Nonlinear diffusion for bacterial traveling wave phenomenon

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Abstract

The bacterial traveling waves observed in experiments are of pulse-type which is different from the monotone traveling waves of the Fisher-KPP equation. For this reason, the Keller-Segel equations are widely used for bacterial waves. Note that the Keller-Segel equations do not contain the population dynamics of bacteria, but the population of bacteria multiplies and plays a crucial role in wave propagation. In this paper, we consider the singular limits of a linear system with active and inactive cells together with bacterial population dynamics. Eventually, we see that if there are no chemotactic dynamics in the system, we only obtain a monotone traveling wave. This is evidence that chemotaxis dynamics are needed even if population growth is included in the system.

1 Introduction

We consider the singular limits of a reaction-diffusion system,

$$(\mathcal{P}_\varepsilon^\mu) \qquad \begin{cases} a_t = d_a\Delta a + rna + \frac{1}{\varepsilon}(\beta(n)w - \alpha(n)a) & \text{in } Q_T,\\ w_t = \mu\Delta w + rnw + \frac{1}{\varepsilon}(\alpha(n)a - \beta(n)w) & \text{in } Q_T,\\ n_t = d_n\Delta n - n(a+w) & \text{in } Q_T,\\ \partial_\nu a = \partial_\nu w = \partial_\nu n = 0 & \text{on } \Gamma_T,\\ a(x,0) = a_0(x), \quad w(x,0) = w_0(x),\ n(x,t) = n_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with dimension $N \geq 1$, $Q_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$, and ν the outer normal vector on the boundary. This system models the evolution of a bacterial species consisting of two phenotypes (or states), a and w, which are the densities of active and inactive cells, respectively. The other unknown n is the density of the nutrient consumed by the bacteria. The diffusivity d_a of the active cells is greater than the diffusivity μ of the inactive cells. The coefficient $r > 0$ is the ratio that the nutrient n turns into the population mass. The two coefficients $\alpha(n)$ and $\beta(n)$ are the conversion rates of the two phenotypes to each other, which are functions of n and satisfy

$$(H_{\alpha,\beta}) \qquad \begin{cases} \alpha, \beta \in C^1([0,\infty)) \cap L^\infty([0,\infty)),\\ \alpha(0) = \alpha_0, \quad \alpha'(n) \leq 0, \quad \text{and} \quad \alpha(n) > 0 \quad \text{for } n > 0,\\ \beta(0) = 0, \quad \beta'(n) \geq 0, \quad \text{and} \quad \beta(n) > 0 \quad \text{for } n \geq 0. \end{cases}$$

The monotonicity of the conversion rates implies that bacteria becomes active if the nutrient is abundant and become inactive otherwise. For initial values, we assume

$$(H_{IC}) \qquad a_0, w_0, n_0 \in C^1(\overline{\Omega}) \quad \text{and} \quad a_0, w_0, n_0 \geq 0 \quad \text{in } \overline{\Omega}.$$

The purpose of the paper is to prove the convergence of solutions as $\varepsilon \to 0$ and $\mu \to 0$ and show the global well-posedness of the problems obtained from the two singular limits.

There are two small parameters in the problem ε and μ. If we take the singular limit as $\mu \to 0$ with a fixed $\varepsilon > 0$, we obtain

$$(\mathcal{P}_\varepsilon^0) \qquad \begin{cases} a_t = d_a \Delta a + rna + \frac{1}{\varepsilon}(\beta(n)w - \alpha(n)a) & \text{in } Q_T, \\ w_t = rnw + \frac{1}{\varepsilon}(\alpha(n)a - \beta(n)w) & \text{in } Q_T, \\ n_t = d_n \Delta n - n(a+w) & \text{in } Q_T, \\ \partial_\nu a = \partial_\nu w = \partial_\nu n = 0 & \text{on } \Gamma_T, \\ a(x,0) = a_0(x), \quad w(x,0) = w_0(x),\ n(x,0) = n_0(x) & \text{on } \Omega. \end{cases}$$

If we take the limit as $\varepsilon \to 0$ with a fixed $\mu > 0$, then the limit of the solution formally satisfy a relation

$$\beta(n)w = \alpha(n)a.$$

If the first two equations in $(\mathcal{P}_\varepsilon^\mu)$ are added together with the relation, we obtain

$$(\mathcal{P}_0^\mu) \qquad \begin{cases} b_t = \Delta(\gamma_\mu(n)b) + rnb & \text{in } Q_T, \\ n_t = d_n \Delta n - nb & \text{in } Q_T, \\ \partial_\nu b = \partial_\nu n = 0 & \text{on } \Gamma_T, \\ b(x,0) = b_0(x) := a_0(x) + w_0(x),\ n(x,0) = n_0(x) & \text{on } \Omega, \end{cases}$$

where $b = a + w$ is the total cell density. Note that the diffusion of the total cell density is nonlinear with the motility $\gamma_\mu(n)$ given by

$$\gamma_\mu(n) = \frac{\mu\alpha(n) + d_a\beta(n)}{\alpha(n) + \beta(n)}. \tag{1.1}$$

Then, since we assume the diffusivity of active cell is greater than the one of inactive cells,

$$\gamma_\mu'(n) = \frac{(\mu - d_a)\beta\alpha' + (d_a - \mu)\alpha\beta'}{(\alpha+\beta)^2} > 0, \tag{1.2}$$

under the hypothesis $(H_{\alpha,\beta})$. Hence, $\gamma_\mu(n)$ is an increasing function of n and takes its minimum $\gamma_\mu(0) = \mu > 0$ when there is no nutrient left. Note that similar diffusion models have been used in cell aggregation models (see [6, 10, 21]). The difference is that the motility functions in those cases are decreasing functions of a signaling chemical.

The first main result of the paper is Theorem 3.7 which shows the existence of the unique weak solution of $(\mathcal{P}_\varepsilon^0)$. We show that, for any $T > 0$, the classical solution of $(\mathcal{P}_\varepsilon^\mu)$ converges to the unique weak solution of $(\mathcal{P}_\varepsilon^0)$ as $\mu \to 0$. The second main result is Theorem 4.3 on the existence of the unique weak solution of $(\mathcal{P}_0^\mu)$. We show that, for any $T > 0$, the classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ of $(\mathcal{P}_\varepsilon^\mu)$ converges to a limit (a^μ, w^μ, n^μ) as $\varepsilon \to 0$ and $(a^\mu + w^\mu, n)$ is the unique weak solution of $(\mathcal{P}_0^\mu)$.

In Section 5, we study the traveling wave solution of the problem $(\mathcal{P}_0^\mu)$ with appropriate boundary conditions. In Theorem 5.2, we show the existence of traveling wave solution for all speed $c \geq c^* = \sqrt{r\gamma_0(n_+)n_+}$, where the minimum wave speed is decided by n_+, the amount of the resource before consumption. In Theorem 5.1, we show that the traveling wave solution is monotone. These observations imply that the traveling wave solution of $(\mathcal{P}_0^\mu)$ is of Fisher-KPP type, but not of a chemotactic one. In the accompanying modeling paper [15], authors numerically show that adding chemotaxis to active cells, can yield pulse-type bacterial traveling waves.

Throughout the paper, constant C denotes a generic positive constant that can vary from line to line. Also, we shall omit the superscript μ and the subscript ε of solutions when it is clear from the context.

2 Preliminaries

The solutions of $(\mathcal{P}_\varepsilon^0)$ and $(\mathcal{P}_0^\mu)$ are defined in a very weak sense. Note that the solutions of $(\mathcal{P}_\varepsilon^0)$ can be defined in a weak sense. However, the solution of $(\mathcal{P}_0^\mu)$ need to be defined in a very weak sense. For the consistency, we define both solutions in a very weak sense. For the notational convenience, we respectively denote the reaction terms for a, w, and n by

$$\begin{aligned} f_a(a,w,n) &= rna + \frac{1}{\varepsilon}(\beta(n)w - \alpha(n)a), \\ f_w(a,w,n) &= rnw + \frac{1}{\varepsilon}(\alpha(n)a - \beta(n)w), \\ f_n(a,w,n) &= -n(a+w). \end{aligned}$$

Definition 2.1 *Let $\varepsilon > 0$. A triple of functions $a, w, n \in L^2(Q_T)$ is called a weak solution of $(\mathcal{P}_\varepsilon^0)$ if $f_a(a,w,n), f_w(a,w,n), f_n(a,w,n) \in L^1(Q_T)$ and*

$$-\iint_{Q_T} a\psi_t - \int_\Omega a_0\psi(\cdot,0) = \iint_{Q_T} d_a a\Delta\psi + \iint_{Q_T} f_a(a,w,n)\psi, \tag{2.1}$$

$$-\iint_{Q_T} w\psi_t - \int_\Omega w_0\psi(\cdot,0) = \iint_{Q_T} f_w(a,w,n)\psi, \tag{2.2}$$

$$-\iint_{Q_T} n\psi_t - \int_\Omega n_0\psi(\cdot,0) = \iint_{Q_T} d_n n\Delta\psi + \iint_{Q_T} f_n(a,w,n)\psi \tag{2.3}$$

for all $\psi \in C^{2,1}(\overline{Q}_T)$ such that $\psi(x,T) = 0$ in Ω and $\partial_\nu\psi = 0$ on $\partial\Omega \times [0,T]$.

