HETEROGENEOUS DISCRETE-TIME RANDOM WALK AND REFERENCE POINT DEPENDENCY

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Abstract. There have been discussions and debates on the correct diffusion law for a long time when diffusivity varies in space. We consider heterogeneous random walk system in the paper and show that the heterogeneity in the walk-length behaves very differently in comparison with the one in the sojourn time. As a conclusion, we will see that the diffusion law cannot be given in terms of diffusivity alone and hence, we need a diffusion law with two components.

1. Introduction

Random walk models are often used in explaining the diffusion coefficient and diffusion equation satisfied by Brownian particle systems. In a position jump model, particles choose the moving direction randomly every sojourn time $\triangle t$ and jump the walk-length $\triangle x$. Then, in n-space dimensions, the diffusivity is given by

$$D = \frac{|\triangle x|^2}{2n\triangle t}. \tag{1.1}$$

In a homogeneous environment, the parameters are assumed to be constant and the diffusion equation is given by

$$u_t = D\Delta u, \tag{1.2}$$

where $u = u(t, x)$ is the particle density, $\Delta u := \sum_{i=1}^n \partial_{x_i}^2 u$ is the Laplacian, $t > 0$ is the time variable, and $x \in \mathbb{R}^n$ is the space variable.

The paper aims to introduce a discrete-time, continuous-space random walk model in a heterogeneous environment and show the corresponding diffusion equation by taking the diffusion limit of the solution. We will see how the diffusion equation is decided in the process. In a heterogeneous environment, the two parameters, $\triangle x$ and $\triangle t$, vary spatially, and hence the diffusivity D is not constant. It should be noted that (1.1) is the corresponding diffusivity, but (1.2) is not the correct diffusion equation. We will see that the diffusivity D is decided by the heterogeneity, but the diffusion equation is decided how to use it, i.e., by the choice of reference point.

Date: September, 2022.

Fick's diffusion equation has been widely used as a heterogeneous diffusion law which is the given by

$$u_t = \nabla \cdot (D\nabla u). \tag{1.3}$$

However, it fails to explain observed phenomenon in many situations (see [2, 8, 16, 23]). The temperature difference is the one studied the most with the longest history (see [13, 17, 18]). The particles in a warm region move more actively than the ones in a cold region. We may model such heterogeneity by taking nonconstant parameters,

$$\triangle x\big|_{\text{at } p} = \epsilon\ell(p), \quad \triangle t\big|_{\text{at } p} = \epsilon^2\tau(p), \tag{1.4}$$

where the small parameter $\epsilon > 0$ is to take a parabolic scale limit and ℓ and τ are given functions. However, the relation (1.4) does not provide a complete information and the way to take the heterogeneities in ℓ and τ should be specified to complete a random walk model. For example, suppose that a particle jumps from a departure point x to an arrival point y. Then, one may take the walk length and the sojourn time as

$$\triangle x\big|_{x\to y} = \epsilon\ell(ay + (1-a)x), \quad \triangle t\big|_{x\to y} = \epsilon^2\tau(by + (1-b)x), \tag{1.5}$$

where $a, b \in [0, 1]$ are the parameters that decide reference points. If $a = b = 0$, the reference point is the departure point x, if $a = b = 1$, the arrival point y, and if $a = b = \frac{1}{2}$, the middle point between x and y. One of the central aspect of the paper is that the obtained diffusion equation depends on the parameter a, but not on b. In other words, the heterogeneity in $\triangle t$ involved in the random walk system very differently from the one in $\triangle x$.

1.1. **Langevin equation.** Einstein [5] introduced the idea of discrete-time random walk and explained the diffusion phenomenon using the Brownian particle motion. Three years later, Langevin [12] introduced the stochastic differential equation (or SDE for brevity) and obtained the same result in a simpler way. The theory of SDE has been developed a lot and is now used widely for random phenomena (see [7, 20]). In a heterogeneous environment, the Langevin equation becomes a nonlinear stochastic differential equation,

$$dx = g(x)dW. \tag{1.6}$$

where $x = x(t)$ is the position of a particle at time t and $dW = dW(t)$ is a stationary stochastic process. The drift term has been dropped from the equation since we consider the model without advection. However, this equation is meaningless until the way of integration is specified, and there have been debates about the right way of handling it (see [15, 19]).

The issue becomes more clear if (1.6) is rewritten in a Riemann-Stieltjes integration form,

$$x(t) = x(0) + \sum_{i=1}^{N} g(p_i)(W(t_i) - W(t_{i-1})), \tag{1.7}$$

where $0 = t_0 < t_1, \cdots < t_N = t$ is a partition of the interval $[0,t]$. The function $g(x)$ is the deviation if $dW(t)$ is the Gaussian white noise and plays the role of the walk-length $\triangle x$ in our case. However, this relation has meaning only when $p_i \in [t_{i-1}, t_i]$ is specified, which plays the role of reference point in (1.5). If the choice is $p_i = x(t_{i-1})$, (1.7) is called an Ito type and the probability density function of the stochastic process satisfies a Fokker-Plank equation

$$u_t = \Delta(Du), \tag{1.8}$$

where the diffusivity D is given by the same formula (1.1) with $\triangle x = g(x)$ and $\triangle t = 1$. This is the same diffusion equation derived by Chapman [1]. If $p_i = x(\frac{t_i+t_{i-1}}{2})$, (1.7) is called a Stratonovich type and the probability density function satisfies a different Fokker-Plank equation

$$u_t = \nabla \cdot (\sqrt{D}\nabla(\sqrt{D}u)), \tag{1.9}$$

which is the same diffusion equation derived by Wereide [21]. In other words, the well-known "Itô versus Stratonovich" controversy [15, 19] is about how to set the reference point for the stochastic integration, which corresponds to taking a reference point of a random walk for the nonconstant walk length in our problem.

We will derive a diffusion equation in a general form with the heterogeneity in $\triangle x$ only. The equation depends on the reference point parameter a in (1.5) and the two diffusion equations, (1.8) and (1.9), are special cases of it. However, if the heterogeneity in $\triangle t$ is included, the two are not special case any more since there is no component corresponding to $\triangle t$ in stochastic differential equation (1.6). We will see that the heterogeneity in $\triangle t$ has the property of Itô stochastic integration (see Conclusion).

1.2. **Model and results.** We briefly introduce main models and results of the paper. Details and complete arguments are given in the following sections rigorously. The first case is when the sojourn time is constant $\tau = \tau_0$ and the heterogeneity is in the walk-length $\ell(x)$ only. For simplicity, we consider the one space dimension. The discrete-time, continuous-space random walk model of the paper is given by a recursion relation,

$$u^\epsilon(t+\epsilon^2\tau_0, x) = \frac{1}{2}u^\epsilon(t, B_-(x))B'_-(x) + \frac{1}{2}u^\epsilon(t, B_+(x))B'_+(x),$$

where $B_\pm$ are given implicitly by

$$B_\pm(x) = x \mp \epsilon\ell(ax + (1-a)B_\pm(x)), \quad a \in [0,1]. \tag{1.10}$$

In words, $B_\pm(x)$ are the two departure points of a particle when x is the arrival point, and $B_+(x) < x < B_-(x)$. The reference point is $p = ax + (1-a)B_\pm(x)$. The first conclusion of the paper is Theorem 4.1, which shows the uniform convergence of the solution u^ϵ as $\epsilon \to 0$ to the solution of a diffusion equation,

$$u_t = \frac{1}{2\tau_0}(\ell^{2a}(\ell^{2-2a}u)_x)_x = (D^a(D^{1-a}u)_x)_x, \tag{1.11}$$

where $D = \frac{\ell^2(x)}{2\tau_0}$ and $a \in [0,1]$. If a=0, the reference is the departure point of Itô type, and (2.1) becomes (1.8). If $a = \frac{1}{2}$, the reference is the middle point of Stratonovich type, and (2.1) becomes (1.9). If $a = 1$, the reference is the arrival point, and (2.1) becomes Fick's law [6]. Hence, the walk-length $\ell(x)$ plays the same role of $g(x)$ in (1.6).

