BIOLOGICAL INVASION WITH A POROUS MEDIUM TYPE DIFFUSION IN A HETEROGENEOUS SPACE

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ABSTRACT. The knowledge of traveling wave solutions is the main tool in the study of wave propagation. However, in a spatially heterogeneous environment, traveling wave solutions do not exist and a different approach is needed. In this paper, we study the generation and the propagation of hyperbolic scale singular limits of a KPP-type reaction-diffusion equation when the carrying capacity is spatially heterogeneous and the diffusion is of a porous medium equation type. We show that the interface propagation speed varies according to the carrying capacity.

1. INTRODUCTION

The purpose of this paper is to understand the effect of spatial heterogeneity on the invasion speed in KPP-type reaction-diffusion equations. More specifically, we show the generation of a sharp interface of the solution of the initial value problem,

(1) EqnU
$$\begin{cases} U_t(x,t) = \varepsilon \Delta U^\ell + \frac{1}{\varepsilon} U\big(1 - \frac{U}{m}\big), & (x,t) \in D \times \mathbb{R}^+, \\ \frac{\partial U}{\partial \nu} = 0, & (x,t) \in \partial D \times \mathbb{R}^+, \\ U(x,0) = U_0(x) \geq 0, & x \in D, \end{cases}$$

and then obtain the propagation speed of the interface when $\varepsilon \to 0$. The solution $U(x,t)$ is the population density of a single species, the domain $D \subset \mathbb{R}^N$ is smooth and bounded, and the vector ν is the outward unit normal vector on the boundary of the domain. In this model, we take the nonlinear diffusion with a constant exponent $\ell \geq 2$. The spatial heterogeneity is placed in the carrying capacity, $m = m(x) > 0$, which satisfies

(2) m1
$$m \in C^2(D), \quad c_m \leq m \text{ and } \; m + |\nabla m| + |\Delta m| \leq C_m$$

for some constants $C_m, c_m > 0$.

The problem (1) is obtained after a hyperbolic scaling, $x \to \varepsilon x$ and $t \to \varepsilon t$, of a multi-scale problems, where the heterogeneity in $m(x)$ is of macroscopic scale. Evans and Sougandis [6, Eq. (1.1)] considered such hyperbolic multi-scale problem for a general heterogeneous reaction function. However, the reaction function in (1) does not satisfy their assumptions. Since the wave speed of the problem is invariant under the hyperbolic scaling of the problem, this approach provides the wave propagation speed in a heterogeneous environment. Hilhorst *et al.* [8] considered a homogeneous case with $m(x) = 1$ and showed that the solution $U(x,t)$ converges to 0 or $m(x)$ as $\varepsilon \to 0$ and the interface moves with a constant speed to the normal direction, i.e.,

(3) Vn0
$$V_n = c_0,$$

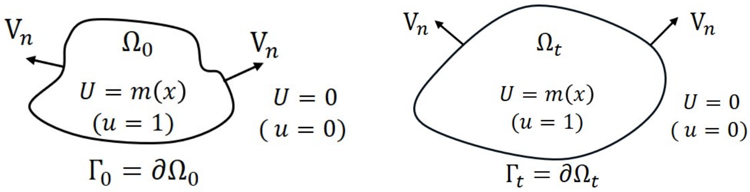

FIGURE 1. A diagram for interface propagation of the singular limit.

⟨fig1⟩

where V_n denotes the speed of the propagating interface in the normal direction and $c_0 > 0$ is the constant wave speed of a traveling wave solution in one space dimension (see Figure 1). The constant speed c_0 depends on the nonlinearity ℓ, but not on the space dimension $d > 0$.

In this paper, we extend the result of the homogeneous case to a heterogeneous one and show that

(4) Vn
$$V_n = c_0 m^p \quad \text{for} \quad p := \frac{\ell - 1}{2},$$

where the invasion speed is not constant anymore. There are three interesting observations in this relation. First, even if there is no traveling wave solution in the heterogeneous case, the traveling wave speed c_0 of the homogeneous case with $m = 1$ still plays a key role. Second, if (4) holds for the linear diffusion case $\ell = 1$, which is beyond the parameter regime of the paper, the invasion speed is same as the homogeneous case when $\ell = 1$. It is related to the well-known fact that the invasion speed is decided by the first order term of the reaction, and the coefficient of the first order term of our model is constant. Third, the relation (4) says that such a well-known fact is true only for the linear diffusion case and the wave speed depends on the second order term for the nonlinear diffusion case.

To obtain the nonconstant invasion speed in a heterogeneous environment, we first transfer the spatial heterogeneity in the reaction function to the diffusion operator and obtain a reaction function that satisfies the hypotheses of Evans and Sougandis [6, (1.2)–(1.4)]. To that purpose we rewrite the equation in terms of the ratio

$$u(x,t) := \frac{U(x,t)}{m(x)}.$$

This ratio is the population per unit resource and has been used as a starvation measure in [9]. Then, (1) becomes

(5) eqn
$$\begin{cases} u_t(x,t) = \frac{\varepsilon}{m}\Delta(mu)^\ell + \frac{1}{\varepsilon}u(1-u), & (x,t) \in D \times \mathbb{R}^+, \\ \frac{\partial (mu)^\ell}{\partial \nu} = 0, & (x,t) \in \partial D \times \mathbb{R}^+, \\ u(x,0) = u_0(x) := \frac{U_0(x)}{m(x)}, & x \in D. \end{cases}$$

Here, we assume that the initial value is uniformly bounded by

(6) u01
$$0 \le u_0(x) \le 1,$$

has a smooth and simply connected compact support such that

(7) u02
$$\Omega_0 := \{x \in D : u_0(x) > 0\} \subset\subset D \quad \text{and} \quad \Gamma_0 := \partial\Omega_0 \in C^{3+\alpha}$$

for some $0 < \alpha < 1$, and has smoothness and boundary steepness such that for a constant $C_0 > 0$,

$$u_0 \in C^2(D) \ \text{ and } \ \frac{\partial u_0}{\partial \nu_0}(x_0) < -C_0 \ \text{ for } \ x_0 \in \Gamma_0. \tag{8}$$ u032

The vector ν_0 is the outward unit normal vector on the boundary Γ_0 of the support of the initial value.

Notice that the support Ω_0 of the initial value is not assumed to be convex since it does not mean much. For a homogeneous problem case, the interface of the solution moves with constant speed (3) and the convexity of the solution support is preserved. However, for a heterogeneous case, the corresponding flow is (4) and the convexity of the support of the solution may break. Instead of the convexity, the support $\overline{\Omega_0}$ is assumed to be simply connected and hence, the boundary $\Gamma_0 := \partial\Omega_0$ is a simple loop and divides the domain $D \subset \mathbb{R}^d$ into two regions. See section 6 for a numerical example of the interface motion (12).

2. Main results

The main result of the paper consists of two theorems. The first one shows how the interface is created.

⟨thm1⟩ **Theorem 1** (Generation of interface). *Let $m(x)$ satisfy (2) and $u_0(x)$ satisfy (6)–(8). Let $u^\varepsilon(x,t)$ be the solution of (5) in a weak sense and let $0 < \eta_g < 1/4$. Then there exist positive constants ε_0, M_G such that for any $\varepsilon \in (0, \varepsilon_0)$ the followings holds.*

(i) *There exists a positive constant $\eta_\varepsilon := \eta_\varepsilon(\varepsilon)$ such that*

$$0 \le u^\varepsilon(x,t) \le 1+\eta_\varepsilon, \ \ \eta_\varepsilon \downarrow 0 \ \text{as } \varepsilon \downarrow 0. \tag{9}$$ thm11

(ii)

$$u^\varepsilon(x,t^\varepsilon) \ge 1 - \eta_g \quad \text{if} \ \ u_0(x) \ge M_G\varepsilon, \tag{10}$$ thm12

(iii)

$$u^\varepsilon(x,t^\varepsilon) = 0 \quad \text{if} \ \ dist(x,\Omega_0) \ge M_G\varepsilon. \tag{11}$$ thm13

where $t^\varepsilon := \varepsilon|\ln\varepsilon|$.