Definition 2.2 *Let $\mu > 0$. A pair of functions $b, n \in L^2(Q_T)$ is called a weak solution of $(\mathcal{P}_0^\mu)$ if $nb \in L^1(Q_T)$ and*

$$-\iint_{Q_T} b\psi_t - \int_\Omega b_0\psi(\cdot,0) = \iint_{Q_T} \gamma(n)b\Delta\psi + \iint_{Q_T} rnb\psi, \tag{2.4}$$

$$-\iint_{Q_T} n\psi_t - \int_\Omega n_0\psi(\cdot,0) = \iint_{Q_T} d_n n\Delta\psi - \iint_{Q_T} nb\psi \tag{2.5}$$

for all $\psi \in C^{2,1}(\overline{Q}_T)$ such that $\psi(x,T) = 0$ in Ω and $\partial_\nu\psi = 0$ on $\partial\Omega \times [0,T]$.

For the compactness of the problem, we use a similar argument introduced in [7, 8]. The main tool is the Fréchet-Kolmogorov theorem(see [2, Theorem 4.1]), and we present it below in a modified version ([5, Proposition 2.5]).

Lemma 2.3 (Fréchet-Kolmogorov) *Let $\mathscr{F}$ be a bounded subset of $L^p(Q_T)$ with $1 < p < \infty$. Then, $\mathscr{F}$ is precompact in $L^p(Q_T)$ if the following two properties hold.*

(i) For any $\eta > 0$ and any subset $S \subset\subset Q_T$, there exists $\delta > 0$ such that

$$\|f(x+\xi,t) - f(x,t)\|_{L^p(S)} + \|f(x,t+\tau) - f(x,t)\|_{L^p(S)} < \eta$$

for all $f \in \mathscr{F}$ if $|\xi| + |\tau| < \delta$.

(ii) For any $\eta > 0$, there exists a subset $S \subset\subset Q_T$ such that, for all $f \in \mathscr{F}$,

$$\|f\|_{L^p(Q_T \setminus S)} < \eta.$$

The solvability of $(\mathcal{P}_\varepsilon^\mu)$ is from a classical fixed point theory for parabolic semilinear equations.

Lemma 2.4 *For any $\varepsilon, \mu > 0$, the solution of the system $(\mathcal{P}_\varepsilon^\mu)$ exists in the classical sense, $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu) \in \left[C^{2,1}(\overline{\Omega} \times (0,T)) \cap C(\Omega \times [0,T])\right]^3$, and is unique. The solution is uniformly bounded with respect to μ. More precisely, there exists $C_\varepsilon > 0$ independently of μ such that*

$$0 \le a_\varepsilon^\mu, w_\varepsilon^\mu \le C_\varepsilon \quad \textit{and} \quad 0 \le n_\varepsilon^\mu \le \max_{x \in \overline{\Omega}} n_0(x) \quad \textit{in } \overline{\Omega} \times [0,T]. \tag{2.6}$$

Proof. Under the hypotheses $(H_{\alpha,\beta})$, we observe that the nonlinearities f_a, f_w and f_n of $(\mathcal{P}_\varepsilon^\mu)$ are locally Lipschitz continuous with respect to functional arguments and

$$f_a(0,w,n), f_w(a,0,n), f_n(a,w,0) \ge 0 \quad \text{for } (u,v,n) \in \mathbf{R}_+^3.$$

Then, there exists a maximal time of existence $T_{\max} > 0$ such that $(\mathcal{P}_\varepsilon^\mu)$ possesses a unique nonnegative classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ on $\Omega \times (0, T_{\max})$ (see [1, 18]). Thanks to the non-increasing property of n_ε^μ, i.e.,

$$\|n_\varepsilon^\mu(\cdot,t)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}),$$

we have the at most linear growth of nonlinearities such that

$$|f_a(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)| + |f_w(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)| + |f_n(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)| \le C_\varepsilon(1 + a_\varepsilon^\mu + w_\varepsilon^\mu + n_\varepsilon^\mu) \quad \text{on } \Omega \times (0, T_{\max})$$

for some $C_\varepsilon > 0$. Moreover, we have the mass conservation

$$\|a_\varepsilon^\mu + w_\varepsilon^\mu + r n_\varepsilon^\mu\|_{L^1(\Omega)} = \|a_0 + w_0 + r n_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}).$$

Then, by a direct application of Theorem 2.3 in [18], we obtain that for any $T > 0$, $(\mathcal{P}_\varepsilon^\mu)$ possesses a unique nonnegative classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ on $\Omega \times (0,T)$. This completes the proof of global classical solvability. □

3 Singular limit as $\mu \to 0$

In this section, we prove the existence and the uniqueness of the weak solution to $(\mathcal{P}_\varepsilon^0)$ and the convergence of the global classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ of $(\mathcal{P}_\varepsilon^\mu)$ to the weak solution $(a_\varepsilon, \beta_\varepsilon, n_\varepsilon)$ of $(\mathcal{P}_\varepsilon^0)$ as $\mu \to 0$. In this section, the parameter $\varepsilon > 0$ is fixed and the generic constant C should be independent of μ. We start with a few μ-independent estimates.

Lemma 3.1 *Let $T > 0$ and $(a^\mu_\varepsilon, w^\mu_\varepsilon, n^\mu_\varepsilon)$ be the solution to $(\mathcal{P}^\mu_\varepsilon)$. Then, there exists a constant $C > 0$ independent of μ such that*

$$\iint_{Q_T} (a^\mu_\varepsilon)^2, \ \iint_{Q_T} (w^\mu_\varepsilon)^2 \le C, \tag{3.1}$$

$$\iint_{Q_T} |\nabla a^\mu_\varepsilon|^2, \ \mu \iint_{Q_T} |\nabla w^\mu_\varepsilon|^2 \le C. \tag{3.2}$$

Proof. For brevity, we will omit the super- and sub-scripts of the solution. Multiplying the first two equations of $(\mathcal{P}^\mu_\varepsilon)$ by a and w, respectively, and integrating them over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (a^2 + w^2) + d_a \int_\Omega |\nabla a|^2 + \mu \int_\Omega |\nabla w|^2 + \frac{1}{\varepsilon}\int_\Omega (\alpha(n)a^2 + \beta(n)w^2) \\
&= \int_\Omega rn(a^2 + w^2) + \frac{1}{\varepsilon}\int_\Omega (\alpha(n) + \beta(n))aw \\
&\le C_1 \int_\Omega (a^2 + w^2) + \frac{C_2}{\varepsilon}\int_\Omega aw \le \left(C_1 + \frac{C_2}{2\varepsilon}\right)\int_\Omega (a^2 + w^2),
\end{aligned} \tag{3.3}$$

where Young's inequality is used and the two positive constants C_1 and C_2 are induced from the upper bound (2.6) and the hypothesis $(H_{\alpha,\beta})$. Let $y(t) = \int_\Omega (a^2(\cdot, t) + w^2(\cdot, t))$. We infer from (3.3) that there exists $C_3 > 0$ such that

$$y'(t) \le C_3 y(t),$$

which gives

$$y(t) \le y(0)\exp(C_3 T) \quad \text{for any } \ t \in (0, T).$$

Thus by the non-negativity of a and w, we show (3.1). In view of (3.1), (3.2) is a direct consequence of the integration of (3.3) in time. $\square$

Lemma 3.2 *Let $T > 0$ and $(a^\mu_\varepsilon, w^\mu_\varepsilon, n^\mu_\varepsilon)$ be a solution of $(\mathcal{P}^\mu_\varepsilon)$. Then, there exists a constant $C > 0$ such that*

$$\sup_{t \le T} \int_\Omega (n^\mu_\varepsilon(\cdot, t))^2 + \iint_{Q_T} |\nabla n^\mu_\varepsilon|^2 \le C. \tag{3.4}$$

Proof. Multiplying the equation for n of $(\mathcal{P}^\mu_\varepsilon)$ by n and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 + d_n \int_\Omega |\nabla n|^2 = -\int_\Omega (a + w)n^2 \le 0.$$

Integrating the result in time implies (3.4). $\square$

The next step is for the compactness. Denote

$$\Omega_r = \{x \in \Omega | B(x, 2r) \subset \Omega\} \quad \text{and} \quad \Omega'_r = \cup_{x \in \Omega_r} B(x, r),$$

where $B(x, r)$ is the ball in $\mathbb{R}^N$ with its radius r and centered at x. Then, $\Omega_r \subset \Omega'_r \subset \Omega$. We are interested in $r > 0$ small. We have $\Omega_r = \emptyset$ for $r > 0$ large and Ω_r approaches to Ω as $r \to 0$. In order to apply Lemma 2.3, we estimate the differences of time translations in the L^2-norm.

Lemma 3.3 *Let $T > 0$ and $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be a solution to $(\mathcal{P}_\varepsilon^\mu)$. Then, there exists a constant $C = C(\varepsilon, T) > 0$ such that*

$$\int_0^{T-\tau} \int_{\Omega_r} (a_\varepsilon^\mu(x, t+\tau) - a_\varepsilon^\mu(x,t))^2 \leq C\tau, \tag{3.5}$$

$$\int_0^{T-\tau} \int_{\Omega_r} (w_\varepsilon^\mu(x, t+\tau) - w_\varepsilon^\mu(x,t))^2 \leq C\tau, \tag{3.6}$$

$$\int_0^{T-\tau} \int_{\Omega_r} (n_\varepsilon^\mu(x, t+\tau) - n_\varepsilon^\mu(x,t))^2 \leq C\tau \tag{3.7}$$

for all $\tau \in (0, T)$ and $r > 0$.