The second case is when the walk length is constant $\ell = \ell_0$ and the heterogeneity is in sojourn time $\tau(x)$ only. The heterogeneity in the sojourn time is involved in the system differently and makes the problem extremely harder.[1] The only case done is when τ is a step function,

$$\tau(x) = \begin{cases} 1, & x < 0, \\ 2, & x \geq 0, \end{cases} \tag{1.12}$$

and the reference point is taken as the middle point with $b = 0.5$ (see [3]). In the case, the corresponding diffusion equation is

$$u_t = (Du)_{xx}, \qquad D = \frac{\ell_0^2}{2\tau}. \tag{1.13}$$

In Lemma 5.1, it is proved that the maximum variation of the total sojourn time with respect to the parameter b is of order $O(\epsilon^2)$ as $\epsilon \to 0$ for any given sample path. Therefore, the diffusion equation should be independent of the parameter b, and hence (1.13) is the diffusion equation not just for $b = 0.5$ case, but for all $b \in [0,1]$.

We expect that (1.13) holds for general sojourn time $\tau(x)$, but we don't know how to work with general nonconstant sojourn time. Instead, we interpret the departing rate as the reciprocal of the sojourn time, i.e.,

$$\gamma(x) := \frac{\tau_0}{\tau(x)}, \quad 0 < \tau_0 \leq \inf_x \tau(x). \tag{1.14}$$

The concept of departing rate has been widely used without connecting it to the sojourn time (see [14, 22]). Then, we can handle the heterogeneity in $\triangle x$ and $\triangle t$ together. Using $B_\pm$ in (1.10) and the departing rate γ in (1.14), one take a discrete-time model

$$u^\epsilon_{n+1} = \frac{1}{2}\gamma(B_+)u^\epsilon_n(B_+)B'_+ + \frac{1}{2}\gamma(B_-)u^\epsilon_n(B_-)B'_- + (1-\gamma)u^\epsilon_n.$$

In this model, the reference point of the departing rate γ is taken as the departure point since the sojourn time is placed as in (1.13) for the special case of (1.12) and the diffusion equation is independent of the reference point parameter b.

The second conclusion of the paper is Theorem 6.1, which shows the uniform convergence of the solution u^ϵ as $\epsilon \to 0$ to the solution of a diffusion

[1] "A spatial jump is trivial when compared with the unknowns of time travel. One is like sliding along the ice and the other is akin to descending blindly into the depths of the freezing water and reappearing as an acorn." Sir Reginald Hargreeves in The Umbrella Academy, the second episode of the first season (run boy run).

equation,

$$u_t = \frac{1}{2\tau_0}(\ell^{2a}(\ell^{2-2a}\gamma u)_x)_x = \Big(\frac{\ell^{2a}}{2}\Big(\frac{\ell^{2-2a}}{\tau}u\Big)_x\Big)_x. \tag{1.15}$$

If we write the equation in terms of the diffusivity D given by (1.1), the diffusion equation (1.15) cannot be written in terms of the diffusivity D only since the heterogeneities in ℓ and τ are involved in the equation differently. If we rewrite (1.15) in terms of D and τ, we obtain

$$u_t = \Big(\tau^a D^a\Big(\frac{D^{1-a}}{\tau^a}u\Big)_x\Big)_x. \tag{1.16}$$

The heterogeneities in $\ell(x)$ and $\tau(x)$ are involved in the diffusion equation differently, and the diffusivity splits into two parts as in (1.15) or (1.16). There can be even more components involved in the real life diffusion. However, even if there are more components involved, the diffusion equation could be eventually written as

$$u_t = (K(Mu)_x)_x, \quad D = KM, \tag{1.17}$$

where K is called the diffusion conductivity and M the motility. If the reference point is chosen as (1.5), the resulting diffusion equation is (1.16) and the two coefficients are

$$K(x) = \tau^a D^a \quad \text{and} \quad M(x) = \tau^{-a}D^{1-a}.$$

If we write then in terms of ℓ and τ,

$$K(x) = \ell^{2a}(x)/2 \quad \text{and} \quad M(x) = \ell^{2-2a}(x)/\tau(x).$$

Remark 1.1. *A random walk model is called reversible (or revertible) if expectation $E(X_2) = X_0$, when the random variable X_2 is the position of a particle after two walks of the opposite direction and X_0 is the initial position (see* [10]. *If $a = \frac{1}{2}$, the random walk system is reversible and we obtain the same diffusion equation derived in* [9]. *We denote the speed of particle as $v = \frac{\triangle x}{\triangle t} = \frac{\ell}{\tau}$ and the turning frequency as $\mu = \frac{2}{\triangle t}$. Therefore, K and M of the $a = \frac{1}{2}$ case are written as*

$$K(x) = \frac{\ell}{2} = \frac{v(x)}{\mu(x)}, \quad M(x)\frac{\ell(x)}{\tau(x)} = v(x).$$

This is the same case obtained in [9, (32)].

In the paper, we study a discrete-time random walk system when the jumping distance $\triangle x$ and the jumping time $\triangle t$ are spatially nonconstant. For a continuous-time random walk model case has been considered in a companying paper [4] and the same diffusion equations (1.15) or (1.16) are obtained. These results show that the obtained diffusion equation and the role of reference points are robust. On the other hand, the two models have different the technical issues and advantages. We may also see different aspects of heterogeneous diffusion from the two models. To handle spatially nonconstant jumping distance properly, we take a continuous-space random walk system. To handle spatially nonconstant jumping time, we replace the sojourn time with departing rate.

The rest of the paper is organized as follows. In Section 2, a discrete-time, continuous-space random walk model is introduced when the sojourn time is constant. Basic properties of the model is discussed. The diffusion equation (1.15) is formally derived in Section 3 and then uniform convergence of the discrete solution to the solution of the diffusion equation is proved in 4 (Theorem 4.1). In Section 5, we show that the diffusion limit of the random walk model is independent of the reference point of the sojourn time (Lemma 5.1), but dependent of the walk length (Lemma 5.2). In Section 6

Since the sojourn time is heterogeneous, the obtained random walk system is not Markovian. The main effort of the section is to transform it into Markovian. In Section 3, we construct a Lipschitz continuous interpolation of the discrete solution of the random walk problem, and find uniform estimates using difference quotients. These estimates give a convergent subsequence of the interpolation. The convergence to the weak solution is finally proved in Section 4. In Section 5, Green's function for the diffusion equation (**??**) is obtained explicitly. It is compared with the discrete random walk system. Three Monte Carlo simulations are given in Section 6 which show the behavior of Green's function and steady states. We also test the reference point independency in the sojourn time. The computation codes of these numerical simulations are given in Appendix.

2. Spatially heterogeneous walk-length

In this section, we introduce a discrete-time, continuous-space random walk system. If the walk-length is spatially heterogeneous, we need to take a continuous-space random walk system to handle such a spatial heterogeneity properly. We start with a heterogeneity in $\triangle x$ first and consider a case with constant jumping time $\tau = \tau_0$. Denote the time mesh as

$$T_\epsilon := \epsilon^2 \tau_0 \mathbb{N} = \{0, \epsilon^2\tau_0, 2\epsilon^2\tau_0, 3\epsilon^2\tau_0, \cdots\}$$

for $\epsilon > 0$ small. Note that we have chosen a parabolic scale ϵ^2 in time to obtain a diffusion limit. For a simpler presentation, we consider diffusion in one space dimension and take the space domain as

$$\mathbb{X} := [0,1] \subset \mathbb{R}$$

with the periodic boundary condition, i.e., $\mathbb{X}$ is a circle.