This theorem provides inner and outer envelopes for the graph of the solution $u^\varepsilon(\cdot, t^\varepsilon)$. The estimate (10) with the boundary steepness (8) and the equality (11) imply that a transition layer of thickness $O(\varepsilon)$ is developed along the boundary Γ_0 of the initial support Ω_0 at the moment of $t^\varepsilon = \varepsilon|\ln\varepsilon|$. If the initial value is larger than $M_G\varepsilon$ at a point $x \in \Omega_0$, $u^\varepsilon(x,t^\varepsilon)$ is between 1 and $1-\eta_g$ by (10). Due to the assumption (8), the layer inside the boundary Γ_0 is of order $\mathcal{O}(\varepsilon)$. Eq. (11) implies that the solution remains equal to zero $u(x,t^\varepsilon) = 0$ on the outside of the layer. Therefore, after taking the limit as $\varepsilon \to 0$, $u^\varepsilon(x,t^\varepsilon)$ converges to a step function, which is 1 for $x \in \Omega_0$ and 0 for $x \notin \Omega_0$. The boundary Γ_0 is the initial interface of discontinuity of this singular limit.

The second theorem is to show that the interface of the step function $u(x,t) := \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ moves according to the relation (4). To make the statement explicit, we first construct a step function with its interface moving according to (4) and then show that $u^\varepsilon(x,t)$ converges to the constructed step function. The boundary $\Gamma_0 := \partial\Omega_0$ divides D into two regions, where the bounded region is called inside and the other outside. We consider a collection of interfaces Γ_t indexed with the time variable $0 \le t \le T$ which is given by the mean curvature flow in heterogeneous field,

$$V_n(x) = c_0 m^p(x) \text{ for } x \in \Gamma_t, \quad \Gamma_t|_{t=0} = \Gamma_0, \tag{12}$$

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where $V_n(x)$ is the speed of the moving interface in the outward normal direction at position $x \in \Gamma_t$ and at time $0 < t < T$. The coefficient c_0 is the traveling wave speed for the homogeneous case with $m = 1$. Under the regularity of m given in (2), the flow is defined well, see Section 3.

We divide the domain D into two parts

$$D = \Omega_t \cup \Omega_t^c,$$

where Ω_t^c is the region bounded by Γ_t and ∂D and Ω_t is the set inside the interface Γ_t. Note that Ω_0 is the support of the initial value $u(x,0) = u_0(x)$. However, Ω_t is not the support of $u(x,t)$. It is simply the interior region bounded by the Γ_t and Γ_t is given by the heterogeneous curvature-flow (4). Finally, using these two sets, we define a signed distance function $\tilde{d}(x,t)$;

$$\begin{cases} \tilde{d}(x,t) = \ \ dist(x,\Gamma_t) & \text{if } \ x \in \Omega_t, \\ \tilde{d}(x,t) = -dist(x,\Gamma_t) & \text{if } \ x \in \Omega_t^c. \end{cases} \tag{13}$$

?tilded?

Note that $\tilde{d}(x,t) \le 0$ if x is in the outer region.

The second theorem is about the propagation of the interface.

⟨thm2⟩ **Theorem 2** (Propagation of interface)**.** *Let $0 < \eta_p < 1/4$ be arbitrary and $t^\varepsilon > 0$ be a constant defined in Theorem 1, $m(x)$ satisfy (2), and $u_0(x)$ satisfy (6)–(8). Then, for a weak solution $u^\varepsilon(x,t)$ of (5), there exist positive constants $\varepsilon_0, M_P \in \mathbb{R}^+$ such that, for all $0 < \varepsilon < \varepsilon_0$, $x \in D$, and $t^\varepsilon < t < T$,*

$$\begin{cases} 0 \le u^\varepsilon(x,t) \le 1{+}\eta_\varepsilon, & \\ u^\varepsilon(x,t) \ge 1 - \eta_g & \text{if } \tilde{d}(x,t) \ge \ \ M_P\varepsilon, \\ u^\varepsilon(x,t) = 0 & \text{if } \tilde{d}(x,t) \le -M_P\varepsilon. \end{cases} \tag{14}$$

thm21

The result (14) in Theorem 2 implies that the interface is generated and propagated following the motion equation (12), with width of $\mathcal{O}(\varepsilon)$. Moreover, by using the similar proof of Theorem 2, one can also conclude that

$$u^\varepsilon(x,t) \to \begin{cases} 1, & x \in \Omega_t, \\ 0, & x \in \Omega_t^c, \end{cases} \quad \text{as } \ \varepsilon \to 0, \tag{15}$$

thm22

see Remark 1 Furthermore, since the boundary $\partial\Omega_t$ is defined by the relation (12) from the beginning, we have obtained the claim of the paper that wave moves with the speed as given in (4).

Note that the solution in the theorem exists for a time interval $[0,T]$, which is only a local solution. For a homogeneous case, the solution support is convex if the initial support is also convex. Hence, we may construct a solution until the interface touches the domain boundary ∂D. If the domain is $\mathbb{R}^d$, the solution exists for all $t > 0$. However, convexity of the solution support is not preserved for a heterogeneous case and the method of the paper fails as soon as the interface Γ_t touches itself.

3. Preliminaries

⟨sec.3⟩ In this section we give the solution definition and some preliminaries which is used in the proof of theorems.

3.1. **Weak solution.** The solution, the super-solution, and the sub-solution of the perturbed problem (5) are defined in a weak sense as follows;

Definition 1. *A function $u : D \times [0,T] \to \mathbb{R}$ is called a super-solution of the singularly perturbed problem* (5) *if*

(i) $mu \in W^{1,\ell}(D \times [0,T])$,
(ii) For any nonnegative test function $\phi \in C_c^1(D \times [0,T]), \phi \geq 0$,

(16) eqn_porous_supersub
$$\int_D u(x,T)\phi(x,T) \geq \int_D u(x,0)\phi(x,0) + \int_0^T \int_D \left(u\phi_t - \varepsilon \nabla (mu)^\ell \cdot \nabla \frac{\phi}{m} + \frac{1}{\varepsilon} u(1-u)\phi \right).$$

If (16) *is satisfied with the opposite inequality, u is called a sub-solution. The function u is called a solution if u is super-solution and sub-solution at the same time.*

Note that the product mu is in $W^{1,\ell}$, not the solution u. Next, we introduce two basic lemmas. The first lemma is a classical comparison principle (see [10]).

⟨lemma1⟩ **Lemma 1** (Comparison principle)**.** *Let $\overline{u}$ and $\underline{u}$ be super- and sub-solution of* (5)*, respectively. If $\overline{u}(x,0) \geq \underline{u}(x,0)$ for all $x \in D$. Then*

$$\overline{u}(x,t) \geq \underline{u}(x,t), \quad (x,t) \in D \times [0,T]$$

For the estimate in the following sections, we construct smooth super- and sub-solutions. The second lemma is to give sufficient conditions for u to be a super- or a sub- solution. Denote

$$\mathcal{L}u := u_t - \frac{\varepsilon}{m}\Delta(mu)^\ell - \frac{1}{\varepsilon}u(1-u).$$

⟨lem_supersub⟩ **Lemma 2.** *Let m satisfy the conditions in* (2) *and u be a differentiable nonnegative function defined on $D \times [0,T]$. Let $D_t^+ := \{x \in D : u(x,t) > 0\}$, n_t be the outward normal vector on ∂D_t^+, and the surface $\cup_{t\in[0,T]} \partial D_t^+ \times \{t\} \subset D \times [0,T]$ be smooth enough. Suppose u satisfies the following three conditions; (i) $u^\ell \in C^1(D \times [0,T])$, (ii) $\dfrac{\partial (mu)^\ell}{\partial n_t} \geq 0$ on ∂D_t^+, and (iii) $\mathcal{L}(u) \geq 0$ in D_t^+. Then, u is a super-solution. If the inequalities in (ii) and (iii) hold in the opposite direction, u is a sub-solution.*