Proof. For the time translation case, the spatial integrations in the lemma can be done over Ω. Hence, Ω_r can be replaced with Ω, which is a stronger argument. Consider the active cell case first. Observe that

$$\begin{aligned}
&\int_0^{T-\tau} \int_\Omega (a(x,t+\tau) - a(x,t))^2 dxdt \\
&\leq \int_0^{T-\tau} \int_\Omega \left\{ (a(x,t+\tau) - a(x,t)) \int_0^\tau \partial_t a(x, t+s) ds \right\} dxdt \\
&\leq \int_0^\tau \int_0^{T-\tau} \int_\Omega (a(x,t+\tau) - a(x,t)) \partial_t a(x,t+s) dxdtds \\
&\leq \int_0^\tau \int_0^{T-\tau} \int_\Omega (a(x,t+\tau) - a(x,t))(d_a \Delta a(x,t+s) + f_a(a,w,n)(x,t+s)) dxdtds.
\end{aligned}$$

Let

$$\begin{aligned}
I_1 &:= d_a \int_0^\tau \int_0^{T-\tau} \int_\Omega (a(x,t+\tau) - a(x,t)) \Delta a(x,t+s) dxdtds, \\
I_2 &:= \int_0^\tau \int_0^{T-\tau} \int_\Omega (a(x,t+\tau) - a(x,t)) f_a(a,w,n)(x,t+s) dxdtds.
\end{aligned}$$

Integration by parts, the gradient estimate of a in (3.2), and Hölder's inequality give

$$\begin{aligned}
I_1 &= -d_a \int_0^\tau \int_0^{T-\tau} \int_\Omega \nabla(a(x,t+\tau) - a(x,t)) \cdot \nabla a(x,t+s) dxdtds \\
&\leq 2 d_a \tau \iint_{Q_T} |\nabla a(x,t)|^2 dxdt \quad \leq C_1 \tau
\end{aligned}$$

for some $C_1 > 0$. On the other hand, the upper bounds of the solutions obtained in Lemma 2.4 imply that

$$I_2 \leq C_2 \tau$$

for some $C_2 > 0$. This completes the proof of (3.5). Similarly, we can prove (3.6) and (3.7). □

Next, we obtain the L^2-differences of the space translations. Indeed, the estimates of a_ε^μ and n_ε^μ are independent of μ, whereas that for w_ε^μ is not.

Lemma 3.4 *Let $T > 0$, $r > 0$, and $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be a solution to $(\mathcal{P}_\varepsilon^\mu)$. Then, there exists a constant $C > 0$ such that*

$$\int_0^T \int_{\Omega_r} (a_\varepsilon^\mu(x+\xi,t) - a_\varepsilon^\mu(x,t))^2 dx \le C|\xi|^2, \tag{3.8}$$

$$\int_0^T \int_{\Omega_r} (w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t))^2 dx \le \frac{C}{\mu}|\xi|^2, \tag{3.9}$$

$$\int_0^T \int_{\Omega_r} (n_\varepsilon^\mu(x+\xi,t) - n_\varepsilon^\mu(x,t))^2 dx \le C|\xi|^2 \tag{3.10}$$

for all $\xi \in \mathbb{R}^N$ with $|\xi| \le r$.

Proof. For a solution (a, w, n) of $(\mathcal{P}_\varepsilon^\mu)$, we observe that

$$\begin{aligned}
\mu \int_0^T \int_{\Omega_r} & (w(x+\xi,t) - w(x,t))^2 dxdt \\
&= \mu \int_0^T \int_{\Omega_r} \left(\int_0^1 \nabla w(x+s\xi,t) \cdot \xi ds \right)^2 dxdt \\
&\le \mu|\xi|^2 \int_0^1 \int_0^T \int_{\Omega_r'} |\nabla w(x+s\xi,t)|^2 dxdtds \\
&\le \mu|\xi|^2 \int_0^T \int_{\Omega_r'} |\nabla w(x,t)|^2 dxdt \quad \le C|\xi|^2,
\end{aligned}$$

where we used the gradient estimate of w in (3.2). This completes the proof of (3.9). Similarly, we can prove (3.8) and (3.10). In particular, each positive constant C in (3.8) and (3.10) depends on d_a and d_n, respectively. □

The above L^2 estimate for w_ε^μ depends on μ. However, we can obtain a uniform L^1-estimate using the idea from the proof of [8, Lemma 3.7].

Lemma 3.5 *Let $T > 0$, $r > 0$, and $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be a solution to $(\mathcal{P}_\varepsilon^\mu)$. Then, there exists a positive function $p_\varepsilon(\xi)$ such that $p_\varepsilon(\xi) \to 0$ as $|\xi| \to 0$ and*

$$\int_0^T \int_{\Omega_r} |w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t)| \le p_\varepsilon(\xi), \tag{3.11}$$

for all $\xi \in \mathbb{R}^N$ with $|\xi| \le r$.

Proof. In this proof, we denote

$$\begin{array}{lll}
a_\xi = a(x+\xi,t), & w_\xi = w(x+\xi,t), & n_\xi = n(x+\xi,t), \\
a = a(x,t), & w = w(x,t), & n = n(x,t), \\
\hat{a} = a_\xi - a, & \hat{w} = w_\xi - w, & \hat{n} = n_\xi - n, \\
\overline{a} = a_\xi + a, & \overline{w} = w_\xi + w, & \overline{n} = n_\xi + n.
\end{array}$$

Let $m : \mathbb{R} \to \mathbb{R}_+$ be a smooth convex function such that

$$m \ge 0, \quad m(0) = 0 \quad \text{and} \quad m(r) = |r| - \frac{1}{2} \quad \text{for} \quad |r| > 1.$$

Then for $h > 0$, $m_h(r) := hm(r/h)$ is an approximation of m satisfying

$$m_h(r) \to |r| \quad \text{and} \quad m_h'(r) \to \operatorname{sgn}(r) \quad \text{as} \;\; h \to 0$$

with

$$\operatorname{sgn}(r) = \begin{cases} \;\;1 & \text{if} \;\; r > 0, \\ \;\;0 & \text{if} \;\; r = 0, \\ -1 & \text{if} \;\; r < 0. \end{cases}$$

Moreover, we define a function k such that

$$k \in C_0^\infty(\Omega_r'), \;\; 0 \le k(x) \le 1 \;\text{ in }\; \Omega_r', \;\; k(x) = 1 \;\text{ in }\; \Omega_r,$$
$$|\nabla k|, \;\; |\Delta k| \le C(r).$$

From the second equation of $(\mathcal{P}_\varepsilon^\mu)$, we obtain

$$\hat{w}_t = \mu \Delta \hat{w} + f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n).$$

Multiplying it by $km_h'(\hat{w})$ and integrating over Ω_r', we have

$$\begin{aligned}
\int_{\Omega_r'} \hat{w}_t k m_h'(\hat{w}) =& \mu \int_{\Omega_r'} \Delta \hat{w} k m_h'(\hat{w}) + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}) \\
=& -\mu \int_{\Omega_r'} \nabla \hat{w} \cdot \nabla (k m_h'(\hat{w})) + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}) \\
=& -\mu \int_{\Omega_r'} m_h'(\hat{w}) \nabla \hat{w} \cdot \nabla k - \mu \int_{\Omega_r'} k m_h''(\hat{w}) |\nabla \hat{w}|^2 \\
& + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}) \\
\le& -\mu \int_{\Omega_r'} \nabla m_h(\hat{w}) \cdot \nabla k + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}) \\
=& \mu \int_{\Omega_r'} m_h(\hat{w}) \Delta k + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}),
\end{aligned}$$

where we used integration by parts and the convexity of m_h. Since k is independent of t, we can write

$$\frac{d}{dt} \int_{\Omega_r'} m_h(\hat{w}) k \le \mu \int_{\Omega_r'} m_h(\hat{w}) \Delta k + \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}).$$

Then integrating in time implies

$$\begin{aligned}
\int_{\Omega_r'} m_h(\hat{w})(t) k \le& \int_{\Omega_r'} m_h(\hat{w})(0) k + \mu \int_0^t \int_{\Omega_r'} m_h(\hat{w}) \Delta k \\
& + \int_0^t \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k m_h'(\hat{w}).
\end{aligned} \tag{3.12}$$

By taking the limit $h \to 0$ on the both sides of (3.12), the Lebesgue dominiated convergence theorem yields

$$\begin{aligned}
\int_{\Omega_r'} |\hat{w}(t)| k \le& \int_{\Omega_r'} |\hat{w}(0)| k + \mu \int_0^t \int_{\Omega_r'} |\hat{w}| \Delta k \\
& + \int_0^t \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n)) k \operatorname{sgn}(\hat{w}).
\end{aligned} \tag{3.13}$$

As to the term containing $|\Delta k|$ in (3.13), we deduce from Hölder's inequality and (3.9) that for any $t \in (0,T)$

$$\begin{aligned}\mu \int_0^t \int_{\Omega_r'} |\hat{w}| \Delta k \leq& \sqrt{\mu} \left(\mu \int_0^t \int_{\Omega_r'} |\hat{w}|^2 \right)^{1/2} \left(\int_0^t \int_{\Omega_r'} |\Delta k|^2 \right)^{\frac{1}{2}} \\ \leq& C_1 \sqrt{T} \|\Delta k\|_{L^2(\Omega_r')} |\xi|\end{aligned} \tag{3.14}$$

for some $C_1 > 0$. Letting $C_2(r) := C_1 \sqrt{T} \|\Delta k\|_{L^2(\Omega_r')}$, (3.13) becomes

$$\begin{aligned}\int_{\Omega_r'} |\hat{w}(t)| k \leq& C_2(r)|\xi| + \int_{\Omega_r'} |\hat{w}(0)| k \\ &+ \int_0^t \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a,w,n)) k \operatorname{sgn}(\hat{w}).\end{aligned} \tag{3.15}$$