The particle density is a function $u : T_\epsilon \times \mathbb{X} \to \mathbb{R}$, where $u(n\epsilon^2\tau_0, x)$ denotes the particle density at position $x \in \mathbb{X}$ and time $n\epsilon^2\tau_0$. If needed, we denote the ϵ dependency explicitly as u^ϵ. We also denote the density distribution at nth time step as $u_n(x) = u(n\epsilon^2\tau_0, x)$. To derive a discrete-time and continuous-space random walk model, we first find a recursive relation that computes $u_{n+1}(x)$ from $u_n(x)$.

In a random walk system, a particle placed at x moves to right or left at the next time step with equal probability. Let $N_+(x) > x$ and $N_-(x) < x$ be the two possible positions at the 'N'ext time step when x is the departing position. Let y be the arrival point, i.e., $y = N_+(x)$ or $y = N_-(x)$. In a homogeneous random walk system, the walk-length $|y-x|$ is a given constant. In our case, the walk-length depends on the space variable and given by a smooth function $\ell \in C^2(\mathbb{X})$. Since ℓ is smooth and the domain $\mathbb{X}$ is compact, ℓ and its derivatives are uniformly bounded, i.e., there exist constants $\ell_{\min}$, $\ell_{\min}$, and $M > 0$ such that

$$\begin{cases} 0 < \ell_{\min} \le \ell(x) \le \ell_{\max} < \infty, \\ |\ell'(x)|, |\ell''(x)| < M, \end{cases} \qquad x \in \mathbb{X}. \tag{2.1}$$

Even if the domain is not compact, we may proceed the estimate under the assumptions in (2.1).

The reference points are taken according to (**??**), where, for the walk from x to y, the walk-length is given by

$$|y - x| = \epsilon\ell(ay + (1-a)x).$$

The arrival points $y = N_\pm(x)$ is decided by implicitly by

$$\begin{cases} N_+(x) = x + \epsilon\ell(aN_+(x) + (1-a)x), \\ N_-(x) = x - \epsilon\ell(aN_-(x) + (1-a)x). \end{cases} \tag{2.2}$$

If $\epsilon > 0$ is small enough, $N_\pm$ are uniquely decided by the implicit relations due to the uniform bounds in (2.1).

2.1. **Algorithm with volume factor.** Since ℓ is smooth and $\epsilon > 0$ is small, $N_\pm$ are differentiable and invertible. We denote the inverse functions of $N_\pm$ as $B_\pm$. In other words, $B_\pm(x)$ are the possible two positions that a particle was placed one step 'B'efore the the position one step before the current position x. Then, for $N_\pm$ in (2.2),

$$\begin{cases} B_+(y) = y - \epsilon\ell(ay + (1-a)B_+(y)), \\ B_-(y) = y + \epsilon\ell(ay + (1-a)B_-(y)). \end{cases} \tag{2.3}$$

Note that both $N_\pm$ and $B_\pm$ depend on ϵ. If needed, we may denote them by $N^\epsilon_\pm$ and $B^\epsilon_\pm$ to show the ϵ dependency explcitly. Let $\Omega \subset \mathbb{X}$ be a region. Then, the population in the region Ω at the next time step is given by

$$\begin{aligned} \int_\Omega u^\epsilon_{n+1}(x)dx &= \frac{1}{2}\int_{B_-(\Omega)} u^\epsilon_n(x)dx + \frac{1}{2}\int_{B_+(\Omega)} u^\epsilon_n(x)dx \\ &= \frac{1}{2}\int_\Omega u^\epsilon_n(B_-(x))B'_-(x)dx + \frac{1}{2}\int_\Omega u^\epsilon_n(B_+(x))B'_+(x)dx, \end{aligned}$$

where the second equality is by change of variables. Since the relation holds for any region Ω, the density u^ϵ_n satisfies

$$\begin{cases} u^\epsilon_{n+1}(x) = \frac{1}{2}u^\epsilon_n(B_+(x))B'_+(x) + \frac{1}{2}u^\epsilon_n(B_-(x))B'_-(x), \\ u^\epsilon_0(x) \;\; = u_0(x), \end{cases} \tag{2.4}$$

where $u_0(x)$ is the initial distribution. This is the discrete-time and continuous-space recursive relation.

One of the main differences of the model in comparison with discrete-space random walk systems is the presence of the volume factor, $B'_\pm(x)$. Since u_n^ϵ is the density, not the population, the volume factor appears. In a homogeneous random walk system with a constant walk-length $\triangle x$, $B_\pm(x) = x \mp \triangle x$ and hence $B'_\pm(x) = 1$. Therefore, the continuous-space random walk system takes the same recursive relation as the discrete-space random walk system when the model is spatially homogeneous.

2.2. **Formulation of heterogeneous nonlocal diffusion.** A nonlocal diffusion equation is often given in a convolution form using an integration kernel. The recursive relation (2.4) can be similarly written using a convolution kernel. However, since it is a heterogeneous model, we need a kernel with two independent variables,

$$K(x,y) = \frac{1}{2}\delta(y - N_-(x)) + \frac{1}{2}\delta(y - N_+(x)). \tag{2.5}$$

The kernel $K(x,y)$ gives the migration rate from departing position x to arrival point y. Consider a discrete-time random walk model in a convolution form,

$$u_{n+1}(x) = \int K(y,x)u_n(y)dy. \tag{2.6}$$

If we substitute the kernel (2.5), it is written as

$$u_{n+1}(x) = \frac{1}{2}\int \delta(x - N_-(y))u_n(y)dy + \frac{1}{2}\int \delta(x - N_+(y))u_n(y)dy.$$

After change the variables with $p = N_\pm(y)$, the integrals become

$$\int \delta(x - N_-(y))u_n(y)dy = \int \delta(x-p)u_n(B_-(p))B'_-(p)dp = u(B_-(x))B'_-(x).$$

$$\int \delta(x - N_+(y))u_n(y)dy = \int \delta(x-p)u_n(B_+(p))B'_+(p)dp = u(B_+(x))B'_+(x).$$

Therefore, after substitution, we return to the recursive formula (2.4),

$$u_{n+1}(x) = \frac{1}{2}u_n(B_-(x))B'_-(x) + \frac{1}{2}u_n(B_+(x))B'_+(x).$$

In other words, the kernel given in (2.5) contains the volume factor correctly inside.

Remark 2.1. *One might take*

$$K(x,y) = \frac{1}{2}\delta(B_-(y) - x) + \frac{1}{2}\delta(B_+(y) - x)$$

as a kernel. However, it does not give the same random walk. If we write (2.6) *in a recursive formula, we obtain*

$$u_{n+1}(x) = \frac{1}{2}u_n(B_-(x)) + \frac{1}{2}u_n(B_+(x)),$$

which is different from our recursive realtion (2.4).

3. Formal derivation

In this section, we formally show that the recursive relation (2.3)–(2.4) converges to a diffusion equation,

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2\tau_0}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x \\ u(0,x) \;= u_0(x) \end{cases} \tag{3.1}$$

as $\epsilon \to 0$. The exponent a in the equation is from the choice of the reference point given by $p = ay + (1-a)x$ when a particle jumps from x to y.