Proof. We will prove the theorem only for a super-solution case. The sub-solution case can be proved similarly. For a nonnegative test function $\phi \in C_c^1(D \times [0,T])$, we have

$$\frac{d}{dt}\left(\int_D u\phi\right) = \frac{d}{dt}\left(\int_{D_t^+} u\phi\right) = \int_{D_t^+}(u\phi_t + u_t\phi) + \int_{\partial D_t^+} u\phi V_t$$
$$= \int_{D_t^+}(u\phi_t + u_t\phi),$$

where V_t is the speed of the propagating interface ∂D_t^+ in the outward normal direction. The last equality holds since $u = 0$ on ∂D_t^+. Integrating both sides over $[0, T]$ gives

$$\int_0^T \int_{D_t^+} u\phi_t = -\int_0^T \int_{D_t^+} u_t\phi + \int_D u(T)\phi(T) - \int_D u(0)\phi(0).$$

Then,

$$\begin{aligned}
\int_D u(T)\phi(T) &= \int_D u_0\phi(0) + \int_0^T \int_{D_t^+} u\phi_t + \int_0^T \int_{D_t^+} u_t\phi \\
&\geq \int_D u_0\phi(0) + \int_0^T \int_{D_t^+} u\phi_t + \int_0^T \int_{D_t^+} \left(\frac{\varepsilon}{m}\Delta(mu)^\ell + \frac{1}{\varepsilon}u(1-u)\right)\phi \\
&= \int_D u_0\phi(0) + \int_0^T \int_{D_t^+} u\phi_t + \int_0^T \varepsilon\frac{\phi}{m}\nabla(mu)^\ell \cdot n_t\Big|_{\partial D_t^+} \\
&\qquad - \int_0^t \int_{D_t^+} \varepsilon\nabla\frac{\phi}{m} \cdot \nabla(mu)^\ell + \int_0^T \int_{D_t^+} \frac{1}{\varepsilon}u(1-u)\phi \\
&\geq \int_D u_0\phi(0) + \int_0^T \int_D \left(u\phi_t - \varepsilon\nabla(mu)^\ell \cdot \nabla\frac{\phi}{m} + \frac{1}{\varepsilon}u(1-u)\phi\right).
\end{aligned}$$

Therefore, u is a super-solution in the weak sense. □

3.2. **Traveling wave solution.** The traveling wave solution for a homogeneous case still plays a key role for a heterogeneous case. Consider a homogeneous reaction-diffusion equation in one space dimension,

$$v_t = (v^\ell)_{xx} + v(1-v).$$

Let $v(x,t) = U(x + c_0 t)$ be a traveling wave solution with the minimum speed $c_0 > 0$. Here, the traveling wave moves from right to left. There exists a traveling wave solution for each $c \geq c_0$ which is unique upto a translation. The support of the traveling wave solution is the whole real line $\mathbb{R}$ if $c > c_0$. However, for the traveling wave solution with the minimum speed c_0, the support is a half line $[x_0, \infty)$ for some $x_0 \in \mathbb{R}$ and we may set $x_0 = 0$ after a translation. Let $z = x + c_0 t$. Then, the traveling wave solution satisfies

(17) ?eqn_tw?
$$\begin{cases} U^\ell_{zz}(z) - c_0 U_z(z) + U(1-U) = 0, \\ \lim_{z\to\infty} U(z) = 1, \\ U(z) > 0 \quad \text{for } z > 0, \\ U(z) = 0 \quad \text{for } z \leq 0. \end{cases}$$

Consider a two parameters family of perturbations of the traveling wave solution U given by

$$V(z;\delta,\zeta) := (1+\delta)U\big((1+\delta)^{1-\frac{\ell}{2}}\zeta z\big), \tag{18}$$ V

where the parameters are bounded by $-\frac{1}{2} < \delta < \frac{1}{2}$ and $\min_{x\in D} \frac{1}{m^p(x)} \le \zeta \le \max_{x\in D} \frac{1}{m^p(x)}$. Then, the perturbed wave V satisfies

$$\begin{cases} V^{\ell}_{zz} - c(\delta,\zeta)V_z + \zeta^2 V(1+\delta-V) = 0 & \text{for } z\in\mathbb{R}, \\ \lim_{z\to\infty} V(z;\delta,\zeta) = 1+\delta, \\ V(z;\delta,\zeta) > 0 & \text{for } z>0, \\ V(z;\delta,\zeta) = 0 & \text{for } z\le 0, \end{cases} \tag{19}$$ eqn_tw_V

where

$$c(\delta,\zeta) = c_0(1+\delta)^{\frac{\ell}{2}}\zeta.$$

The perturbed waves are used to construct super- and sub-solutions in the proof of Theorem 2. The following lemma consists of the properties of the perturbed wave V which will be used in the proof.

⟨lem_tw⟩ **Lemma 3.** *The perturbed wave has the regularity $V \in C^2(\mathbb{R}^+)\cap C(\mathbb{R})$ and satisfies*

$$V_\zeta = \frac{z}{\zeta}V_z, \quad V_{\zeta\zeta} = \left(\frac{z}{\zeta}\right)^2 V_{zz}, \quad V_{z\zeta} = \frac{1}{\zeta}V_z + \frac{z}{\zeta}V_{zz} \tag{20}$$ lem_tw_i

$$V_\delta = \frac{V}{1+\delta} + \frac{2-\ell}{2(1+\delta)}zV_z, \quad V_z > 0 \ \text{ for } z>0 \tag{21}$$ lem_tw_iii

There exists a generic constant $C_V > 0$ independent of δ and ζ such that

$$|c(0,\zeta) - c(\delta,\zeta)| \le C_V|\delta\zeta| \tag{22}$$ lem_tw_ii

$$0 < 1+\delta - V \le C_V e^{-\beta z} \tag{23}$$ lem_tw_iv

$$V^{\ell}_z \le C_V V \tag{24}$$ lem_tw_v

$$|zV_z| + |V^{\ell}_{zz}| + |zV^{\ell}_{zz}| \le C_V(V+V_z) \tag{25}$$ lem_tw_vii

for $z>0$.

Proof. The relations in (20) and (21) are directly obtained from the formula in (18). The estimate (22) is from definition of $c(\delta,\zeta)$. We will show the rest for the case with $\delta = 0$ only and the general case is obtained by the continuous dependence of $c(\delta,\zeta)$ and by taking the generic constant C_V larger. Estimates in (23) and (24) can be found in [8]. And in the same reference we know that

$$|zV_z| \le C_V V$$

for $z \ge 1$ and some positive constant C_V. And since

$$|zV_z| \le C_V V_z$$

for $0 < z < 1$, thus we obtain (25) for $|zV_z|$ since $V, V_z \ge 0$. Also, by (19) one can also obtain (25) for $|V_{zz}|$. Also, by (23) we have

$$z(1+\delta-V) \le C$$

for some positive constant C since $ze^{-\beta z} \le \beta^{-1}$ for $z > 0$. This implies that

$$|zV^{\ell}_{zz}| \le c|zV_z| + |z(1+\delta - V)|V \le C_V(V + V_z).$$

Also, since $|V^{\ell}_{zz}| \le cV_z + |\zeta^2(1+\delta-V)|V$, the inequality also holds for V^{ℓ}_{zz} as well. □

3.3. **Signed distance function.** In this section we consider properties of the signed distance function $\tilde{d}(x,t)$ in a neighborhood of the surface that consists of the interfaces Γ_t for $t \in [0,T]$ in $D \times [0,T]$ space. Denote

$$N(r,\tau) := \{(x,t) \in D \times [0,\tau] : |\tilde{d}(x,t)| \le r\}.$$

Then, clearly, $\cup_{t\in[0,\tau]}\Gamma_t \times \{t\} \subset N(r,\tau)$ for all $r > 0$, i.e., $N(r,\tau)$ is a neighborhood of the surface that consists of interface Γ_t.