By the similar argument above with (3.8) and (3.10), we can find $C_3(r)$ and $C_4(r) > 0$ such that

$$\begin{aligned}\int_{\Omega_r'} |\hat{a}(t)| k \leq& C_3(r)|\xi| + \int_{\Omega_r'} |\hat{a}(0)| k \\ &+ \int_0^t \int_{\Omega_r'} (f_a(a_\xi, w_\xi, n_\xi) - f_a(a,w,n)) k \operatorname{sgn}(\hat{a}),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\int_{\Omega_r'} |\hat{n}(t)| k \leq& C_4(r)|\xi| + \int_{\Omega_r'} |\hat{n}(0)| k \\ &+ \int_0^t \int_{\Omega_r'} (f_n(a_\xi, w_\xi, n_\xi) - f_n(a,w,n)) k \operatorname{sgn}(\hat{n}).\end{aligned} \tag{3.17}$$

Now we focus on the nonlinear terms in (3.15)–(3.17). We observe that

$$\begin{aligned}&f_a(a_\xi, w_\xi, n_\xi) - f_a(a,w,n) \\ &= r(n_\xi a_\xi - na) + \frac{1}{\varepsilon}\left((\beta(n_\xi)w_\xi - \alpha(n_\xi)a_\xi) - (\beta(n)w - \alpha(n)a)\right) \\ &= \frac{r}{2}\Big\{(n_\xi + n)(a_\xi - a) + (n_\xi - n)(a_\xi + a)\Big\} \\ &\quad + \frac{1}{\varepsilon}\Big\{(\beta(n_\xi) - \beta(n))w_\xi + \beta(n)(w_\xi - w) - (\alpha(n_\xi) - \alpha(n))a_\xi - \alpha(n)(a_\xi - a)\Big\}, \\ &f_w(a_\xi, w_\xi, n_\xi) - f_w(a,w,n) \\ &= \frac{r}{2}\Big\{(n_\xi + n)(w_\xi - w) + (n_\xi - n)(w_\xi + w)\Big\} \\ &\quad + \frac{1}{\varepsilon}\Big\{(\alpha(n_\xi) - \alpha(n))a_\xi + \alpha(n)(a_\xi - a) - (\beta(n_\xi) - \beta(n))w_\xi - \beta(n)(w_\xi - w)\Big\}, \\ &f_n(a_\xi, w_\xi, n_\xi) - f_n(a,w,n) \\ &= -\frac{1}{2}\Big\{(n_\xi + n)(a_\xi - a + w_\xi - w) + (n_\xi - n)(a_\xi + a + w_\xi + w)\Big\}.\end{aligned}$$

Sequentially, by the hypotheses $(H_{\alpha,\beta})$, we obtain

$$\begin{aligned}
&\int_0^t \int_{\Omega_r'} (f_a(a_\xi, w_\xi, n_\xi) - f_a(a, w, n))k \operatorname{sgn}(\hat{a}) \\
&= \int_0^t \int_{\Omega_r'} k \left\{ \frac{r}{2}\overline{n}|\hat{a}| + \frac{r}{2}\hat{n}\overline{a} \operatorname{sgn}(\hat{a}) + \frac{1}{\varepsilon}(\beta(n_\xi) - \beta(n))w_\xi \operatorname{sgn}(\hat{a}) + \frac{1}{\varepsilon}\beta(n)\hat{w} \operatorname{sgn}(\hat{a}) \right. \\
&\quad \left. - \frac{1}{\varepsilon}(\alpha(n_\xi) - \alpha(n))a_\xi \operatorname{sgn}(\hat{a}) - \frac{1}{\varepsilon}\alpha(n)|\hat{a}| \right\} \\
&\leq \int_0^t \int_{\Omega_r'} C_5^\varepsilon \left(|\hat{a}| + |\hat{w}| + |\hat{n}|\right) k,
\end{aligned} \tag{3.18}$$

where $C_5^\varepsilon > 0$ is determined by the Lipschitz continuity of α and β, and the uniform boundedness of a, w, n in Lemma 2.4. Similarly, we find $C_6^\varepsilon, C_7^\varepsilon > 0$ such that

$$\begin{aligned}
&\int_0^t \int_{\Omega_r'} (f_w(a_\xi, w_\xi, n_\xi) - f_w(a, w, n))k \operatorname{sgn}(\hat{w}) \\
&= \int_0^t \int_{\Omega_r'} k \left\{ \frac{r}{2}\overline{n}|\hat{w}| + \frac{r}{2}\hat{n}\overline{w} \operatorname{sgn}(\hat{w}) + \frac{1}{\varepsilon}(\alpha(n_\xi) - \alpha(n))a_\xi \operatorname{sgn}(\hat{w}) + \frac{1}{\varepsilon}\alpha(n)\hat{a} \operatorname{sgn}(\hat{w}) \right. \\
&\quad \left. - \frac{1}{\varepsilon}(\beta(n_\xi) - \beta(n))w_\xi \operatorname{sgn}(\hat{w}) - \frac{1}{\varepsilon}\beta(n)|\hat{w}| \right\}. \\
&\leq \int_0^t \int_{\Omega_r'} C_6^\varepsilon \left(|\hat{a}| + |\hat{w}| + |\hat{n}|\right) k
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\int_0^t \int_{\Omega_r'} (f_n(a_\xi, w_\xi, n_\xi) - f_n(a, w, n))k \operatorname{sgn}(\hat{n}) \\
&= \int_0^t \int_{\Omega_r'} k \left\{ -\frac{1}{2}\overline{n}(\hat{a} - \frac{1}{2}\hat{w}) \operatorname{sgn}(\hat{n}) - \frac{1}{2}\hat{b}(\overline{a} + \overline{w}) \operatorname{sgn}(\hat{n}) \right\}. \\
&\leq \int_0^t \int_{\Omega_r'} C_7^\varepsilon \left(|a| + |w| + |n|\right) k.
\end{aligned} \tag{3.20}$$

Combining (3.15)–(3.20), we infer that for any $t \in (0, T)$

$$\begin{aligned}
\int_{\Omega_r'} (|\hat{a}(t)| + |\hat{w}(t)| + |\hat{n}(t)|)k \leq & C_8(r)|\xi| + \int_{\Omega_r'} (|\hat{a}(0)| + |\hat{w}(0)| + |\hat{n}(0)|)k \\
& + \int_0^t \int_{\Omega_r'} C_9^\varepsilon \left(|\hat{a}| + |\hat{w}| + |\hat{n}|\right) k,
\end{aligned}$$

which, together with the Grönwall inequality, implies

$$\int_0^T \int_{\Omega_r'} (|\hat{a}| + |\hat{w}| + |\hat{n}|)k \leq \left(C_8(r)|\xi| + \int_{\Omega_r'} (|\hat{a}(0)| + |\hat{w}(0)| + |\hat{n}(0)|)k \right) \exp(C_9^\varepsilon T).$$

Let $p_0(\xi)$ be a positive function such that $p_0(\xi) \to 0$ as $|\xi| \to 0$ and

$$\int_{\Omega_r'} (|\hat{a}(0)| + |\hat{w}(0)| + |\hat{n}(0)|)k \leq p_0(\xi)$$

for each $x \in \Omega_r'$ and $\xi \in \mathbb{R}^n$ with $|\xi| \le r$. Then, we conclude that

$$\int_0^T \int_{\Omega_r} (|\hat{a}| + |\hat{w}| + |\hat{n}|) \le \int_0^T \int_{\Omega_r'} (|\hat{a}(t)| + |\hat{w}(t)| + |\hat{n}(t)|)k$$
$$\le C_{10}\,(p_0(\xi) + |\xi|)\exp(C_9^\varepsilon T).$$

Choosing $p_\varepsilon(\xi) := C_{10}\,(p_0(\xi) + |\xi|)\exp(C_9^\varepsilon T)$, we find that $p_\varepsilon(\xi) \to 0$ as $|\xi| \to 0$. This completes the proof. □

Lemma 3.6 *Let $T > 0$ and $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be a solution of $(\mathcal{P}_\varepsilon^\mu)$. Then, for each fixed $\varepsilon > 0$, $\{a_\varepsilon^\mu\}$, $\{w_\varepsilon^\mu\}$ and $\{n_\varepsilon^\mu\}$ are relatively compact in $L^2(Q_T)$ as $\mu \to 0$.*

Proof. To show the compactness, we apply Lemma 2.3 for $p = 2$. By (3.5)–(3.7), the L^2-differences of time translations of a_ε^μ, w_ε^μ and n_ε^μ tend to zero as the parameter $\tau \to 0$. On the other hand, by (3.8) and (3.10), the L^2-differences of space translations of a_ε^μ and n_ε^μ tend to zero uniformly in μ as $\xi \to 0$. As to the L^2-differences of space translations of w_ε^μ, we infer from (2.6) that for any $t \in (0, T]$

$$\int_{\Omega_r} (w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t))^2 \le 2\sup_\Omega w_\varepsilon^\mu(x,t) \int_{\Omega_r} |w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t)|$$
$$\le C_1 \int_{\Omega_r} |w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t)|,$$

where the constant $C_1 > 0$ is independent of μ. Then, in view of (3.11), we obtain

$$\int_0^T \int_{\Omega_r} |w_\varepsilon^\mu(x+\xi,t) - w_\varepsilon^\mu(x,t)| \le 2C_1 p(\xi),$$

where the right hand side tends to zero uniformly in μ as $\xi \to 0$. It remains to show that condition (ii) in Lemma 2.3 holds. Thanks to the boundedness (2.6), we obtain

$$\int_{T-\tau}^T \int_{\Omega_r} (a_\varepsilon^\mu)^2 + (w_\varepsilon^\mu)^2 + (n_\varepsilon^\mu)^2 \le C_2|\Omega|\tau,$$

$$\int_0^T \int_{\Omega\setminus\Omega_r} (a_\varepsilon^\mu)^2 + (w_\varepsilon^\mu)^2 + (n_\varepsilon^\mu)^2 \le C_3 T|\partial\Omega| r,$$

where the constants $C_2, C_3 > 0$ are independent of μ. Therefore, the relative compactness of the sequences $\{a_\varepsilon^\mu\}$, $\{w_\varepsilon^\mu\}$ and $\{n_\varepsilon^\mu\}$ in $L^2(Q_T)$ is from Lemma 2.3. □

We now prove the first main theorem on the convergence as $\mu \to 0$ using previous a priori estimates.