We start the discussion with the case $a = 1$, i.e., the reference point is the arrival point $p = y$, and then extend it to the general case. If $a = 1$, $B_\pm(x)$ in (2.3) are explicitly given by

$$B_+(x) = x - \epsilon\ell(x), \qquad B_-(x) = x + \epsilon\ell(x). \tag{3.2}$$

Then, (2.4) gives

$$\begin{aligned}\tau_0 \frac{u^\epsilon_{n+1}(x) - u^\epsilon_n(x)}{\epsilon^2\tau_0} =& \frac{1}{2\epsilon^2}\Big(u^\epsilon_n(x+\epsilon\ell(x))(1+\epsilon\ell'(x)) \\ &+ u^\epsilon_n(x-\epsilon\ell(x))(1-\epsilon\ell'(x)) - 2u^\epsilon_n(x)\Big).\end{aligned}$$

The left side converges to the time derivative $\tau_0 u_t$ as $\epsilon \to 0$. The Taylor expansion of the right side gives

$$\begin{aligned}&\frac{1}{2\epsilon^2}\Big(u^\epsilon_n(x+\epsilon\ell(x))(1+\epsilon\ell'(x)) + u^\epsilon_n(x-\epsilon\ell(x))(1-\epsilon\ell'(x)) - 2u^\epsilon_n(x)\Big) \\ &\quad = \frac{1}{2\epsilon^2}\Big((u^\epsilon_n)_{xx}\epsilon^2\ell^2 + 2(u^\epsilon_n)_x\epsilon^2\ell\ell_x + O(\epsilon^3)\Big) \\ &\quad = \frac{1}{2}(\ell^2(u^\epsilon_n)_x)_x + O(\epsilon).\end{aligned}$$

Therefore, after taking $\epsilon \to 0$ limit, we formally obtain Fick's diffusion law,

$$u_t = \frac{1}{2\tau_0}(\ell^2 u_x)_x, \tag{3.3}$$

which is the equation (3.1) with $a = 1$. Since τ_0 is constant, we may place it inside the derivatives. For example, we may write it as

$$u_t = \frac{1}{2}\Big(\ell^2\Big(\frac{1}{\tau_0}u\Big)_x\Big)_x,$$

which is the special case of (1.15) when $a = 1$ and $\tau = \tau_0$.

Now we consider the general case that the reference point is given by

$$p = ay + (1-a)x$$

when a particle jumps from x to y. In the following Lemma, we show that $N_\pm(x)$ is invertible if ϵ is small enough and find the Taylor expansion of the inverse $B_\pm(x)$.

Lemma 3.1. *Let $\mathbb{X} \subset \mathbb{R}$ be a closed interval with periodic boundary condition and $\ell \in C^2(\mathbb{X})$ satisfies the uniform bounds in (2.1). Then, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $N_\pm(x)$ are invertible and their inverse functions $B_\pm(x)$ satisfy the following expansion;*

$$B_\pm(x) = x \mp \epsilon\ell(x) + (1-a)\epsilon^2\ell(x)\ell'(x) + O(\epsilon^3) \tag{3.4}$$

Proof. First, we show that $N_\pm$ are invertible if $\epsilon > 0$ is small enough. Let $y = N_\pm(x)$. Then, by a differentiation,

$$\frac{dy}{dx} = 1 \pm \epsilon\ell'(ay + (1-a)x)\left(a\frac{dy}{dx} + 1 - a\right).$$

Then,

$$\frac{dy}{dx}(1 \mp \epsilon\ell'(a^2y + (1-a)ax)) = 1 \pm \epsilon\ell'(ay + (1-a)x)(1-a).$$

Since $|\ell'|$ is bounded, there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$,

$$(1 \mp \epsilon\ell'(a^2y + (1-a)ax)) > 0, \quad 1 \pm \epsilon\ell'(ay + (1-a)x)(1-a) > 0.$$

Therefore, $N_\pm(x)$ are monotone and invertible.

The inverse functions $B_\pm$ are given explicitly as in (3.2) when the reference point is the arrival point, i.e., $a = 1$. In general, the inverse functions are not given explicitly. However, in order to find the parabolic scaling limit, we only need the Taylor expansion of $B_\pm$. Since $|y - x| = \epsilon|\ell(ay + (1-a)x)| \le \epsilon\ell_{\max}$, we have

$$\begin{aligned} y - x &= \pm\epsilon\ell(ay + (1-a)x) \\ &= \pm\epsilon\left(\ell(y) + (1-a)\ell'(y)(x-y) + O(\epsilon^2)\right). \end{aligned}$$

If we rewrite it for $y - x$, we obtain

$$y - x = \pm\frac{\epsilon\ell + O(\epsilon^3)}{1 \pm \epsilon(1-a)\ell'} = \pm\epsilon\ell(y) - (1-a)\epsilon^2\ell(y)\ell'(y) + O(\epsilon^3).$$

Finally, we obtain

$$x = B_\pm(y) = y \mp \epsilon\ell(y) + (1-a)\epsilon^2\ell(y)\ell'(y) + O(\epsilon^3).$$

Therefore, $B_\pm(x)$ satisfies the expansion (3.4). □

We may follow the formal computation of the case with $a = 1$ by substituting the expansion formula (3.4) to the recursive relation (2.4) to obtain the equation (3.1), which is left to readers of the paper. In below, we present another formal calculation.

Fix $x_0 \in \mathbb{X}$ and define $U(x)$ for $x > x_0$ as

$$U(x) = \int_{x_0}^{x} u(y)dy,$$

and $B(x,\epsilon)$ for $|\epsilon|<\epsilon_0$ as

$$B(x,\epsilon)=\begin{cases}B_+(x), & \text{if } \epsilon>0,\\ x, & \text{if } \epsilon=0,\\ B_-(x), & \text{if } \epsilon<0.\end{cases}$$

Then, $U(x)$ is the total population in the region $[x_0,x]$ and $B(x,\epsilon)$ is smooth with respect to ϵ at least twice due to the expansion (3.4). By expanding $U(B(x,\epsilon))$ with respect to ϵ with fixed x, we obtain

$$\begin{aligned}U\left(B(x,\epsilon)\right)&=U(x)+\epsilon u\left(B(x,\epsilon)\right)\partial_\epsilon B\Big|_{\epsilon=0}\\&\quad+\frac{1}{2}\epsilon^2\left(u_x\left(B(x,\epsilon)\cdot(\partial_\epsilon B)^2+u\left(B(x,\epsilon)\right)\partial_{\epsilon\epsilon}B\right)\Big|_{\epsilon=0}+O(\epsilon^3)\\&=U(x)-\epsilon u(x)\ell(x)\\&\quad+\frac{1}{2}\epsilon^2\left(u_x(x)\ell(x)^2+2(1-a)u(x)\ell(x)\ell'(x)\right)+O(\epsilon^3).\end{aligned}$$

By differentiating the result with respect to x, we obtain

$$\begin{aligned}u\left(B(x,\epsilon)\right)B_x(x,\epsilon)&=u(x)-\epsilon(u(x)\ell(x))_x\\&\quad+\frac{1}{2}\epsilon^2\left(u_x(x)\ell(x)^2+2(1-a)u(x)\ell(x)\ell'(x)\right)_x+O(\epsilon^3).\end{aligned}$$

Then, using the formula, we obtain

$$\begin{aligned}&u\left(B(x,\epsilon)\right)B_x(x,\epsilon)+u\left(B(x,-\epsilon)\right)B_x(x,-\epsilon)-2u(x)\\&\quad=\epsilon^2\left(u_x\ell^2+2(1-a)u\ell\ell'\right)_x+O(\epsilon^3)\\&\quad=\epsilon^2\left(\ell^2u_x+\ell^{2a}(\ell^{2-2a})_xu\right)_x+O(\epsilon^3)\\&\quad=\epsilon^2\left(\ell^{2a}(\ell^{2-2a}u)_x\right)_x+O(\epsilon^3).\end{aligned}$$

For u_n^ϵ in (2.4), we obtain

$$\tau_0\frac{u_{n+1}^\epsilon(x)-u_n^\epsilon(x)}{\epsilon^2\tau_0}=\frac{1}{2\epsilon^2}\left(\epsilon^2\left(\ell^{2a}(\ell^{2-2a}u)_x\right)_x+O(\epsilon^3)\right). \tag{3.5}$$