⟨lem_d⟩ **Lemma 4.** *There exist positive constants d_0, T, C_d such that for all $(x,t) \in N(2d_0,T)$ the following holds;*

(26) lem_d1 $$\tilde{d} \in C^{2,1}(N(2d_0,T)), \quad |\nabla \tilde{d}| = 1, \quad |\tilde{d}_t(x,t) - c_0 m^p(x,t)| \le C_d|\tilde{d}|.$$

Proof. Under the assumption $\Gamma_0 \in C^{3+\alpha}$ in (7), we can follow the construction of the interface motion equation in [4] and rewrite the interface flow (12) in terms of a partial differential equation for $\tilde{d}$,

(27) Eq17 $$\tilde{d}_t(x,t) = c_0 m^p(y(x,t)),$$

in a neighborhood of $\Gamma_t \times \{t\}$. In this formula, the heterogeneity in m is taken from $y(x,t) \in \Gamma_t$ that satisfies

$$dist(y(x,t),x) = |\tilde{d}(x,t)|.$$

Such a point $y(x,t)$ exists uniquely if the interface is smooth enough and satisfies

$$y(x,t) = x - \tilde{d}\nabla\tilde{d}$$

(see [7, Section 14.6]). Thus we obtain a partial differential equation for $\tilde{d}$,

(28) eqn_IM_signed_pde $$\tilde{d}_t(x,t) = c_0 m^p \left(x - \tilde{d}(x,t)\nabla\tilde{d}(x,t)\right).$$

The conditions in (2) and (8), and Theorem 2 in [5, Section 3.2] imply the existence of the solution (28) in a set $N(2d_0,T)$ for some positive constants d_0, T with the regularity $\tilde{d} \in C^{2,1}(N(2d_0,T))$.

Note that, since the initial interface is smooth enough, $\Gamma_0 \in C^{3+\alpha}$, we have $|\nabla\tilde{d}(x,0)| = 1$ for $x \in \{x \in D : |\tilde{d}| \le 2d_0\}$ by taking smaller d_0 if needed. Let

$w(x,t) := |\nabla\tilde{d}|^2 - 1$. Using (28) we obtain

$$\begin{aligned}
w_t = 2\nabla\tilde{d}\cdot\nabla\tilde{d}_t &= \sum_{i=1}^{N} 2\partial_{x_i}\tilde{d}\; c_0\nabla m^p\cdot(1^i - \partial_{x_i}\tilde{d}\nabla\tilde{d} - \tilde{d}\partial_{x_i}\nabla\tilde{d}) \\
&= 2c_0\sum_{i=1}^{N}\partial_{x_i}\tilde{d}\partial_{x_i}m^p - 2c_0\nabla m^p\cdot\nabla\tilde{d}\sum_{i=1}^{N}\partial_{x_i}\tilde{d}^2 \\
&\quad - 2c_0\tilde{d}\sum_{i=1}^{N}\partial_{x_i}\tilde{d}\nabla m^p\cdot\partial_{x_i}\nabla\tilde{d} \\
&= 2c_0\nabla m^p\cdot\nabla\tilde{d}\;(1-|\nabla\tilde{d}|^2) - 2c_0\tilde{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\partial_{x_i}\tilde{d}\partial_{x_j}m^p\partial_{x_i}\partial_{x_j}\tilde{d} \\
&= -2c_0\nabla m^p\cdot\nabla\tilde{d}\;w - c_0\tilde{d}\sum_{i=1}^{N}\sum_{j=1}^{N}\partial_{x_j}m^p\partial_{x_j}(\partial_{x_i}\tilde{d}^2) \\
&= -2c_0\nabla m^p\cdot\nabla\tilde{d}\;w - c_0\tilde{d}\sum_{j=1}^{N}\partial_{x_j}m^p\partial_{x_j}|\nabla\tilde{d}|^2 \\
&= -2c_0\nabla m^p\cdot\nabla\tilde{d}\;w - c_0\tilde{d}\nabla m^p\cdot\nabla w,
\end{aligned}$$

which is a first order partial differential equation of w. By the characteristic technique with the initial value $w(x,0)=0$, we obtain $w(x,t)=0$ on $N(2d_0,T)$.

Using the relation (27), we have

$$|\tilde{d}_t(x,t) - c_0m^p(x)| = c_0|m^p(y(x,t)) - m^p(x)|.$$

As m^p is Lipschitz continuous, there exists a constant $C_d>0$ such that

$$|\tilde{d}_t(x,t) - c_0m^p(x)| \le C_d\; dist(y(x,t),x) = C_d|\tilde{d}(x,t)|,$$

which is the third inequality in (26). ☐

Next, we construct a cut-off distance function. Let $h:\mathbb{R}\to\mathbb{R}$ be a $C^2(\mathbb{R})$ function that satisfies

$$h(s) = \begin{cases} s & \text{if } |s|\le d_0, \\ 2d_0 & \text{if } s\ge 2d_0, \\ -2d_0 & \text{if } s\le -2d_0. \end{cases}$$

The cut-off distance $d: D\times[0,T]\to\mathbb{R}$ is defined by

$$d(x,t) := h(\tilde{d}(x,t)).$$

Then, by Lemma 4, we have, the cut-off distance function satisfies

$$|1-|\nabla d|^2| \le C_d|d|, \quad |d_t - c_0m^p| \le C_d|d|, \quad |\nabla d| + |\Delta d| \le C_d \tag{29}$$

by choosing $C_d>0$ large enough if necessary.

4. Generation of the interface

In this section we prove the generation of the interface stated in Theorem 1. The uniform estimate (9) is obtained by the comparison principle in Lemma (1) and the initial condition (6). The other two estimates, (10) and (11), are obtained by comparison principle after constructing appropriate super- and sub-solutions. Note that the reaction term in (5) dominates the dynamics in the first stage of interface generation and hence it is natural we construct the super- and sub-solutions using the solution of the ordinary differential equation with the reaction terms,

$$\begin{cases} \frac{d}{d\tau}Y(\tau,\xi;\delta) = Y(1-Y+\delta), \\ Y(0,\xi;\delta) = \xi, \end{cases}$$

where $|\delta| < \frac{1}{2}$. The spatial heterogeneity of the original problem (1) in the reaction term has been moved to the diffusion term as in (5). Hence, the obtain solution Y which is similar to to the one in [1]. The property of the solution Y is from this paper.

⟨lem_ODE_Y⟩ **Lemma 5.** *Let $0 < \eta_g < \frac{1}{2}$. Then there exists a positive constant $C_Y = C_Y(\eta_g)$ that satisfies the following estimates for all $|\delta| < \frac{1}{2}$.*

- *(i) For $\xi > 0$ and $\tau > 0$, we have $0 < Y_\xi(\tau,\xi;\delta) \le C_Y e^{(1+\delta)\tau}$.*
- *(ii) For $\xi > 0$ and $\tau > 0$, we have $\left|\frac{Y_{\xi\xi}(\tau,\xi;\delta)}{Y_\xi(\tau,\xi;\delta)}\right| \le C_Y\left(e^{(1+\delta)\tau} - 1\right)$.*
- *(iii) For all $\tau > 0$ we have $Y(\tau,\xi;\delta) \le 1+\delta$ if $\xi \le 1+\delta$ and $Y(\tau,\xi;\delta) \le 0$ if $\xi \le 0$*
- *(iv) There exists a positive constant ε_Y such that for all $\varepsilon \in (0,\varepsilon_Y)$ we have $Y(|\ln\varepsilon|,\xi;\delta) \ge 1+\delta-\eta_g$ if $\xi \ge C_Y\varepsilon, |\delta| \le \varepsilon$.*

Now we prove the generation of interface in Theorem 1.