Theorem 3.7 (Convergence as $\mu \to 0$) *Let $T > 0$ and $(H_{\alpha,\beta})$ and (H_{IC}) be satisfied. The weak solution of $(\mathcal{P}_\varepsilon^0)$ exists and is unique. The classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ of $(\mathcal{P}_\varepsilon^\mu)$ converges to the unique weak solution of $(\mathcal{P}_\varepsilon^0)$ as $\mu \to 0$.*

Proof. Let $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be a classical solution to $(\mathcal{P}_\varepsilon^\mu)$. Then by Lemma 3.6, we can pick a sequence $\mu = \mu_j$ such that for any $T > 0$, there exists $(a_\varepsilon, w_\varepsilon, n_\varepsilon) \in [L^\infty(Q_T)]^3$ such that

$$(a_\varepsilon^{\mu_j}, w_\varepsilon^{\mu_j}, n_\varepsilon^{\mu_j}) \to (a_\varepsilon, w_\varepsilon, n_\varepsilon) \quad \text{in } [L^2(Q_T)]^3 \quad \text{and a.e. in } Q_T \tag{3.21}$$

as $\mu = \mu_j \to 0$. Along such $\mu = \mu_j \to 0$, thanks to (3.2) and (3.4), we have

$$(a_\varepsilon^{\mu_j}, n_\varepsilon^{\mu_j}) \rightharpoonup (a, n) \ \text{ in } \ [L^2(0,T;H^1(\Omega))]^2. \tag{3.22}$$

We note that the classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ to $(\mathcal{P}_\varepsilon^\mu)$ satisfies the following integral identities

$$\begin{aligned}
-\iint_{Q_T} a_\varepsilon^\mu \psi_t - \int_\Omega a_0\psi(\cdot,0) &= \iint_{Q_T} d_a a_\varepsilon^\mu \Delta\psi + \iint_{Q_T} f_a(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)\psi, && (3.23)\\
-\iint_{Q_T} w_\varepsilon^\mu \psi_t - \int_\Omega w_0\psi(\cdot,0) &= \iint_{Q_T} \mu w_\varepsilon^\mu \Delta\psi + \iint_{Q_T} f_w(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)\psi, && (3.24)\\
-\iint_{Q_T} n_\varepsilon^\mu \psi_t - \int_\Omega n_0\psi(\cdot,0) &= \iint_{Q_T} d_n n_\varepsilon^\mu \Delta\psi + \iint_{Q_T} f_n(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)\psi && (3.25)
\end{aligned}$$

for all $\psi \in C^{2,1}(\overline{Q}_T)$ such that $\psi(x,T) = 0$ in Ω and $\partial_\nu \psi = 0$ on $\partial\Omega \times [0,T]$. Then, the convergences (3.21) and (3.22) show that one can pass to the limit $\mu = \mu_j \to 0$ in the weak formulation (3.23)–(3.25) to obtain (2.1)–(2.3).

To show the uniqueness of the solution, we assume that there exist two solutions (a_1, w_1, n_1) and (a_2, w_2, n_2) of $(\mathcal{P}_\varepsilon)$. Let $\tilde{a} = a_1 - a_2$, $\tilde{w} = w_1 - w_2$ and $\tilde{n} = n_1 - n_2$. Then, $(\tilde{a}, \tilde{w}, \tilde{n})$ solves

$$\begin{cases}
\tilde{a}_t = d_a\Delta\tilde{a} + r(n_1a_1 - n_2a_2) + \frac{1}{\varepsilon}(\beta(n_1)w_1 - \beta(n_2)w_2 - \alpha(n_1)a_1 + \alpha(n_2)a_2) & \text{in } Q_T,\\
\tilde{w}_t = r(n_1w_1 - n_2w_2) + \frac{1}{\varepsilon}(\alpha(n_1)a_1 - \alpha(n_2)a_2 - \beta(n_1)w_1 + \beta(n_2)w_2) & \text{in } Q_T,\\
\tilde{n}_t = d_n\Delta\tilde{n} - n_1(a_1 + w_1) + n_2(a_2 + w_2) & \text{in } Q_T.
\end{cases} \tag{3.26}$$

By the μ-independent estimates (2.6) and Lemma 3.6, we infer that

$$0 \le a_1, a_2, w_1, w_2 \le C \quad \text{and} \quad 0 \le n_1, n_2 \le \max_{x\in\overline{\Omega}} n_0(x). \tag{3.27}$$

Multiplying the equations for $\tilde{a}$, $\tilde{w}$ and $\tilde{n}$ of (3.26) by $\tilde{a}$, $\tilde{w}$ and $\tilde{n}$, respectively and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{a}^2 + \int_\Omega d_a|\nabla\tilde{a}|^2\\
&= \int_\Omega \Big\{ r(n_1\tilde{a}^2 + a_2\tilde{n}\tilde{a}) + \frac{1}{\varepsilon}(\beta(n_1)w_1 - \beta(n_1)w_2 + \beta(n_1)w_2 - \beta(n_2)w_2)\tilde{a}\\
&\quad - \frac{1}{\varepsilon}(\alpha(n_1)a_1 - \alpha(n_1)a_2 + \alpha(n_1)a_2 - \alpha(n_2)a_2)\tilde{a} \Big\}\\
&\le C_1^\varepsilon \int_\Omega \left(\tilde{a}^2 + |\tilde{n}\tilde{a}| + |\tilde{w}\tilde{a}| + |\tilde{n}\tilde{a}| + \tilde{a}^2 + |\tilde{n}\tilde{a}|\right),
\end{aligned}$$

where we used the boundedness (3.27) and the hypotheses $(H_{\alpha,\beta})$. Thus, Young's inequality implies

$$\frac{d}{dt}\int_\Omega \tilde{a}^2 \le C_2^\varepsilon \int_\Omega \left(\tilde{a}^2 + \tilde{w}^2 + \tilde{n}^2\right).$$

Repeating the similar procedure for $\tilde{w}$ and $\tilde{n}$, we obtain

$$\frac{d}{dt}\int_\Omega \left(\tilde{a}^2 + \tilde{w}^2 + \tilde{n}^2\right) \le C_3^\varepsilon \int_\Omega \left(\tilde{a}^2 + \tilde{w}^2 + \tilde{n}^2\right).$$

Thus, Grönwall's inequality completes the proof for the uniqueness. $\square$

4 Singular limit as $\varepsilon \to 0$ for $\mu > 0$ fixed

In this section, we prove the existence and the uniqueness of a weak solution (b^μ, n^μ) to $(\mathcal{P}_0^\mu)$. The solution pair (b^μ, n^μ) are obtained by taking a subsequential limit so that

$$(a^\mu_{\varepsilon_j} + w^\mu_{\varepsilon_j}, n^\mu_{\varepsilon_j}) \to (b^\mu, n^\mu) \quad \text{as} \ \ \varepsilon_j \to 0$$

for some $\varepsilon_j \in (0,1)$. To this end, we need ε-independent estimates on $(a^\mu_\varepsilon, w^\mu_\varepsilon, n^\mu_\varepsilon)$. In this section, the parameter $\mu > 0$ is fixed and the generic constant C is independent of ε. Using the self-adjoint realization of the Laplace operator, the following lemma provides an L^2-estimate for a^μ_ε and w^μ_ε (see also [11, 12, 13]).