After taking the limit as $\epsilon\to0$, we obtain the diffusion equation,

$$\partial_t u=\frac{1}{2\tau_0}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x.$$

4. Uniform convergence

In this section, we show that the solution u_n^ϵ of the recursive relation (2.4), converges uniformly to the solution u of the diffusion equation (3.1) as $\epsilon\to0$. To do that u^ϵ is extended to a function defined on the continuous time space by a linear interpolation,

$$u^\epsilon(t,x)=\frac{t-n\epsilon^2\tau_0}{\epsilon^2\tau_0}u_{n+1}^\epsilon(x)+\frac{(n+1)\epsilon^2\tau_0-t}{\epsilon^2\tau_0}u_n^\epsilon(x) \tag{4.1}$$

for $t\in[n\epsilon^2\tau_0,(n+1)\epsilon^2\tau_0]$. Then, $u_n^\epsilon(x)=u^\epsilon(n\epsilon^2\tau_0,x)$ and the continuous-time function $u^\epsilon(t,x)$ and the discrete-time function $u_n^\epsilon(x)$ agree on the discrete-time grids $T_\epsilon = \{0, \epsilon^2\tau_0, 2\epsilon^2\tau_0, 3\epsilon^2\tau_0, \cdots\}$. The main convergence theorem is as follows.

Theorem 4.1 (Uniform convergence)**.** *Let $\mathbb{X} \subset \mathbb{R}$ be a closed interval with periodic boundary condition and $\ell \in C^2(\mathbb{X})$ satisfies the uniform bounds in* (2.1)*. Let the initial condition $u_0(x)$ be smooth, bounded, and positive. Let u^ϵ be the linear interpolation* (4.1) *of the solution u_n^ϵ of* (2.4) *with the initial condition $u_0(x)$. Then, for a fixed $T > 0$, u^ϵ converges to the solution u of* (3.1) *uniformly, i.e.,*

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{X}\times[0,T])} \to 0 \quad as \quad \epsilon \to 0.$$

The uniqueness and the existence of solutions to an explicit recursive relation such as (2.4) is obvious. We will show a comparison principle and use it for the proof of Theorem 4.1. For the comparison principle, we start with super- and sub-solutions of a discrete-time model in a general setting, which is a simple extension of the ones from continuous-time models.

Definition 4.2. *An operator $K : L^1(\mathbb{X}) \to L^1(\mathbb{X})$ is called nonnegative if $Ku \in L^1(\mathbb{X})$ is nonnegative for all nonnegative $u \in L^1(\mathbb{X})$.*

For a given operator K, an initial function $u_0 \in L^1(\mathbb{X})$, and a sequence of functions $f_n \in L^1(\mathbb{X})$, we may defined a sequence of functions recursively as

$$u_{n+1} = Ku_n + f_n, \quad n \geq 0. \tag{4.2}$$

Definition 4.3. *A sequence of $v_n \in L^1(\mathbb{X})$ is called a super-solution of the relation* (4.2) *if*

$$\begin{aligned} v_0(x) &\geq u_0(x) \\ v_{n+1}(x) &\geq Kv_n(x) + f_n(x). \end{aligned}$$

The sequence v_n is called a sub-solution if the inequalities are reversed.

Theorem 4.4 (Comparison property)**.** *Suppose $\{w_n\}$ and $\{v_n\}$ are respectively super- and sub-solutions with initial values, w_0 and v_0, and source terms, $\{f_n\}$ and $\{g_n\}$. If K is a nonnegative linear operator, $w_0 \geq v_0$, and $f_n \geq g_n$ for each n, then $w_n \geq v_n$ for all n.*

Proof. The two sequences satisfy

$$\begin{aligned} w_{n+1}(x) &\geq Kw_n(x) + f_n(x), \\ v_{n+1}(x) &\leq Kv_n(x) + g_n(x). \end{aligned}$$

Subtract the second equation from the first one and obtain

$$\begin{aligned} w_{n+1}(x) - v_{n+1}(x) &\geq Kw_n(x) - Kv_n(x) + f_n(x) - g_n(x) \\ &= K(w_n(x) - v_n(x)) + f_n(x) - g_n(x). \end{aligned}$$

Therefore, $w_n - v_n$ is a super-solution with a nonnegative initial value and source terms. Hence, $w_n \geq v_n$. □

Corollary 4.5. *Let $K \geq 0$, $u_0 \geq 0$, $f_n \geq 0$, and $u_n(x)$ be a super-solution. Then, $u_n \geq 0$ for all n.*

By Lemma 3.1, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $N_\pm$ are invertible and the time discrete density distribution u_n^ϵ satisfies the recursive relation,

$$u_{n+1}(x) = \frac{1}{2}u_n(B_+(x))B'_+(x) + \frac{1}{2}u_n(B_-(x))B'_-(x), \tag{4.3}$$

where $B_\pm$ are the inverse functions of $N_\pm$. The next position function $N_\pm$ and the previous position function $B_\pm$ are given by relations (2.2) and (2.3). We define the operator K^ϵ as

$$K^\epsilon u := \frac{1}{2}u(B_+(x))B'_+(x) + \frac{1}{2}u(B_-(x))B'_-(x). \tag{4.4}$$

Then, the operator is linear and positive. Note that the operator K^ϵ depends on the parameter ϵ, where the functions $B_\pm$ and $N_\pm$ depend on ϵ, too.

Using the operator K^ϵ, we may write the discrete-time model (4.3) in a simplified form

$$\begin{aligned} u_{n+1}^\epsilon(x) &= K^\epsilon u_n^\epsilon(x), \\ u_0^\epsilon(x) &= u_0(x). \end{aligned} \tag{4.5}$$

Lemma 4.6. *Let $u^\epsilon(t,x)$ be the linear interpolation given by* (4.1)*. Then,*

$$u^\epsilon(t+\epsilon^2\tau_0, x) = K^\epsilon u^\epsilon(t,x). \tag{4.6}$$

Proof. Since K^ϵ is a linear operator,

$$\begin{aligned} K^\epsilon u^\epsilon(t,x) &= \frac{t-n\epsilon^2\tau_0}{\epsilon^2\tau_0}K^\epsilon u_{n+1}^\epsilon(x) + \frac{(n+1)\epsilon^2\tau_0 - t}{\epsilon^2\tau_0}K^\epsilon u_n^\epsilon(x) \\ &= \frac{t+\epsilon^2\tau_0-(n+1)\epsilon^2\tau_0}{\epsilon^2\tau_0}u_{n+2}^\epsilon(x) + \frac{(n+2)\epsilon^2\tau_0-(t+\epsilon^2\tau_0)}{\epsilon^2\tau_0}u_{n+1}^\epsilon(x) \\ &= u^\epsilon(t+\epsilon^2\tau_0, x), \end{aligned}$$

which completes (4.6). □

The following lemma corresponds to the formal calculation in Section 3.