Proof. (of Theorem 1). In the proof we choose a constant $0 < \varepsilon_0 < \min\{\varepsilon_Y, e^{-1}\}$ so that

$$\varepsilon_0 \max_{x\in D} \frac{|\Delta(m(x))^\ell|}{m(x)} \left(\frac{3}{2}\right)^{\ell-1} < \frac{\eta_g}{2}.$$

We first prove (9). The inequality $0 \le u^\varepsilon$ is easily obtained since the function $w^-(x,t) \equiv 0$ is a sub-solution of u^ε. Let $w^+(x,t) \equiv 1+\eta_\varepsilon$ where

$$\eta_\varepsilon = \varepsilon^2 \max_{x\in D} \frac{|\Delta(m(x))^\ell|}{m(x)} \left(\frac{3}{2}\right)^{\ell-1} < \frac{\eta_g}{2}$$

Direct computations of $\mathcal{L}w^+$ give

$$\begin{aligned} \mathcal{L}w^+ &= -\frac{\varepsilon}{m}(1+\eta_\varepsilon)^\ell \Delta(m)^\ell + \frac{1}{\varepsilon}(1+\eta_\varepsilon)\eta_\varepsilon \\ &= \frac{1+\eta_\varepsilon}{\varepsilon}\left(\eta_\varepsilon - \frac{\Delta m}{m}\varepsilon^2(1+\eta_\varepsilon)^{\ell-1}\right) \\ &\ge \frac{1+\eta_\varepsilon}{\varepsilon}\left(\eta_\varepsilon - \frac{\Delta m}{m}\varepsilon^2\left(\frac{3}{2}\right)^{\ell-1}\right). \end{aligned}$$

Thus, by the definition of η_ε, w^+ is a super solution which implies (9).

Next, we prove (10) and (11). First, we extend the initial value u_0 to a C^2 function $\overline{u}_0 : D \to \mathbb{R}$, which is available by Whitney extension theorem. Moreover, by condition (8) we can find a positive constant $\overline{d} < \min\{d_0, 1\}$ such that

$$\overline{u}_0 \leq \overline{d}d(x,0) \text{ if } -\overline{d} < d(x,0) < 0.$$

Then, we let $\sigma : D \to [0,1]$ be a smooth function satisfying

$$\sigma(x) \in \begin{cases} 1 & \text{if } d(x,0) > -\dfrac{\overline{d}}{2} \\ (0,1) & \text{if } -\overline{d} < d(x,0) \leq -\dfrac{\overline{d}}{2} \\ 0 & \text{if } d(x,0) \leq -\overline{d}. \end{cases}$$

With these functions in hand, we define $\tilde{u}_0 : D \to \mathbb{R}$ as follows

$$\tilde{u}_0(x) := \sigma(x)\overline{u}_0 - (1 - \sigma(x)).$$

Then we obtain

$$\tilde{u}_0(x) \leq \max\{\overline{c}d(x,0), -2\overline{c}\overline{d}\} \text{ if } d(x,0) < 0. \tag{30}$$

With this extended function we construct functions $w^{\pm}(x,t)$ as

$$w^{\pm}(x,t) = Y\left(\frac{t}{\varepsilon}, \max(\tilde{u}_0 \pm \varepsilon^2 P(t), 0); \pm\varepsilon\right), \; P(t) = K(e^{t/\varepsilon} - 1)$$

where K is a positive constant. We will show that if K is chosen appropriately, we have

$$w^-(x,t) \leq u^\varepsilon(x,t) \leq w^+(x,t) \tag{31}$$

for the solution u^ε of (5). First, the initial values of the two functions are

$$w^{\pm}(x,0) = \max(\tilde{u}_0, 0) = u_0(x),$$

i.e., $w^{\pm}$ and u share the same initial value of the solution u^ε. Therefore, if we show w^+ is a super-solution and w^- is a sub-solution, the claim (31) is obtained. We first show w^+ is a super-solution.

The conditions (i) and (ii) of Lemma 2 follows from the definitions of Y and $\tilde{u}_0$. Direct computations of $\mathcal{L}w^+$ on the support of w^+ give

$$\begin{aligned}
\mathcal{L}w^+(x,t) &= \left(\frac{1}{\varepsilon}Y_\tau + K'\varepsilon e^{t/\varepsilon}Y_\xi\right) - \frac{1}{\varepsilon}Y(1-Y) \\
&\quad - \frac{\varepsilon}{m}\left((w^+)^\ell \Delta m^\ell + 2\nabla m^\ell \cdot \nabla (w^+)^\ell + m^\ell \Delta (w^+)^\ell\right) \\
&= K\varepsilon e^{t/\varepsilon}Y_\xi + Y - \frac{\varepsilon}{m}\Big(Y^\ell \Delta m^\ell + 2\nabla m^\ell \cdot (\ell Y^{\ell-1}Y_\xi \nabla \tilde{u}_0) \\
&\quad + m^\ell[\ell(\ell-1)Y^{\ell-2}Y_\xi^2 |\nabla \tilde{u}_0|^2 + \ell Y^{\ell-1}(Y_{\xi\xi}|\nabla \tilde{u}_0|^2 + Y_\xi \Delta \tilde{u}_0)]\Big) \\
&\geq \varepsilon Y_\xi \Bigg(K'e^{t/\varepsilon} - \frac{\ell Y^{\ell-2}}{m}\Bigg(\left[\left|\frac{Y_{\xi\xi}}{Y_\xi}\right| Y + (\ell-1)Y_\xi\right] |\nabla \tilde{u}_0|^2 \\
&\quad + Y|\Delta \tilde{u}_0| + Y|\nabla m^\ell \cdot \nabla \tilde{u}_0|\Bigg)\Bigg) + Y\left(1 - \varepsilon\frac{\Delta m^\ell}{m}Y^{\ell-1}\right).
\end{aligned}$$

The last term in the inequality becomes by the choice of ε_0. Moreover, since we choose $\varepsilon_0 < e^{-1}$, for $0 < \tau < t^\varepsilon$ we obtain

$$e^{(1+\varepsilon)\tau/\varepsilon} \leq Ce^{\tau/\varepsilon},$$

for some positive constant C. Then by conditions (2), (8) for the m and u_0 and Lemma 5, we can choose K large enough such that

$$\mathcal{L}w^+ \geq 0$$

for $0 < t \leq \varepsilon|\ln\varepsilon|$. Therefore, w^+ is a super-solution. We can similarly show that w^- is a sub-solution.

Next, we take $M_G < \overline{d}^{-1}(C_Y + K)$ and choose ε_0 smaller enough such that $(C_Y + K)\varepsilon_0 < 1$. For any point $x \in D$ satisfying $u_0(x) \geq M_G\varepsilon$ we have

$$\tilde{u}_0(x) - \varepsilon^2 P(t^\varepsilon) \geq M_G\varepsilon - K\varepsilon(1-\varepsilon) \geq C_Y\varepsilon,$$

where the last inequality holds since $M_G < \overline{d}^{-1}(C_Y + K) < C_Y + K$. Thus, by Lemma 5 we obtain

$$u(x, t^\varepsilon) \geq Y(|\ln\varepsilon|, \tilde{u}_0(x) - K\varepsilon(1-\varepsilon); -\varepsilon) \geq 1 - \eta_g.$$

And by (30), for any $x \in D$ satisfying $d(x, 0) \leq -M_G\varepsilon$ we have

$$\begin{aligned}\tilde{u}_0(x) + \varepsilon^2 P(t^\varepsilon) &\leq \max\{\overline{d}d(x,0), -1\} + K\varepsilon(1-\varepsilon)\\ &\leq \max\{-\overline{d}M_G\varepsilon, -1\} + K\varepsilon(1-\varepsilon) \leq -C_Y\varepsilon \leq 0.\end{aligned}$$

Thus, by Lemma 5 we obtain

$$u^\varepsilon(x, t^\varepsilon) \leq w^+(x, t^\varepsilon) = 0.$$

□

5. Propagation of the interface

We prove Theorem 2 in this section. We construct a pair of functions $u^\pm$super- and sub-solutions as

$$u^\pm(x,t) = V\left(\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}; \pm q(t), \frac{1}{m^p(x)}\right),$$

where

$$p(t) = -\frac{C_V C_m^p + 1}{\sigma}e^{-\sigma t/\varepsilon} + K_1 e^{Lt},\ q(t) = \frac{\eta_p}{4}e^{-\sigma t/\varepsilon} + K_2\varepsilon.$$

Here C_m is the constant defined in (2), C_V is the constant appeared in Lemma 3, K_1, K_2, σ are some positive constants and $0 < \eta_p < 1/4$. And we choose $\sigma \in (0,1)$ small enough such that