Lemma 4.1 *Let $T > 0$ and $(a^\mu_\varepsilon, w^\mu_\varepsilon, n^\mu_\varepsilon)$ be solutions to $(\mathcal{P}^\mu_\varepsilon)$. Then, there exists a constant $C > 0$ such that*

$$\iint_{Q_T} (a^\mu_\varepsilon)^2 + (w^\mu_\varepsilon)^2 \le C. \tag{4.1}$$

Proof. Let $\mathcal{A}$ be a self-adjoint realization of $-\Delta$ defined on

$$D(\mathcal{A}) := \{\phi \in H^2(\Omega) | \int_\Omega \phi = 0 \ \text{ and } \ \frac{\partial \phi}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega\}.$$

Multiplying the equation for n of $(\mathcal{P}^\mu_\varepsilon)$ by r and adding the result to the equations for a and w, we have

$$(a + w + rn)_t = \Delta(d_a a + \mu w + r d_n n). \tag{4.2}$$

Letting $\rho := a + w + rn$ and $M := \frac{d_a a + \mu w + r d_n n}{a + w + rn}$, (4.2) turns into

$$\rho_t = \Delta(M\rho) \tag{4.3}$$

Since $\partial_\nu(M\rho) = 0$ in Γ_T, by integrating (4.3) over Ω, we obtain

$$\overline{\rho}(\cdot, t) = \frac{1}{\Omega}\int_\Omega \rho(\cdot, t) = \frac{1}{\Omega}\int_\Omega \rho_0 = \overline{\rho}_0 \quad \text{for any} \ \ t > 0.$$

Using $\mathcal{A}$, we rewrite (4.3) as

$$(\rho - \overline{\rho})_t = -\mathcal{A}(M\rho - \overline{M\rho}). \tag{4.4}$$

Multiplying (4.4) by $\mathcal{A}^{-1}(\rho - \overline{\rho})$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(\rho - \overline{\rho})|^2 &= -\int_\Omega \mathcal{A}\left(M\rho - \overline{M\rho}\right) \cdot \mathcal{A}^{-1}(\rho - \overline{\rho}) \\
&= -\int_\Omega \left(M\rho - \overline{M\rho}\right) \cdot (\rho - \overline{\rho}) \\
&= -\int_\Omega M(\rho - \overline{\rho})^2 - \overline{\rho}\int_\Omega M(\rho - \overline{\rho}).
\end{aligned}$$

Let $M_1 := \min\{d_a, \mu, r d_n\}$ and $M_2 := \max\{d_a, \mu, r d_n\}$. Since $0 < M_1 \le M \le M_2$, we infer that

$$\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(\rho - \overline{\rho})|^2 + 2\int_\Omega M(\rho - \overline{\rho})^2 = -2\overline{\rho}\int_\Omega M(\rho - \overline{\rho}) \le 2\overline{\rho}^2 \int_\Omega M \le C_1, \tag{4.5}$$

where $C_1 = 2\overline{\rho}^2|\Omega|M_2$. Integrating (4.5) in time, we obtain

$$\iint_{Q_T} M(\rho - \overline{\rho})^2 \leq \frac{1}{2}\left(C_1 T + \int_\Omega |\mathcal{A}^{-\frac{1}{2}}(\rho_0 - \overline{\rho_0})|^2\right) \leq C_2(T+1) \tag{4.6}$$

for some $C_2 > 0$ depending on ρ_0. It follows from (4.5), (4.6) and the inequality $a^2 \leq C_3((a-b)^2 + b^2)$ for sufficiently large $C_3 > 0$ that

$$\iint_{Q_T} M\rho^2 \leq C_3 \iint_{Q_T} M(\rho - \overline{\rho})^2 + C_3 \iint_{Q_T} M\overline{\rho}^2 \leq C_2 C_3 (T+1) + \frac{C_1 C_3 T}{2}.$$

Since $M\rho^2 = (d_a a + \mu w + r d_n n)(a + w + rn)$, the nonnegativity of a and w implies (4.1). This completes the proof. □

Lemma 4.2 *Let $T > 0$ and $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ be solutions to $(\mathcal{P}_\varepsilon^\mu)$. Then, there exists a constant $C > 0$ such that*

$$\int_0^T \|\partial_t n_\varepsilon^\mu\|^2_{L^2(H^1(\Omega))^*} \leq C. \tag{4.7}$$

Proof. Given $\varphi \in H^1(\Omega)$, multiplying the equation for n of $(\mathcal{P}_\varepsilon^\mu)$ by φ and integrating by parts, there exists $C_1 > 0$ such that

$$\begin{aligned}\int_\Omega \partial_t n\varphi &= -\int_\Omega \nabla n \cdot \nabla\varphi - \int_\Omega n(a+w)\varphi \\ &\leq C_1\left(\|\nabla n\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}\right)\|\varphi\|_{H^1(\Omega)},\end{aligned}$$

where we used the boundedness of n in (2.6). It follows from (3.4) and(4.1) that

$$\int_0^T \|\partial_t n\|^2_{(H^1(\Omega))^*} \leq C_1\left(\int_0^T\int_\Omega |\nabla n|^2 + \int_0^T\int_\Omega (a^2 + w^2)\right) \leq C_2(T+1)$$

for some $C_2 > 0$. This completes the proof. □

We now prove the second main theorem on the convergence as $\varepsilon \to 0$ using previous a priori estimates.

Theorem 4.3 (Convergence as $\varepsilon \to 0$) *Let $T > 0$ and $(H_{\alpha,\beta})$ and (H_{IC}) be satisfied. The weak solution of $(\mathcal{P}_0^\mu)$ exists and is unique. The classical solution $(a_\varepsilon^\mu, w_\varepsilon^\mu, n_\varepsilon^\mu)$ of $(\mathcal{P}_\varepsilon^\mu)$ converges to a limit (a^μ, w^μ, n^μ) as $\varepsilon \to 0$ and $(a^\mu + w^\mu, n)$ is the unique weak solution of $(\mathcal{P}_0^\mu)$.*

Proof. From (2.6), (3.4), (4.1) and (4.7), we deduce that

$$\begin{cases} a_\varepsilon^\mu,\ w_\varepsilon^\mu \in L^2(Q_T), \\ n_\varepsilon^\mu \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*). \end{cases}$$

Since the Sobolev space H^1 is compactly embedded in L^2, and L^2 is continuously embedded in $(H^1(\Omega))^*$, by the Aubin-Lions lemma (Theorem 2.3 in [19]), we find that $\{n_\varepsilon^\mu\}_{\varepsilon_j \in (0,1)}$ is strongly precompact in $L^2(Q_T)$. Therefore, there exist (a^μ, w^μ, n^μ) satisfying the following convergences

$$a^\mu_\varepsilon \rightharpoonup a^\mu \qquad \text{in } \ L^2(Q_T), \tag{4.8}$$
$$w^\mu_\varepsilon \rightharpoonup w^\mu \qquad \text{in } \ L^2(Q_T), \tag{4.9}$$
$$n^\mu_\varepsilon \to n^\mu \qquad \text{in } \ L^2(Q_T) \ \text{ and } \ a.e. \ \text{ in } \ Q_T, \tag{4.10}$$

along a subsequence $\varepsilon = \varepsilon_{j_i}$, $i \to \infty$. Then, we infer from (4.10) that

$$\alpha(n^\mu_\varepsilon) \to \alpha(n^\mu), \quad \beta(n^\mu_\varepsilon) \to \beta(n^\mu) \quad \text{in } L^2(Q_T) \ \text{ and } \ a.e. \ \text{ in } \ Q_T \tag{4.11}$$

Multiplying the equation for a^μ_ε in $(\mathcal{P}^\mu_\varepsilon)$ by $\varphi \in C_0^\infty(Q_T)$ and integrating over Q_T, we obtain

$$-\varepsilon \iint_{Q_T} a^\mu_\varepsilon \varphi_t = \varepsilon \left(\iint_{Q_T} d_a a^\mu_\varepsilon \Delta\varphi + \iint_{Q_T} r n^\mu_\varepsilon a^\mu_\varepsilon \varphi \right) + \iint_{Q_T} (\beta(n^\mu_\varepsilon) w^\mu_\varepsilon - \alpha(n^\mu_\varepsilon) a^\mu_\varepsilon)\varphi,$$

which by taking the limit as $\varepsilon = \varepsilon_j \to 0$ turns into

$$0 = \iint_{Q_T} (\beta(n^\mu) w^\mu - \alpha(n^\mu) a^\mu)\varphi,$$

where we used the convergence (4.8)–(4.11). Thus, $\beta(n^\mu)w^\mu = \alpha(n^\mu)a^\mu$ $a.e.$ in Q_T. Now, we multiply the equations for a^μ_ε and w^μ_ε in $(\mathcal{P}^\mu_\varepsilon)$ by $\psi \in C^{2,1}(\overline{Q}_T)$ such that $\psi(x,T) = 0$ in Ω and $\partial_\nu \psi = 0$ on $\partial\Omega \times [0,T]$. Adding the results, we have

$$-\iint_{Q_T} (a^\mu_\varepsilon + w^\mu_\varepsilon)\psi_t - \int_\Omega (a_0 + w_0)\psi(\cdot,0) = \iint_{Q_T} (d_a a^\mu_\varepsilon + \mu w^\mu_\varepsilon)\Delta\psi + \iint_{Q_T} r n^\mu_\varepsilon (a^\mu_\varepsilon + w^\mu_\varepsilon)\psi.$$

Taking the limit as $\varepsilon = \varepsilon_j \to 0$, we obtain

$$-\iint_{Q_T} b^\mu \psi_t - \int_\Omega b_0 \psi(\cdot,0) = \iint_{Q_T} \gamma_\mu(n^\mu) b^\mu \Delta\psi + \iint_{Q_T} r n^\mu b^\mu \psi,$$

where $b^\mu = a^\mu + w^\mu$ and γ_μ is given by (1.1). As to the equation for n^μ_ε in $(\mathcal{P}^\mu_\varepsilon)$, we have

$$-\iint_{Q_T} n^\mu_\varepsilon \psi_t - \int_\Omega n_0 \psi(\cdot,0) = \iint_{Q_T} d_n n^\mu_\varepsilon \Delta\psi - \iint_{Q_T} n^\mu_\varepsilon (a^\mu_\varepsilon + w^\mu_\varepsilon)\psi$$

for $\psi \in C^{2,1}(\overline{Q}_T)$ such that $\psi(x,T) = 0$ in Ω and $\partial_\nu \psi = 0$ on $\partial\Omega \times [0,T]$. Again taking the limit as $\varepsilon = \varepsilon_j \to 0$ and using (4.8)–(4.10), we obtain

$$-\iint_{Q_T} n^\mu \psi_t - \int_\Omega n_0 \psi(\cdot,0) = \iint_{Q_T} d_n n^\mu \Delta\psi - \iint_{Q_T} n^\mu b^\mu \psi.$$

The uniqueness can be proved by a similar manner in the proof of Theorem 3.7. $\square$

5 Traveling wave solution

We consider traveling wave solution of $(\mathcal{P}_0^\mu)$ with $\mu = 0$ in one space dimension,

$$\begin{cases} b_t = (\gamma_0(n)b)_{xx} + rnb \\ n_t = -nb \end{cases} \qquad \text{for } x \in \mathbf{R},\ t > 0, \tag{5.1}$$

where we take the diffusivity of the resource as $d_n = 0$ as Keller and Segel did in their seminal paper [14]. Note that Keller-Segel equations do not contain a population growth term and the traveling wave phenomenon is driven by an advection term. However, Eq. (5.1) contains a reaction term and a traveling wave phenomenon appears by an interaction between the Fokker-Planck type diffusion and the reaction.