Lemma 4.7 (Formal convergence)**.** *Let $B_\pm$ be given by* (2.3) *and define two operators*

$$Lu := \frac{1}{2\tau_0}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x, \quad \textit{and} \quad L^\epsilon u := \frac{K^\epsilon u - u}{\tau_0\epsilon^2} \tag{4.7}$$

for $a \in [0,1]$ and $u \in C^2(\mathbb{X})$. Then,

$$L^\epsilon u = Lu + O(\epsilon) \quad \textit{as} \quad \epsilon \to 0.$$

Proof. It is obtained in (3.5) that, for each choice of reference $a \in [0,1]$,

$$L^\epsilon u = \frac{1}{\epsilon^2\tau_0}\left(\frac{1}{2}u(B_+(x),t)B'_+(x) + \frac{1}{2}u(B_-(x),t)B'_-(x) - u(t,x)\right)$$

is a smooth function for small ϵ and satisfies

$$L^\epsilon u = Lu + O(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

By Taylor's formula, the difference satisfies

$$\begin{aligned}|L^\epsilon u(x) - Lu(x)| &= |Lu(\tilde{x}) - Lu(x)| \\ &= |(Lu)_x(\tilde{\tilde{x}})| \cdot |x - \tilde{x}|\end{aligned}$$

for some $\tilde{x}, \tilde{\tilde{x}} \in (B_+(x), B_-(x))$ and the distance is of order $\epsilon\|\ell\|$. Once we assume smoothness and boundedness of u, ℓ, and their derivatives, the difference is bounded by $C\epsilon$ with $C = C(u, \ell, a)$ is constant determined by u, ℓ, and the reference point parameter $a \in [0, 1]$. □

Using the operator L given in (4.7), the diffusion equation (3.1) is written as

$$\partial_t u(t,x) = Lu(t,x). \tag{4.8}$$

Lemma 4.8. *If $\ell \in C^3(\mathbb{X})$, there exists $\epsilon_0 > 0$ small such that*

$$L^\epsilon(1) = \frac{1}{2\epsilon^2\tau_0}(B'_+ + B'_- - 2)$$

is uniformly bounded for $a \in [0, 1]$, $x \in \mathbb{X}$, and $\epsilon \in [0, \epsilon_0]$.

Proof. $N_\pm$ is defined for small $\epsilon > 0$. By Lemma 4.7, for a small ϵ and $u = 1$,

$$\begin{aligned}|L^\epsilon(1)| &\le |L(1)| + |(L(1))_x| \cdot \|\ell\|\epsilon \\ &= |\frac{1}{2\tau_0}\left(\ell^{2a}(\ell^{2-2a})_x\right)_x| + |\frac{1}{2\tau_0}\left(\ell^{2a}(\ell^{2-2a})_x\right)_{xx}| \cdot \|\ell\|\epsilon.\end{aligned}$$

Since ℓ and its derivataives are uniformly bounded, the above is uniformly bounded for a fixed a and ϵ. Then, by the continuity along a and ϵ, it is uniformly bounded for a and ϵ, too. □

Note that if the arrival point is taken as the reference point, i.e., $a = 1$, then we have

$$L^\epsilon(1) = \frac{1}{2\tau_0}\left(1 + \epsilon\ell' + 1 - \epsilon\ell' - 2\right) = 0.$$

Next, we prove the uniform convergence of the main theorem.

Proof of Theorem 4.1. Denote $w^\epsilon := u^\epsilon - u$ and take samples,

$$w^\epsilon_n(x) = w^\epsilon(n\epsilon^2\tau_0, x) = u^\epsilon_n(x) - u(n\epsilon^2\tau_0, x).$$

Then, from the recursive relation and the diffusion equation, we have

$$\begin{aligned}w^\epsilon_{n+1}(x) &= u^\epsilon_{n+1}(x) - u((n+1)\epsilon^2\tau_0, x) \\ &= K^\epsilon u^\epsilon_n(x) - u(n\epsilon^2\tau_0, x) - \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} Lu(s,x)ds \\ &= K^\epsilon w^\epsilon_n(x) + K^\epsilon u(n\epsilon^2\tau_0, x) - u(n\epsilon^2\tau_0, x) - \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} Lu(s,x)ds \\ &= K^\epsilon w^\epsilon_n(x) + \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} \left(Lu(n\tau_0\epsilon^2, x) - Lu(s,x) + O(\epsilon)\right) ds.\end{aligned}$$

Denote

$$F_{\epsilon,n}(u;x) = \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} \left(Lu(n\tau_0\epsilon^2, x) - Lu(s,x) + O(\epsilon)\right) ds.$$

Then,

$$\begin{aligned} |F_{\epsilon,n}(u;x)| &\le \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} (s - n\tau_0\epsilon^2)\|L^2u\|_\infty + O(\epsilon) ds \\ &\le \|L^2u\|_\infty \tau_0^2\epsilon^4 + O(\epsilon^3) \le \epsilon^2\theta(\epsilon) \end{aligned}$$

for some $\theta(\epsilon)$ which depends only on ϵ and of order $O(\epsilon)$ as $\epsilon \to 0$. Then, $\{w_n^\epsilon(x)\}$ satisfies a recursive relation with a source term,

$$\begin{cases} w_{n+1}(x) = K^\epsilon w_n(x) + F_{\epsilon,n}(u;x), \\ w_0(x) = 0. \end{cases} \tag{4.9}$$

Next, we construct a super-solution of the recursion relation (4.9). First, let

$$\eta(\epsilon) = \max_{x\in\mathbb{X}} \frac{1}{\epsilon^2\tau_0}|K^\epsilon(1) - 1| = \max_{x\in\mathbb{X}} \frac{1}{2\epsilon^2\tau_0}|B'_+(x) + B'_-(x) - 2|,$$

which is bounded for $\epsilon \in (0, \epsilon_0]$. Define a function of $\epsilon > 0$ as

$$\bar{w}_n(\epsilon) = \begin{cases} \frac{\theta(\epsilon)}{\eta(\epsilon)}\left(\left(1 + \epsilon^2\eta(\epsilon)\right)^n - 1\right), & \eta(\epsilon) \ne 0, \\ \theta(\epsilon)n\epsilon^2, & \eta(\epsilon) = 0, \end{cases}$$

where the case with $\eta(\epsilon) = 0$ is extended continuously from the other case. We can easily check that $\{\bar{w}_n(\epsilon)\}$ is the solution of a recursive relation

$$\begin{cases} \bar{w}_{n+1} = \left(1 + \eta(\epsilon)\epsilon^2\right)\bar{w}_n + \epsilon^2\theta(\epsilon) \\ \bar{w}_0 = 0. \end{cases}$$

Then,

$$\begin{aligned} \bar{w}_{n+1}(\epsilon) &= (1 + \epsilon^2\eta(\epsilon))\bar{w}_n(\epsilon) + \epsilon^2\theta(\epsilon) \\ &\ge K^\epsilon\bar{w}_n(\epsilon) + \epsilon^2\theta(\epsilon) \\ &\ge K^\epsilon\bar{w}_n(\epsilon) + F_{\epsilon,n}(u;x). \end{aligned}$$

Therefore, $\bar{w}_n(\epsilon)$ is a super-solution of (4.9). By the comparison property,

$$-\bar{w}_n \le w_n^\epsilon \le \bar{w}_n.$$

Now we prove the uniform convergence $w^\epsilon \to 0$. We show the convergence for discrete-time steps first and then extend it to continuous-time using the linear interpolation. Let $T > 0$ be a given finite time and $N(\epsilon)$ be the number of steps to cover upto T in the sense that $[0, T + \epsilon^2\tau_0] \cap T_\epsilon = \{0, \epsilon^2\tau_0, \cdots, N(\epsilon)\epsilon^2\tau_0\}$. Then,

$$\frac{T}{\epsilon^2\tau_0} < N(\epsilon) \le \frac{T + \epsilon^2\tau_0}{\epsilon^2\tau_0}.$$

Since $\bar{w}_n(\epsilon)$ is an increasing sequence, for $n \le N(\epsilon)$,

$$\begin{aligned}\bar{w}_n(\epsilon) &\le \bar{w}_{n(\epsilon)}(\epsilon)\\ &= \begin{cases}\frac{\theta(\epsilon)}{\eta(\epsilon)}\left(\left(1+\epsilon^2\eta(\epsilon)\right)^{N(\epsilon)}-1\right), & \eta(\epsilon)\neq 0\\ \theta(\epsilon)N(\epsilon)\epsilon^2, & \eta(\epsilon)=0\end{cases}\\ &\le \begin{cases}\frac{\theta(\epsilon)}{\eta(\epsilon)}\left(\left(1+\epsilon^2\eta(\epsilon)\right)^{\frac{T+\epsilon^2\tau_0}{\epsilon^2\tau_0}}-1\right), & \eta(\epsilon)\neq 0\\ \theta(\epsilon)\frac{T+\tau_0\epsilon^2}{\tau_0}, & \eta(\epsilon)=0\end{cases}\\ &\le \theta(\epsilon)B\end{aligned}$$

for some $B = B(T, \tau_0, \epsilon_0, \eta)$ which uniformly bounded with respect to $\epsilon \in (0, \epsilon_0)$ and converges to T/τ_0 as $\epsilon \to 0$. Since $\theta(\epsilon) \to 0$ as $\epsilon \to 0$, we have

$$\sup_{0\le n\le N(\epsilon)} \|u_n^\epsilon(\cdot) - u(\cdot, n\epsilon^2\tau_0)\|_{L^\infty(\mathbb{X})} \to 0, \tag{4.10}$$

which completes the uniform convergence along discrete-time steps.