(32) cond_sigma $$\sigma\left(1 + \frac{2-\ell}{2}C_V\right) < 1,\ \ \frac{2-\ell}{8}\sigma C_V < 1$$

We make the following additional assumptions on K_1, K_2, ε_0 that

(33) cond_e0 $$\left(\frac{C_V C_m^p + 1}{\sigma} + K_1 e^{LT}\right)\varepsilon_0 < d_0,\ \ K_2\varepsilon_0 < \frac{\eta_p}{4}$$

which can be obtained by choosing $\varepsilon_0 > 0$ small enough. Then these implies that

(34) cond_pq
$$\varepsilon_0 |p(t)| \leq d_0, \quad q(t) \leq \frac{\eta_p}{2}.$$

Note that, if $\ell > 2$ we have $2 - \ell < 0$ which means that (32) holds for any $\sigma \in (0,1)$. We first prove that $u^{\pm}$ are a pair of sub- and super-solutions.

propagation_supersub⟩ **Proposition 1.** *Let $K_1 > 1$. Then there exist positive constants K_2, L, ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, $(x,t) \in D \times [0, T - t^{\varepsilon}]$ we have*

(35) eqn_propagation_supersub
$$\mathcal{L}u^{-}(x,t) \leq 0 \leq \mathcal{L}u^{+}(x,t).$$

Proof. To show (35) we check the conditions $(i) - (iii)$ of Lemma 2 holds. The support of $u^{\pm}$ is equal to $\{(x,t) \in D \times [0,T],\ d \pm \varepsilon p(t) > 0\}$, so its boundary in $D \times [0,T]$ is smooth by Lemma 4. With this, and by lemmas 4 and 3 we can see conditions (i) and (ii) holds. For (iii), we only show for u^{+}; one can use the same method for u^{-} to show the condition (iii). For simplicity, we define $z_d = d(x,t) + \varepsilon p(t)$. Direct computation gives

$$\begin{aligned}
u_t^{+} &= \left(\frac{d_t}{\varepsilon} + p'(t)\right) V_z + q'(t) V_z \\
\nabla V^{\ell} &= \frac{\nabla d}{\varepsilon} V_z^{\ell} + \nabla \frac{1}{m^p} V_{\zeta}^{\ell} = \left[\frac{\nabla d}{\varepsilon} + \frac{z_d}{\varepsilon} \nabla \frac{1}{m^p}\right] V_z^{\ell} \\
\Delta V^{\ell} &= \left|\frac{\nabla d}{\varepsilon}\right|^2 V_{zz}^{\ell} + 2 \frac{\nabla d}{\varepsilon} \cdot \nabla \frac{1}{m^p} V_{z\zeta}^{\ell} + \left|\nabla \frac{1}{m^p}\right|^2 V_{\zeta\zeta}^{\ell} \\
&\quad + \nabla \cdot \left[\frac{\nabla d}{\varepsilon} + \frac{z_d}{\varepsilon} \nabla \frac{1}{m^p}\right] V_z^{\ell} \\
&= \left[\frac{\nabla d}{\varepsilon} + \frac{z_d}{\varepsilon} \nabla \frac{1}{m^p}\right]^2 V_{zz}^{\ell} + \nabla \cdot \left[\frac{\nabla d}{\varepsilon} + \frac{z_d}{\varepsilon} \nabla \frac{1}{m^p}\right] V_z^{\ell} \\
&\quad + 2 m^p \frac{\nabla d}{\varepsilon} \cdot \nabla \frac{1}{m^p} V_z^{\ell}
\end{aligned}$$

where the equality holds by (20). This implies that

$$\begin{aligned}
\mathcal{L}u^+(x,t) =& \frac{d_t+\varepsilon p'}{\varepsilon}V_z + q'V_\delta \\
&- \frac{\varepsilon}{m}\left(m^\ell \Delta V^\ell + 2\nabla m^\ell \cdot \nabla V^\ell + \Delta m^\ell V^\ell\right) - \frac{1}{\varepsilon}V(1-V) \\
=& \frac{d_t+\varepsilon p'}{\varepsilon}V_z + q'V_\delta \pm \frac{c_\varepsilon m^{2p}}{\varepsilon}V_z - \frac{1}{\varepsilon}V(1-V) \pm \frac{q}{\varepsilon}V \\
&- \frac{\varepsilon}{m}\left(m^\ell\left[\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]^2 V^\ell_{zz} + m^\ell \nabla\cdot\left[\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]V^\ell_z\right. \\
&\left.+2m^{p+\ell}\frac{\nabla d}{\varepsilon}\cdot\nabla\frac{1}{m^p}V^\ell_z + 2\nabla m^\ell\cdot\left[\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]V^\ell_z + \Delta m^\ell V^\ell\right) \\
=& \frac{d_t - c_\varepsilon m^{2p}}{\varepsilon}V_z + p'V_z + \frac{c_\varepsilon m^{2p}}{\varepsilon}V_z - \frac{1}{\varepsilon}V(1+q-V) + \frac{q}{\varepsilon}V + q'V_\delta \\
&- \frac{\varepsilon}{m}m^\ell\frac{|\nabla d|^2}{\varepsilon^2}V^\ell_{zz} - \frac{\varepsilon}{m}m^\ell\frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\cdot\left[2\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]V^\ell_{zz} \\
&- \frac{\varepsilon}{m}\left(m^\ell\nabla\left[\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]V^\ell_z + 2m^{p+\ell}\nabla\frac{d}{\varepsilon}\cdot\nabla\frac{1}{m^p}V^\ell_z\right. \\
&\left.+2\nabla m^\ell\cdot\left[\frac{\nabla d}{\varepsilon} + \frac{m^p z_d}{\varepsilon}\nabla\frac{1}{m^p}\right]V^\ell_z + \Delta m^\ell V^\ell\right),
\end{aligned}$$

where $c_\varepsilon := c\left(q(t), \frac{1}{m^p}\right)$. Using (19) and the fact that $\ell - 1 = 2p$ we can rewrite $\mathcal{L}u^+ = E_1 + E_2 + E_3$, where

$$\begin{aligned}
E_1 &= p'(t)V_z + \frac{c_0 m^p - c_\varepsilon m^{2p}}{\varepsilon}V_z + q'(t)V_\delta + \frac{q(t)}{\varepsilon}V, \\
E_2 &= \frac{d_t - c_0 m^p}{\varepsilon}V_z + \frac{1-|\nabla d|^2}{\varepsilon}m^{2p}V^\ell_{zz} - \frac{\varepsilon}{m}\Delta m^\ell V^\ell, \\
E_3 &= -m^{3p}\nabla\frac{1}{m^p}\cdot\left[2\nabla d + m^p z_d\nabla\frac{1}{m^p}\right]\frac{z_d}{\varepsilon}V^\ell_{zz} - 2m^{3p}\nabla d\cdot\nabla\frac{1}{m^p}V^\ell_z \\
&\quad - \frac{1}{m}\left(m^\ell\nabla\left[\nabla d + m^p z_d\nabla\frac{1}{m^p}\right]V^\ell_z + 2\nabla m^\ell\cdot\left[\nabla d + m^p z_d\nabla\frac{1}{m^p}\right]V^\ell_z\right).
\end{aligned}$$

(i) Estimates of E_1

Since $K_1 > 1$, direct computation gives

$$p'(t)V_z \geq \left(\frac{C_V C_m^p + 1}{\varepsilon}e^{-\sigma t/\varepsilon} + Le^{Lt}\right)V_z. \tag{36}$$

eqn_E1_1

By (21) we have

$$\begin{aligned}
\frac{q(t)}{\varepsilon}V + q'(t)V_\delta =& K_2 V + \frac{\eta_p e^{-\sigma t/\varepsilon}}{4\varepsilon}V \\
&- \sigma\frac{\eta_p e^{-\sigma t/\varepsilon}}{4\varepsilon}\left(\frac{V}{1+q(t)} + \frac{2-\ell}{2(1+q(t))}\frac{z_d}{\varepsilon}V_z\right).
\end{aligned}$$