We introduce a moving frame variable $\xi = x - ct$, where c is a constant wave speed. The traveling wave solution, $(b(x,t), n(x,t)) = (b(x-ct), n(x-ct))$, satisfies

$$\begin{cases} cb' = -(\gamma_0(n)b)'' - rnb \\ cn' = nb \end{cases} \qquad \text{for } \xi \in \mathbf{R}, \tag{5.2}$$

where the notation $'$ denotes the differentiation with respect to ξ. We consider traveling wave solutions that satisfy boundary conditions,

$$b \to b_\pm \geq 0,\ n \to n_\pm \geq 0,\ b' \to 0,\ n' \to 0 \quad \text{as } \xi \to \pm\infty. \tag{5.3}$$

In the next theorem, we show necessary conditions on the boundary values $b_\pm, n_\pm$ for the existence of a traveling wave solution with a positive wave speed. We also find the monotonicity of traveling wave solutions.

Theorem 5.1 (Monotonicity) *Suppose that there exists a smooth nontrivial traveling wave solution (b, n) of (5.2)–(5.3) with a positive wave speed $c > 0$. (i) The nutrient density n is an increasing function, and the boundary values satisfy*

$$n_- = 0, \quad b_+ = 0, \quad \text{and} \quad b_- = rn_+ > 0. \tag{5.4}$$

(ii) If $\gamma_0''(n) \geq 0$, the cell density b is a decreasing function.

Proof. (i) The monotonicity of n comes from the second equation of (5.2) and hence $n_+ > 0$. Since $n' \to 0$ as $\xi \to \pm\infty$, we should have $n_\pm b_\pm = 0$ and hence $b_+ = 0$ by the second equation of (5.2). To see the boundary values at $-\infty$, integrate the first equation of (5.2). Then, we have

$$c(b_+ - b_-) = -cb_- = -\int_{-\infty}^{\infty} rnb\, d\xi < 0.$$

Therefore, $b_- > 0$ and $n_- = 0$. Since $cn' = nb$, the above relation gives

$$cb_- = \int_{-\infty}^{\infty} crn'\, d\xi = cr(n_+ - n_-) = crn_+.$$

Therefore, $b_- = rn_+$.

(ii) First, we show that $n(\xi) > 0$ for all $\xi \in \mathbf{R}$. Suppose to the contrary that there exists ξ_0 such that $n(\xi_0) = 0$. Then, by the monotonicity of n, $n(\xi) = 0$ for all $\xi \leq \xi_0$. From (5.2), we have

$$cb' = -(\gamma_0(n)b)'' - rcn'.$$

Integrating the result from $-\infty$ to ξ, we obtain

$$c(b-b_-) = -(\gamma_0(n)b)' - rcn = -\frac{d\gamma_0}{dn}n'b - \gamma_0(n)b' - rcn. \tag{5.5}$$

Since $n(\xi)=0$ for $\xi \le \xi_0$ and $\gamma_0(0)=0$, we have $b(\xi)=b_-$ for all $\xi \le \xi_0$. We may consider the second equation of (5.2) as an ODE for n with a given b. Then, by the uniqueness of an initial value problem, the solution should be the trivial one $n(\xi)\equiv 0$ for all $\xi \in \mathbf{R}$.

For the cell density b, we show that $b'(\xi) \le 0$ for $0 < b(\xi) < rn_+$. The first equation of (5.2) gives that

$$\gamma_0(n)b'' + \left(2\frac{d\gamma_0}{dn}n' + c\right)b' + \left(\frac{d^2\gamma_0}{dn^2}(n')^2 + \frac{d\gamma_0}{dn}n'' + rn\right)b = 0. \tag{5.6}$$

Taking the derivative with respect to ξ to the second equation of (5.2), we have

$$cn'' = n'b + nb'. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain

$$\gamma_0(n)b'' + \left(2\frac{d\gamma_0}{dn}n' + \frac{1}{c}\frac{d\gamma_0}{dn}nb + c\right)b' + \left(\frac{d^2\gamma_0}{dn^2}(n')^2 + \frac{1}{c}\frac{d\gamma_0}{dn}n'b + rn\right)b = 0.$$

Then, due to the non-negativity of $\gamma_0'(n), \gamma_0''(n), n'$, and b, and the strict positivity of n we have

$$\gamma_0(n)b'' \le -\left(2\frac{d\gamma_0}{dn}n' + \frac{1}{c}\frac{d\gamma_0}{dn}nb + c\right)b'. \tag{5.8}$$

Next, we assume b is not a decreasing function and derive a contradiction. If so, there exists $\xi_1 \in \mathbb{R}$ such that $b'(\xi_1) > 0$. Since $b_+ = 0 < b(\xi_1)$, there exists $\xi_0 > 0$ such that $b'(\xi_0) = 0$ and $b'(\xi) > 0$ on $[\xi_1, \xi_0)$. There are two possible cases. First, if $b'(\xi) > 0$ on $(-\infty, \xi_0)$, (5.8) implies that $b'' \le 0$ for on $(-\infty, \xi_0)$. However, by the boundary conditions, $b'(\xi) \to 0$ as $\xi \to -\infty$ and as $\xi \to \xi_0$. Concave functions that satisfy the boundary conditions are only constant functions, which is a contradiction. Second, if there exists $\xi < \xi_1$ such that $b'(\xi) \le 0$, there exists $\xi_2 < \xi_1$ such that $b'(\xi) > 0$ on (ξ_2, ξ_0) and $b'(\xi_2) = 0$. Then, by (5.8) again, b is concave on (ξ_2, ξ_0). Therefore, b is a constant function which is a contradiction again. Therefore, there is no point such that $b'(\xi) > 0$ and b is a decreasing function. □

In the next theorem, we show the necessary conditions on the boundary values $b_\pm, n_\pm$ for the existence of a traveling wave solution is also sufficient if the wave speed is greater than a minimum speed.

Theorem 5.2 (Existence) *Suppose that the boundary values satisfy* (5.4). *There exists a nontrivial traveling wave solution* (b, n) *of* (5.2)–(5.3) *for all* $c \ge c^* := 2\sqrt{r\gamma_0(n_+)n_+}$.

In the following discussion, we construct the traveling wave solution using classical phase plane analysis and complete the proof of the theorem. Denote

$$u := \gamma_0(n)b.$$

Then, the first equation of (5.2) is written as

$$cb' = -u'' - rnb = -u'' - crn'.$$

Integrate it over (ξ, ∞) and obtain

$$-cb = u' + cr(n - n_+).$$

Multiply $\gamma_0(n)$ and rewrite it as

$$\gamma_0(n)u' = -cu - c\gamma_0(n)r(n - n_+).$$

The second equation of (5.2) is written as

$$c\gamma_0(n)n' = un.$$

Since $\gamma_0(n) > 0$ and n is monotone, we may define a new variable z such that $\frac{dz}{d\xi} = \frac{1}{\gamma_0(n)}$. From now on, we take the same notation $\dot{\square}$ for the differentiation with respect to z. Then, we obtain

$$\begin{cases} \dot{n} = \frac{1}{c}un, \\ \dot{u} = -cu - cr\gamma_0(n)(n - n_+), \end{cases} \tag{5.9}$$

which takes two critical points,

$$(n, u) = (0, 0) \text{ and } (n_+, 0).$$

Next, we construct a traveling wave trajectory that starts from $(n, u) = (0, 0)$ and heads to $(n, u) = (n_+, 0)$ using phase plane analysis.

