.9), (5.10) and the fact that $1 < a = 1 + q < 3/2$ we obtain

$$\begin{aligned} \varepsilon L(u^{\varepsilon,+}) &\geq U[\varepsilon q'(t) + q - \varepsilon (C_7 + C_8) a^k - \varepsilon C_9 a] \\ &\geq U[\varepsilon q'(t) + q - \varepsilon \bar{C}] \\ &= U[\sigma(-e^{-\frac{t}{\varepsilon}} + \varepsilon^2 L^2 e^{Lt}) + \sigma (e^{-\frac{t}{\varepsilon}} + \varepsilon L e^{Lt}) - \varepsilon \bar{C}] \\ &\geq U[\sigma \varepsilon^2 L^2 e^{Lt} + \sigma \varepsilon L e^{Lt} - \varepsilon \bar{C}] \\ &= \varepsilon U [\sigma \varepsilon L^2 e^{Lt} + \sigma L e^{Lt} - \bar{C}] \\ &\geq 0, \end{aligned}$$

provided that $\sigma L \geq \bar{C}$. This completes the proof of Lemma 5.2.

□

Proof of Theorem 2.2: Choose $\sigma < \min(\sigma_0, \eta_1)$ and let $\eta_0 = \sigma/2$ in Theorem 2.1. Then (2.6) implies that there exists a positive constant ε_0 such that

$$0 \leq u^\varepsilon (x, t^\varepsilon) \leq 1 + \frac{\sigma}{2} < 1 + \eta_1$$

for $x \in \Omega$ and $\varepsilon \in (0, \varepsilon_0)$. Also, by choosing ε_0 small enough we obtain that $\mathcal{L}(0) \leq 0 \leq \mathcal{L}(1 + \eta_1)$ for any $\varepsilon \in (0, \varepsilon_0)$, which in turn imply (2.9).

We now prove (2.10) and (2.11). Note that by (2.4) we have

$$u_0(x) \geq -\frac{M_0}{C} d(x,0)$$

for all $x \in \Omega_0$, where M_0 is the positive constant defined in Theorem 2.1 and $C \geq M_0$ is a large enough positive constant. This inequality implies that

$$u_0(x) \geq M_0 \varepsilon$$

for $x \in \Omega$ satisfying $d(x,0) \leq -C\varepsilon$. Thus by Theorem 2.1 we have

$$u^\varepsilon (x, t^\varepsilon) \leq 1 + \frac{\sigma}{2} \qquad \text{for } x \in \Omega,\ u^\varepsilon(x, t^\varepsilon) = 0 \qquad \text{if } d(x,0) \geq C\varepsilon \tag{5.13}$$

$$u^\varepsilon (x, t^\varepsilon) \geq 0 \qquad \text{for } x \in \Omega,\ u^\varepsilon(x, t^\varepsilon) \geq 1 - \frac{\sigma}{2} \qquad \text{if } d(x,0) \leq -C\varepsilon. \tag{5.14}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Since $q(0) \geq \sigma$, we may fix $K > 0$ large enough such that

$$(1 + q(0)) U(C - K) \geq 1 + \frac{\sigma}{2},\ U(-C + K) = 0$$

which is possible by (5.3).

This implies that

$$u^{\varepsilon,+}(x,0) \geq (1+q(0))U(C-K) \geq 1+\frac{\sigma}{2} \qquad \text{if } d(x,0) \leq C\varepsilon \tag{5.15}$$

$$u^{\varepsilon,-}(x,0) \leq (1-q(0))U(-C+K) = 0 \qquad \text{if } d(x,0) \geq -C\varepsilon \tag{5.16}$$