By (25) and (32) we obtain

$$\frac{q(t)}{\varepsilon}V + q'(t)V_\delta \geq K_2 V + \frac{\eta_p e^{-\sigma t/\varepsilon}}{4\varepsilon}\left(V - \sigma\left(V + \frac{2-\ell}{2}C_V(V+V_z)\right)\right)$$

$$\geq K_2 V - \frac{\eta_p e^{-\sigma t/\varepsilon}}{\varepsilon}\frac{2-\ell}{8}\sigma C_V V_z$$

(37) eqn_E1_2
$$\geq K_2 V - \frac{e^{-\sigma t/\varepsilon}}{\varepsilon}V_z. \tag{37}$$

And by (22) we have

$$\frac{V_z}{\varepsilon}(c_0 m^p - c_\varepsilon m^{2p}) \geq -C_V\frac{q(t)}{\varepsilon}m^p V_z$$

$$\geq -C_V C_m^p\left(\frac{\eta_p}{4\varepsilon}e^{-\sigma t/\varepsilon} + K_2\right)V_z$$

(38) eqn_E1_3
$$\geq -C_V C_m^p\left(\frac{e^{-\sigma t/\varepsilon}}{\varepsilon}e^{-\sigma t/\varepsilon} + K_2\right)V_z \tag{38}$$

Thus the inequalities (36), (37) and (38) implies

(39) eqn_E1
$$E_1 \geq K_2 V + (L - C_1 K_2)V_z \tag{39}$$

for some positive constant C_1.

(ii) Estimates of E_2

By (29) we obtain

$$\frac{V_z}{\varepsilon}(d_t - c_0 m^p) + \frac{1-|\nabla d|^2}{\varepsilon}m^{2p}V_{zz}^\ell \geq -C_d\frac{|d|}{\varepsilon}(V_z + C_m^p|V_{zz}^\ell|)$$

$$\geq -C_d\frac{z_d}{\varepsilon}(V_z + C_m^p|V_{zz}^\ell|) - C_d p(t)(V_z + C_m^p|V_{zz}^\ell|)$$

Thus by (25) we have following inequality

(40) eqn_E2
$$E_2 \geq -C_2(V + V_z) \tag{40}$$

for some positive constant C_2.

(iii) Estimates of E_3

Note that $z_d \leq |d| + |p(t)| \leq 2d_0$ by (34). By (2), (25) and (29) we obtain

(41) eqn_E3
$$E_3 \geq -C_3(V + V_z) \tag{41}$$

for some positive constant C_3.

We now show $\mathcal{L}u^+ \geq 0$. By (39),(40) and (41) we have

$$\mathcal{L}u^+ \geq \left(K_2 - \tilde{C}\right)V + (L - C_1 K_2 - \tilde{C})V_z,$$

where $\tilde{C} = C_2 + C_3$. Thus, by choosing L and K_2 large enough we have $\mathcal{L}u^+ \geq 0$.

□

Proof of Theorem 2.

The first inequality of (14) can be obtained by letting $w^- \equiv 0, w^+ \equiv 1+\eta_\varepsilon$ as sub- and super- solutions, where η_ε is a constant introduced in Theorem 1. We prove the rest of the results with $u^\pm$.

For $\eta_g \in (0, \eta_p/2)$ let ε_0, M_G be constants satisfying Theorem 1. By (8) we can find $C > 0$ such that

$$\text{if } d(x,0) \geq C\varepsilon \text{ then } u_0(x) \geq M_G\varepsilon,$$
$$\text{if } d(x,0) \leq -C\varepsilon \text{ then } u_0(x) = 0.$$

With this, and by Theorem 1 we have

$$u^\varepsilon(x,t^\varepsilon) \leq H^+(x) := \begin{cases} 1+\eta_\varepsilon & \text{if } d(x,0) \geq -C\varepsilon \\ 0 & \text{if } d(x,0) < -C\varepsilon \end{cases},$$
$$u^\varepsilon(x,t^\varepsilon) \geq H^-(x) := \begin{cases} 1-\eta_g & \text{if } d(x,0) \geq C\varepsilon \\ 0 & \text{if } d(x,0) < C\varepsilon \end{cases}.$$

Equations (23) and (34) imply

$$V\left(z; q(0), \frac{1}{m^p(x)}\right) \geq 0, \; V\left(z; -q(0), \frac{1}{m^p(x)}\right) \leq 1 - \frac{\eta_p}{2} < 1 - \eta_g,$$

for $x \in D, z \in \mathbb{R}$, where the last inequality holds by the choice of η_g. Moreover, we can fix $K_1 > 0$ large enough such that

$$V\left((-C+K_1); q(0), \frac{1}{m^p}\right) \geq 1+\eta_\varepsilon, \; V\left(C-K_1; -q(0), \frac{1}{m^p}\right) = 0.$$

And these inequalities imply that

$$u^+(x,0) \geq H^+(x), \quad u^-(x,0) \leq H^-(x).$$

Thus, by Proposition 1 and Lemma 2 we have

(42) eqn_prop_comparison $$u^-(x,t) \leq u^\varepsilon(x,t+t^\varepsilon) \leq u^+(x,t) \quad \text{for} \quad x \in D, t \in [0, T-t^\varepsilon].$$

By (23) and (33), we can choose $M_P > 0$ satisfying

$$V\left((M_P - p(t));\; -q(t),\; \frac{1}{m^p}\right) \geq 1 - \eta_p$$
$$V\left((-M_P + p(t));\; q(t),\; \frac{1}{m^p}\right) = 0$$

for any $(x,t) \in D \times [0, T-t^\varepsilon]$. With this, and by (42) we have

$$\text{if } d(x,t) \geq M_P\varepsilon \to u^\varepsilon(x,t+t^\varepsilon) \geq 1-\eta_p$$
$$\text{if } d(x,t) \leq -M_P\varepsilon \to u^\varepsilon(x,t+t^\varepsilon) = 0.$$

Therefore Theorem 2 holds. □

⟨rmk_1⟩ **Remark 1.** *By using the same sub- and super-solution $u^\pm$, we can also prove the convergence result* (15)*. Indeed, by* (23) *we have*

$$V\left(\beta^{-1}|\ln\varepsilon|; -q(t), \frac{1}{m^p(x)}\right) \geq 1 - C_V\varepsilon - q(t).$$

Moreover, we have $q(t) \leq (\eta_p + K_2)\varepsilon$ *for* $t \geq t^\varepsilon = \varepsilon|\ln\varepsilon|$. *Thus we can fix* $C > 0$ *large enough such that for any small enough* $\varepsilon > 0$ *we have*

$$u^\varepsilon(x, t + t^\varepsilon) \geq 1 - C\varepsilon \quad \text{for} \quad d(x,t) \geq C\varepsilon|\ln\varepsilon|, \;\; t \geq t^\varepsilon \tag{43}$$

rmk_11

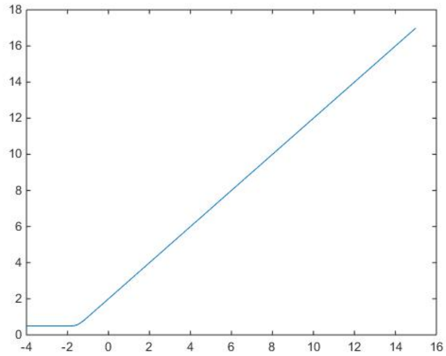
where we used (42). *With this,* (14) *and the fact that* $t^\varepsilon \downarrow 0$ *as* $\varepsilon \downarrow 0$ *implies* (15).

This result gives different description of the solution u^ε. *In view of* (14), *we expect the interface is generated and propagated with width* $\mathcal{O}(\varepsilon)$, *which seems natural since the equation is obtained by the hyperbolic scaling. In view of* (43), *even though the result may not seems related to the hyperbolic scaling, it gives much finer expectation of* u^ε *which allows to see the convergence result* (15).