For the stability analysis, consider the Jacobian matrix of the system,

$$J = \begin{bmatrix} u/c & n/c \\ -cr\gamma_0'(n)(n - n_+) - cr\gamma_0(n) & -c \end{bmatrix}.$$

For the steady state, $(n, u) = (n_+, 0)$, the eigenvalues satisfy

$$\lambda^2 + c\lambda + r\gamma_0(n_+)n_+ = 0.$$

Since $r\gamma_0(n_+)n_+ > 0$ and $c > 0$, the real part of eigenvalues are negative. Hence, $(n, u) = (n_+, 0)$ is a stable steady state. If

$$c > 2\sqrt{r\gamma_0(n_+)n_+} =: c^*,$$

then both eigenvalues are real and negative. If $c < c^*$, the two eigenvalues are complex numbers with negative real parts. Therefore, $(n_+, 0)$ is a spiral and nonnegative traveling wave does not exist. Let $c > c^*$ and λ_1 and λ_2 be the negative eigenvalues such that $\lambda_2 < \lambda_1 < 0$. Then, the corresponding eigenvectors are

$$\mathbf{e}_i = \begin{bmatrix} 1 \\ \frac{c\lambda_i}{n_+} \end{bmatrix}, \quad i = 1, 2.$$

Consider the other steady state, $(n, u) = (0, 0)$. First we know that this is an unstable steady state. Since $\gamma_0(0) = 0$, eigenvalues satisfy

$$\lambda^2 + c\lambda = 0.$$

Denote the two eigenvalues by $\lambda_3 = 0$ and $\lambda_4 = -c$ and the corresponding eigenvectors by $\mathbf{e}_3$, $\mathbf{e}_4$, where

$$\mathbf{e}_3 = \begin{bmatrix} 1 \\ r\gamma_0'(0)n_+ \end{bmatrix} \text{ and } \mathbf{e}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since one of the eigenvalues is zero, the linear analysis does not decide the stability of the steady state. To determine the solution trajectory at $(0,0)$, we consider a normal form by taking a new variable $v := u - r\gamma_0'(0)n_+n$. Then, by (5.9), the equation of n turns into

$$c\dot{n} = nu = n(r\gamma_0'(0)n_+n + v) = r\gamma_0'(0)n_+n^2 + nv.$$

Therefore, v satisfies

$$\begin{aligned} \dot{v} &= -r\gamma_0'(0)n_+\dot{n} + \dot{u} \\ &= -\frac{(r\gamma_0'(0)n_+)^2}{c}n^2 - \frac{r\gamma_0'(0)n_+}{c}nv - cv - cr\gamma_0'(0)n_+n - cr\gamma_0(n)(n-n_+). \end{aligned} \tag{5.10}$$

Thus, the new system in (n,v) is

$$\begin{cases} \dot{n} = \frac{r\gamma_0'(0)n_+}{c}n^2 + \frac{1}{c}nv, \\ \dot{v} = -\frac{(r\gamma_0'(0)n_+)^2}{c}n^2 - \frac{r\gamma_0'(0)n_+}{c}nv - cv - cr\gamma_0'(0)n_+n - cr\gamma_0(n)(n-n_+). \end{cases} \tag{5.11}$$

Linearizing the system at $(n,v) = (0,0)$, we obtain

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix},$$

which enables us to find the same eigenvalues $\lambda_5 = 0, \lambda_6 = -c$ with corresponding eigenvectors

$$\mathbf{e}_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{e}_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Following the center manifold theorem in [4], there is a solution trajectory of (5.10) locally around $(0,0)$, which is approximated by

$$v = h(n) = c_2n^2 + c_3n^3 + O(n^4) \tag{5.12}$$

so that the center manifold is tangent to $\mathbf{e}_5$. Then, by the equation of n in (5.11), we have

$$\dot{v} = h'(n)\dot{n} = (2c_2n + 3c_3n^2 + \cdots)\left(\frac{r\gamma_0'(n)n_+}{c}n^2 + \frac{n}{c}(c_2n^2 + c_3n^3 + \cdots)\right). \tag{5.13}$$

On the other hand, substituting (5.12) into the equation of v in (5.11) and using a Taylor expansion of $\gamma(n)$, we find

$$\begin{aligned} \dot{v} = &-\frac{(r\gamma_0'(0)n_+)^2}{c}n^2 - \frac{r\gamma_0'(0)n_+}{c}n(c_2n^2 + c_3n^3 + \cdots) - c(c_2n^2 + c_3n^3 \cdots) \\ &- cr\gamma_0'(0)n_+n - cr\left(\gamma_0'(0)n + \frac{\gamma_0''(0)}{2}n^2 + \cdots\right)(n - n_+). \end{aligned} \tag{5.14}$$

Equating the coefficients of n^2 in (5.13) and (5.14), we obtain

$$c_2 = -\frac{(r\gamma_0'(0)n_+)^2}{c^2} - r\gamma_0'(0) + \frac{r\gamma_0''(0)n_+}{2}.$$

Thus, the center manifold for (5.11) is

$$v = c_2 n^2 + O(n^3),$$

which determines the center manifold for (5.9) such that

$$u = r\gamma_0'(0)n_+ n + c_2 n^2 + O(n^3).$$

Substituting the results into the equation of n in (5.9), we have

$$\frac{dn}{dz} = \frac{1}{c}\left(r\gamma_0'(0)n_+ n^2 + c_2 n^3 + O(n^4)\right).$$

so that $n \to 0$ as well as $u \to 0$ as $z \to -\infty$. Therefore, the solution trajectory starts from $(0,0)$ along the $\mathbf{e}_3$ direction.

Remark 5.3 *Notice that* $(0,0)$ *is unstable and* $(n_+,0)$ *is stable. Therefore, the traveling wave is a mono-stable one. We expect that there is a minimum wave speed* $c^* > 0$ *and there exists a traveling wave for each* $c \geq c^*$*. The more meaningful traveling wave is the minimum speed one. We should understand the case* $c = c^*$ *more than other cases.*

The phase plane equation of (5.9) is

$$\frac{du}{dn} = \frac{c^2}{un}(-u - r\gamma_0(n)(n - n_+)).$$

Denote the zero slope isocline by

$$u(n) = -r\gamma_0(n)(n - n_+).$$

The monotonicity of the solution switches along this curve. If $c > c^*$, then

$$\left.\frac{du}{dn}\right|_{n=n_+} = -r\gamma_0(n_+) > \frac{c\lambda_2}{n_+},$$

where $c\lambda_2/n_+$ is the slope of the eigenvector $\mathbf{e}_2$.

Now we find an invariant region of the vector field to (5.9). We may consider the triangular region which is bounded by the u-axis, n-axis, and a straight line,

$$u = f(n) := -k(n - n_+),$$

for some $k > 0$. We want to show that there exists $k > 0$ such that, for all $0 < n < n_+$,

$$\left.\frac{du}{dn}\right|_{u=f(n)} < -k. \tag{5.15}$$

The inequality in (5.15) is written as

$$\frac{c^2}{-k(n - n_+)n}\left(k(n - n_+) - r\gamma_0(n)(n - n_+)\right) < -k.$$

Direct calculations show that the above is equivalent to

$$nk^2 - c^2 k + c^2 r\gamma_0(n) < 0 \text{ for } n < n_+. \tag{5.16}$$

Since $\gamma_0(n)$ is an increasing function, (5.16) holds if

$$n_+k^2 - c^2k + c^2r\gamma_0(n_+) < 0. \tag{5.17}$$

If $c > c^*$, then $c^4 - 4c^2r\gamma_0(n_+)n_+ > 0$. Therefore, there exists $k > 0$ such that (5.17) holds. The above results implies that the slope of the vector field on the line $u = f(n)$ is always less than the slope of $u = f(n)$. Furthermore, the slope of the vector field on the u-axis (n-axis) is negative infinite (positive infinite). In conclusion, we obtain the existence of the trajectory from $(0, 0)$ to $(0, n_+)$.

Remark 5.4 *The above existence result has been shown under $\gamma_0(n)n < \gamma_0(n_+)n_+$ for all $n < n_+$. Note that $\gamma_0(n)n < \gamma_0(n_+)n_+$ holds for all $n_+ > 0$ if and only if $\gamma_0(n)n$ is increasing. Hence, the monotonicity of $\gamma_0(n)$ is not essential.*

6 Discussions and Memories

The late Professor Mayan Mimura introduced the model system $(\mathcal{P}_\varepsilon^0)$ to Y.-J. Kim and C. Yoon in 2016 as a mesoscopic-scale level bacterial behavior model that connects microscopic-scale bacterial wave phenomena to a macroscopic-scale nonlinear diffusion model. However, after the first round of discussions at Meiji University in 2017, the project was forgotten. Recently, chemotaxis models with nonlinear diffusion have been actively studied with or without population dynamics (see [3, 6, 9, 16, 17, 20, 21, 22]) and they realized that Mayan's idea provides a theoretical background of such chemotaxis models. The project was brought back to life for the purpose and the first result is this paper.

The original idea was to show that the solutions of $(\mathcal{P}_\varepsilon^0)$ converge to the solutions of $(\mathcal{P}_0^0)$ as $\varepsilon \to 0$. However, the singularity in the problem $(\mathcal{P}_\varepsilon^0)$ did not allow the convergence proof to authors. Instead, they considered a regularized problem $(\mathcal{P}_\varepsilon^\mu)$ and showed convergence as $\varepsilon \to 0$ with fixed $\mu > 0$ and as $\mu \to 0$ with fixed $\varepsilon > 0$. These processes show the dynamics how the linear diffusion in the mesoscopic-scale level can produce a nonlinear diffusion in a macroscopic-scale level. In Section 5, it is shown that the resulting nonlinear diffusion problem provides a traveling wave phenomenon. However, the obtained traveling wave is not of pulse-type, but of Fisher-KPP type. Indeed, only the minimum wave speed, $c^* = \sqrt{r\gamma_0(n_+)n_+}$, is decided, where the boundary condition n_+ is the amount of resource before consumption. Then, there exist traveling wave solutions for all speed $c \geq c^*$. Furthermore, the shape of traveling waves are not of pulse-type, but of front-type with monotonicity. The main conclusion of the paper is that linear diffusion does not produce wave pulse (or band) and the chemotactic wave pulse is an indication of chemotactic mechanisms.

In the accompanying modeling paper [15], two modifications of $(\mathcal{P}_\varepsilon^\mu)$ are considered to explain experimentally observed patterns. In the first modification, chemotaxis has been added to active cells. The only change is the replacement of the linear diffusion $d_a\Delta a$ with $\Delta(\phi(n)a)$ with a decreasing function ϕ. This change brought traveling wave band of the experiments. In the second modification, an extra resource is added which correspond to oxygen in the experiment. This modification gave the second chemotactic wave. These studied are done numerically only. We can see that the linear model $(\mathcal{P}_\varepsilon^\mu)$ can play as a foundation of various models for the study of chemotaxis phenomena.

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