Also by (5.3) we obtain

$$0 \leq u^{\varepsilon,+}(x,0),\ u^{\varepsilon,-}(x,0) \leq 1-q(0) \leq 1-\sigma < 1-\frac{\sigma}{2} \tag{5.17}$$

for $x \in \Omega$. Then (5.13) and (5.15) implies that $u^\varepsilon(x,t^\varepsilon) \leq u^{\varepsilon,+}(x,0)$ for x satisfying $d(x,0) \geq C\varepsilon$ and by (5.13) and (5.17) implies that $u^\varepsilon(x,t^\varepsilon) \leq u^{\varepsilon,+}(x,0)$ for x satisfying $d(x,0) < C\varepsilon$. Thus we obtain $u^\varepsilon(x,t^\varepsilon) \leq u^{\varepsilon,+}(x,0)$ for $x \in \Omega$. With the similar reasoning using (5.14), (5.16) and (5.17) we obtain that $u^{\varepsilon,-}(x,0) \leq u^\varepsilon(x,t^\varepsilon)$ in $x \in \Omega$, thus we have

$$u^{\varepsilon,-}(x,0) \leq u^\varepsilon(x,t^\varepsilon) \leq u^{\varepsilon,+}(x,0) \text{ (Ordering initial data).}$$

Therefore, Lemma 5.2 and the comparison principle imply that

$$u^{\varepsilon,-}(x,t) \leq u^\varepsilon(x,t+t^\varepsilon) \leq u^{\varepsilon,+}(x,t)$$

for $t \in [0, T-t^\varepsilon]$. Then, we choose M_1 large enough such that

$$(1+q(t))U(M_1-p(t)) = 0$$
$$(1-q(t))U(-M_1+p(t)) \geq 1-\eta_1$$

for $t \in [0, T-t^\varepsilon]$ which is possible since $\sigma < \eta_1$ and by (5.3). Thus, from the fact that U is a decreasing function we obtain that

$$u^\varepsilon(x,t) \geq (1-q(t))U(-M_1+p(t)) \geq 1-\eta_1 \qquad \text{if } d(x,t) \leq -\varepsilon M_1,$$
$$u^\varepsilon(x,t) \leq (1+q(t))U(M_1-p(t)) = 0 \qquad \text{if } d(x,t) \geq \varepsilon M_1,$$

which implies (2.10) and (2.11). Therefore we obtain Theorem 2.2.

6. **Appendix.** Here we describe the extension of the initial condition u_0 to $\tilde{u}_0$ which was used in Section 4.2. By Whitney extension theorem one can extend $u_0 \in C^2(\overline{\Omega}_0)$ to $\overline{u}_0 \in C^2(\overline{\Omega})$. Moreover, since it's not guaranteed that $\overline{u}_0 < 0$ in $\Omega \setminus \Omega_0$, we modify further the function $\overline{u}_0$. Note that the condition (2.4) implies that there exists a positive constant $\tilde{d} < 1$ such that

$$\overline{u}_0(x) < -\tilde{d}\ dist(x, \Omega_0) \tag{6.1}$$

for any $x \in \Omega \setminus \Omega_0$ satisfying $0 < dist(x,\Omega_0) < \tilde{d}$. Let $\rho : D \to [0,1]$ be a smooth function satisfying

$$\rho(x) \in \begin{cases} 1 & \text{if } 0 \leq dist(x,\Omega_0) < \tilde{d}/2 \\ (0,1) & \text{if } \tilde{d}/2 \leq dist(x,\Omega_0) < \tilde{d} \\ 0 & \text{if } dist(x,\Omega_0) \geq \tilde{d}. \end{cases}$$

Then we define $\tilde{u}_0 : \Omega \to \mathbf{R}$ by

$$\tilde{u}_0(x) := \rho(x)\overline{u}_0(x) - (1-\rho(x)).$$

Then, since $\rho(x) = 1$ in $\{x \in \Omega, dist(x,\Omega_0) = 0\} = \Omega_0$ we have $\tilde{u}_0 = u_0$ in Ω_0. Moreover, since $\tilde{d} < 1$, we have $\tilde{d}\ dist(x,\Omega_0) < 1$ in $\{x \in \Omega, 0 < dist(x,\Omega_0) < \tilde{d}\}$, which implies in view of (6.1) that

$$\tilde{u}_0(x) < -\rho(x)\tilde{d}\ dist(x,\Omega_0) - (1-\rho(x))\tilde{d}\ dist(x,\Omega_0) = -\tilde{d}\ dist(x,\Omega_0)$$

for $0 < dist(x,\Omega_0) < \tilde{d}$. Moreover, $\tilde{u}_0 = -1$ in $\{x \in \Omega, dist(x,\Omega_0) > \tilde{d}\}$.

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Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.

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