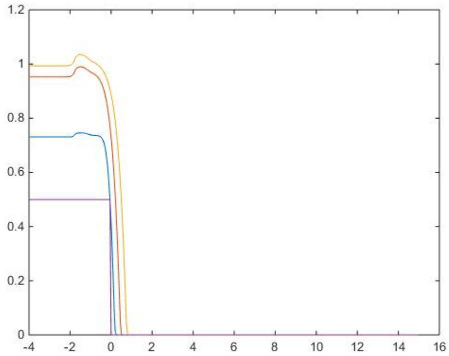
6. Numerical simulation

⟨sect.contacts⟩

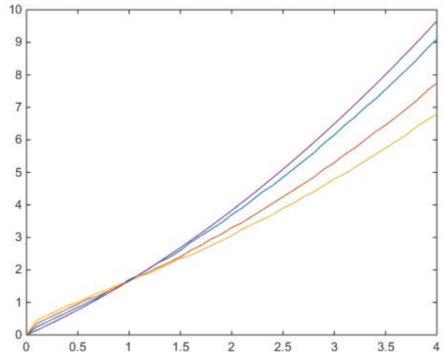
In this section we give a numerical simulation of the solution of (1) and (4). Here we consider $\ell = 2$. Note that $c_0 = 1$ is knwon for $\ell = 2$; see [2]. With such idea, the numerical simulation of (1) and (4) for 1 and 2 dimensional space is given in figures 2 and 3. As we can see in the results, the support of the function $\frac{U}{m}$ can be approximated with the interface following the motion equation (4). The interesting feature of the motion (4) can be observed in Figure 3, especially in $t = 5, 6$. As mentioned in the introduction we no longer expect the interface to be convex, where the non-convexity comes from the heterogeneity of the speed. This expectation can be seen not only in the interface following (4)(the red line in Figure 3) but also to the support of the function U in the numerical simulation.



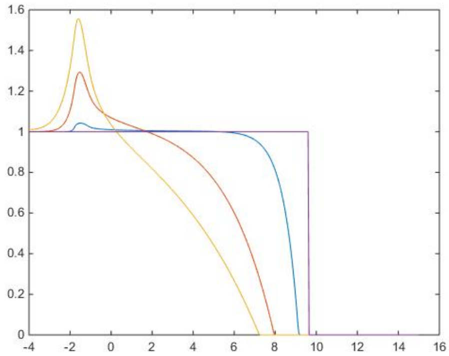
(A) Heterogeneous function m



(B) Generation of interface



(C) Movement of support boundary



(D) $t = 4$ result for $\varepsilon = 0.1, 0.3, 0.5$

FIGURE 2. Numerical simulation of (1) and (4) for $\ell = 2$. Figure (a) is a graph of the function m, which is given as a smooth approximation of $x + 2$. Figure (b) is a numerical simulation of generation of interface. Here we let $\varepsilon = 0.1$ and the initial condition as a step function with 0.5 as maximum value(purple line). Other graphs indicates the function u^ε at time $t = 0.2, 0.4, 0.6$(blue, orange, yellow). Figures (c) and (d) are numerical simulation to see the propagation of u^ε for $\varepsilon = 0.1, 0.3, 0.5$(blue, orange, yellow) and the interface Γ_t(purple). Figure (c) represents the boundary of support u^ε and Γ_t from $t = 0$ to $t = 4$. Figure (d) plots the graphs u^ε at time $t = 4$ and a step function with boundary Γ_t.

⟨fig_2_d1simul⟩

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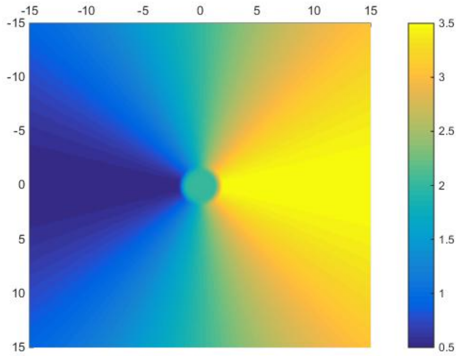
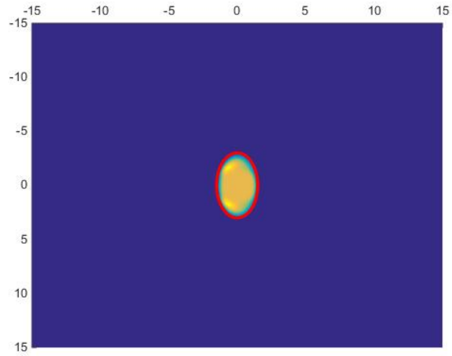
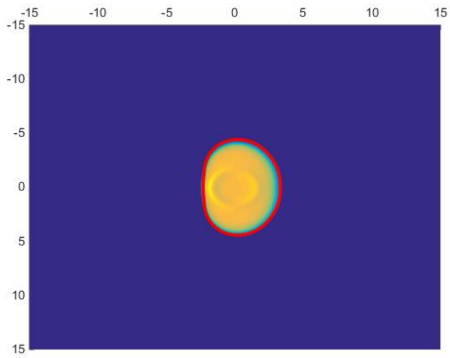


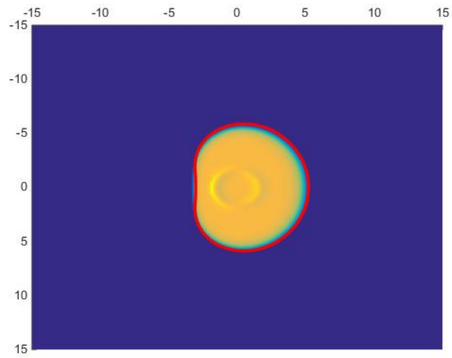
(A) Heterogeneous function m



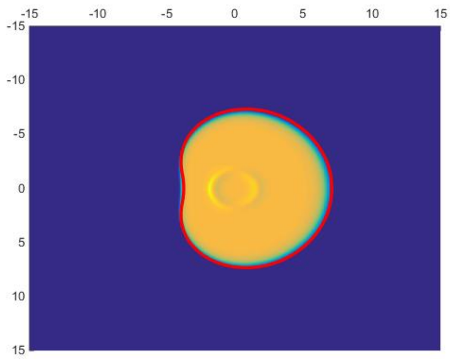
(B) Initial conditions



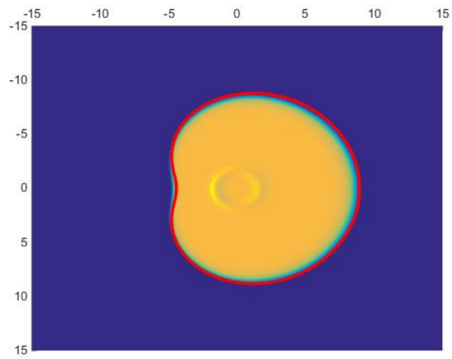
(C) $t = 1$



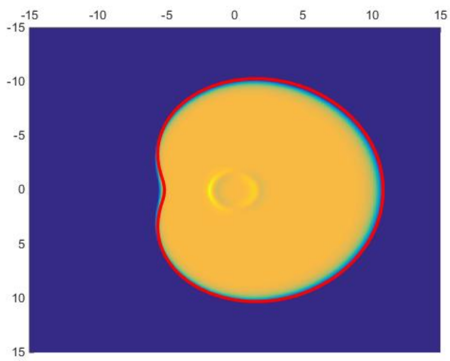
(D) $t = 2$



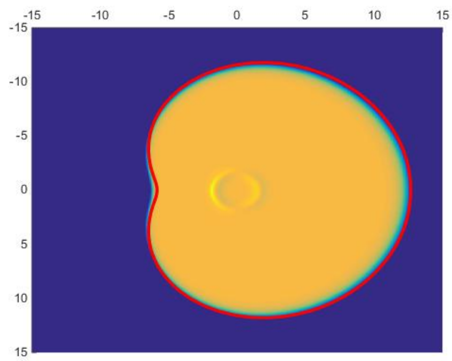
(E) $t = 3$



(F) $t = 4$



(G) $t = 5$



(H) $t = 6$

FIGURE 3. Numerical simulation of (1) and (4) for $\ell = 2$ and $\varepsilon = 0.1$. The color red indicates the interface that moves according to (4) and other colors indicates the value of the function U. The heterogeneous

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