Instead of comparing u^ϵ and u directly, we consider the linear interpolation of u along discrete-time steps with time interval $\tau_0 = \epsilon^2\tau_0$. Let U^ϵ be the linear interpolation

$$U^\epsilon(t,x) = \frac{t-n\epsilon^2\tau_0}{\epsilon^2\tau_0}u((n+1)\epsilon^2\tau_0, x) + \frac{(n+1)\epsilon^2\tau_0 - t}{\epsilon^2\tau_0}u(n\epsilon^2\tau_0, x)$$

for $t \in [n\epsilon^2\tau_0, (n+1)\epsilon^2\tau_0]$. Then, since $u^\epsilon(n\epsilon^2\tau_0, x) = u_n^\epsilon$ and the linear interpolation keeps the limit, the convergence in (4.10) gives

$$\|u^\epsilon - U^\epsilon\|_{L^\infty(\mathbb{X}\times[0,T])} \to 0$$

as $\epsilon \to 0$.

Finally from regularity assumption, u is smooth along time and hence is uniformly approximated by the linear interpolation $U^\epsilon(t,x)$, which gives

$$\|u - U^\epsilon\|_{L^\infty(\mathbb{X}\times[0,T])} \to 0.$$

Hence, the triangle argument gives the uniform convergence

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{X}\times[0,T])} \to 0$$

as $\epsilon \to 0$. □

5. Reference point dependency versus independency

The three diffusion equations, (1.3), (1.8), and (1.9), are based on a belief that the diffusion phenomenon can be explained by the diffusivity $D(x)$ alone even in a heterogeneous environment. On the other hand, the diffusion model (1.17) is based on the idea that it is not. Note that the diffusivity D in (1.1) consists of two components, the walk length $\triangle x$ and the sojourn (or jumping) time $\triangle t$. In this section, we will show that the way to choose the reference point of $\triangle x$ changes the final position of a particle, but the one of $\triangle t$ does not. This difference between the two components tells us that the two components will be separated eventually and we need at least two quantities to explain a heterogeneous diffusion phenomenon.

Let $\tau : X \to \mathbb{R}$ be the function for the heterogeneous jumping time which is smooth and uniformly bounded by

$$0 < \tau_{\min} \le \tau(x) \le \tau_{\max} < \infty. \tag{5.1}$$

The jumping time can be interpreted in two ways. One may consider it as the period of time for a particle to stay until the next jump and then jumps instantaneously to the next position. That's why it's called sojourn time. Or equivalently, it can be considered as the travel time needed for a particle to move to the next position and the particle departs to the next position immediately after the arrival.

The heterogeneity in $\tau(x)$ indicates that it takes different amount of time to jump to the next position. Then, the random walk model is completed after deciding the reference point when a particles jumps from one position to another. Let the reference point for the heterogeneity in the sojourn time be

$$p = by + (1-b)x, \quad 0 \le b \le 1, \tag{5.2}$$

when a particle jumps from x to y. Consider a sample path of a particle $\{x_n : n = 0, 1, \cdots, N\}$, where x_n is the position of the particle at the nth time step. Since we are interested in taking diffusion limits (or parabolic scale limits), we assume that the walk length $|x_{n+1} - x_n|$ is of order ϵ and the sojourn time at each step is of order ϵ^2. The sample path consists of N jumps from the starting point x_0. The total time of the random walk may depend on the parameter b and is defined (or computed) by

$$T(b) = \sum_{n=1}^{N} \epsilon^2 \tau(bx_n + (1-b)x_{n-1}). \tag{5.3}$$

The total number of walks is of order $\frac{t}{\epsilon^2}$ when t is the macroscopic time scale of interest. Then, the total number of walk N is bounded by

$$\frac{t}{\epsilon^2 \tau_{\max}} \le N \le \frac{t}{\epsilon^2 \tau_{\min}}. \tag{5.4}$$

The total time $T(b)$ depends on the parameter b. However, we will show that the difference is trivialized as $\epsilon \to 0$. The simplest case is when τ and the sample x_n are both monotone. Suppose $\tau(x)$ is monotone increasing and $x_n < x_{n+1}$ for all $n = 1, \cdots, N-1$. Then, $T(b)$ has its supremum when $b = 1$ and its infimum when $b = 0$. Therefore, we have

$$\begin{aligned} \sup_{0\le b\le 1} T(b) - \inf_{0\le b\le 1} T(b) &= \sum_{n=1}^{N} \epsilon^2 \tau(x_n) - \sum_{n=1}^{N} \epsilon^2 \tau(x_{n-1}) \\ &= \epsilon^2(\tau(x_n) - \tau(x_0)) \\ &\le \epsilon^2 \tau_{\max}. \end{aligned}$$

Therefore, the difference becomes zero as $\epsilon \to 0$. For a general case, we have the following lemma.

Lemma 5.1 (Reference point independence)**.** *Let τ be twice differentiable and τ'' is uniformly bounded. Let $t > 0$ be fixed and the total number of walks of the sample path $\{x_n : n = 0, 1, \cdots, N\}$ is bounded by (5.4). Then, as $\epsilon \to 0$,*

$$\sup_{0\le b\le 1} T(b) - \inf_{0\le b\le 1} T(b) = O(\epsilon^2).$$

Proof. Denote $a = 1-b$. Consider a jump from x to y. The Taylor expansion gives

$$\tau(by+ax) = \tau(x+b(y-x)) = \tau(x) + b(y-x)\tau'(x) + \frac{1}{2}b^2(y-x)^2\tau''(\tilde{y}),$$

$$\tau(by+ax) = \tau(y+a(x-y)) = \tau(y) + a(x-y)\tau'(y) + \frac{1}{2}a^2(y-x)^2\tau''(\tilde{x}),$$

where $\tilde{x}$ and $\tilde{y}$ are between x and y. By adding the two after multiplying a and b, respectively, we obtain

$$\begin{aligned}\tau(by+ax) &= a\tau(x) + b\tau(y) - ab(x-y)\left(\tau'(x) - \tau'(y)\right) \\ &\qquad + \frac{1}{2}(ba^2+ab^2)(x-y)^2\left(\tau''(\tilde{x}) + \tau''(\tilde{y})\right) \\ &= a\tau(x) + b\tau(y) - ab(x-y)^2\frac{\tau'(x)-\tau'(y)}{x-y} \\ &\qquad + \frac{1}{2}ab(x-y)^2\left(\tau''(\tilde{x}) + \tau''(\tilde{y})\right).\end{aligned}$$

Since the jump distance $|y-x|$ is of order $O(\epsilon)$, there exists $C > 0$ such that $|y-x| < C\epsilon$. Hence, we have

$$\left|\tau(by+ax) - (b\tau(y) + a\tau(x))\right| \le 2abC^2\|\tau''\|_\infty\epsilon^2.$$

Then, for any $b \in [0, 1]$,

$$\begin{aligned}|T(0) - T(b)| &= \epsilon^2\Big|\sum_{n=0}^{N-1}\left(\tau(x_n) - \tau(ax_n + bx_{n+1})\right)\Big| \\ &\le \epsilon^2\Big|\sum_{n=0}^{N-1}\left(\tau(x_n) - b\tau(x_{n+1}) - a\tau(x_n)\right)\Big| + \epsilon^4\sum_{n=0}^{N-1} 2abC^2\|\tau''\|_\infty \\ &= \epsilon^2|b\tau(x_0) - b\tau(x_n)| + \epsilon^4 N 2abC^2\|\tau''\|_\infty.\end{aligned}$$

Since the number of time step is bounded by (5.4), we obtain

$$|T(0) - T(b)| \le \epsilon^2\Big(b\|\tau\|_\infty + 2abC^2\|\tau''\|_\infty\frac{t}{\tau_{\min}}\Big).$$

By using the triangle inequality, we obtain

$$\sup_{0\le b\le 1} T(b) - \inf_{0\le b\le 1} T(b) \le 2\epsilon^2\Big(b\|\tau\|_\infty + 2abC^2\|\tau''\|_\infty\frac{t}{\tau_{\min}}\Big),$$

which completes the proof. ∎

Lemma 5.1 says that the reference point dependency on the total amount of time $T(b)$ is negligible. However, the effect of reference point for the jumping distance is not. To see this, we compute the final position of a particle. This is a process to obtain a sample path when a collection of random numbers $\{W_1, W_2, \cdots, W_n\}$ are given when $W_n = -1$ or 1. Then, depending on the choice of reference points, we may obtain a different sample path. Suppose that the reference point is given by

$$x_{n+1} = x_n + \epsilon\ell(ax_{n+1} + (1-a)x_n)W_{n+1}, \quad n = 1, \cdots, N-1,$$

where the initial point x_0 is given. Define the final position as

$$X(a) = x_n = x_0 + \sum_{n=1}^{N} \epsilon\ell(ax_{n+1} + (1-a)x_n)W_{n+1} \tag{5.5}$$

which depends on the parameter a.

The arguments to compute the final time $T(b)$ does not work in computing the final position $X(a)$. The main reason is that if the reference points are chosen differently, the sample path is changed and the difference may increase rapidly. For example, consider a simple case when the walk-length is given by a function $\ell(x) = x$ with initial position $x = 1$. Suppose further that the particle jumps to right always, i.e., $W_n = 1$ for all $i = 1, \cdots, N$. Denote the sample path as x_n when the reference point is the departure point, i.e., $a = 0$. Then,

$$x_{n+1} = x_n + \epsilon\ell(x_n) = x_n + \epsilon x_n = (1+\epsilon)x_n.$$

Therefore, the final position is $X(0) = x_n = (1+\epsilon)^N x_0 = (1+\epsilon)^N$.

Denote the sample path as y_n when the reference point is the arrival point, i.e., $a = 1$. Then,

$$y_{n+1} = y_n + \ell(y_{n+1}) = y_n + \epsilon y_{n+1}.$$

Therefore,

$$y_{n+1} = \frac{y_n}{1-\epsilon},$$

and the final position is $X(1) = (1-\epsilon)^{-N}$. The difference is

$$X(1) - X(0) = y_n - x_n = (1-\epsilon)^{-N} - (1+\epsilon)^N = N\epsilon^2 + O(\epsilon^3).$$

Since $N \geq \frac{t}{\tau_{\max}\epsilon^2}$, we have

$$X(1) - X(0) \geq \frac{t}{\tau_{\max}} + O(\epsilon^3) \not\to 0$$

as $\epsilon \to 0$.

In the following lemma, we consider a general case. In the lemma, we assume ℓ is monotone decreasing and we can obviously extend the result when ℓ is monotone increasing. If ℓ changes its monotonicity in a macroscopic scale, we may divide the domain. If ℓ changes its monotonicity in a microscopic scale, it has no meaning anyway.

Lemma 5.2 (Reference point dependence). *Let $t > 0$, N be an integer bounded by* (5.4)*, and $\ell(x)$ be twice differentiable and satisfy*

$$0 < \ell_{\min} \leq \ell(\cdot) \leq \ell_{\max} < \infty, \quad -\infty < -\ell'_{\max} \leq \ell'(\cdot) \leq -\ell'_{\min} < 0.$$

For a given sequence $\{W_n\}_{n=1}^N$ with $W_n = -1$ or 1, define $X(a)$ by (5.5)*. Then, for $\epsilon > 0$ small, there exists $\{W_n\}_{n=1}^N$ such that*

$$\sup_{0\leq a\leq 1} X(a) - \inf_{0\leq a\leq 1} X(a) \geq \frac{t}{\tau_{\max}} \ell_{\min}\ell'_{\min}. \tag{5.6}$$

Proof. For a simpler computation, we take a case when the N steps of walks are in the positive direction, i.e., $W_n = 1$ for $n = 1, \cdots, N$. Denote the sample path as x_n when the reference point is the departure point, i.e., $a = 0$. Then, x_k satisfy

$$x_{k+1} = x_k + \epsilon\ell(x_k)$$

and hence

$$x_N = x_0 + \epsilon\ell(x_0) + \cdots + \epsilon\ell(x_{N-1}).$$

Denote the sample path as y_k when the reference point is the arrival point, i.e., $a = 1$. Then, y_k satisfy

$$y_{k-1} = y_k - \epsilon\ell(y_k)$$

and hence

$$y_0 = y_N - \epsilon\ell(y_N) - \epsilon\ell(y_{N-1}) - \cdots - \epsilon\ell(y_1).$$

By definition, one has $x_0 < x_1 < \cdots < x_N$ and $y_0 < y_1 < \cdots < y_N$ with $x_{k+1} - x_k, y_{k+1} - y_k \geq \epsilon\ell_{\min}$. For notational convenience, we set $x_N = y_N$ and estimate $x_0 - y_0$. Then, by comparing y_{N-1} and x_{N-1}, we obtain

$$\begin{aligned} y_{N-1} - x_{N-1} &= y_N - \epsilon\ell(y_N) - x_N + \epsilon\ell(x_{N-1}) \\ &\geq \epsilon\ell(x_{N-1}) - \epsilon\ell(x_N) \\ &\geq \epsilon\ell'_{\min} \cdot |x_N - x_{N-1}| \\ &\geq \epsilon^2\ell_{\min}\ell'_{\min}. \end{aligned}$$

Using inductive argument, we obtain

$$\begin{aligned} y_{N-k} - x_{N-k} &= y_{N-k+1} - x_{N-k+1} - \epsilon\ell(y_{N-k+1}) + \epsilon\ell(x_{N-k}) \\ &\geq (k-1)\epsilon^2\ell_{\min}\ell'_{\min} + \epsilon\ell'_{\min} \cdot |y_{N-k+1} - x_{N-k}| \\ &= (k-1)\epsilon^2\ell_{\min}\ell'_{\min} + \epsilon\ell'_{\min} \cdot |y_{N-k+1} - x_{N-k+1} + x_{N-k+1} - x_{N-k}| \\ &\geq (k-1)\epsilon^2\ell_{\min}\ell'_{\min} + \epsilon\ell'_{\min} \cdot |x_{N-k+1} - x_{N-k}| \\ &\geq k\epsilon^2\ell_{\min}\ell'_{\min}, \end{aligned}$$

and hence

$$y_0 - x_0 \geq N\epsilon^2\ell_{\min}\ell'_{\min}.$$

Since $\frac{t}{\tau_{max}} \geq N\epsilon^2$, we have

$$y_0 - x_0 \geq N\epsilon^2\ell_{\min}\ell'_{\min} \geq \frac{t}{\tau_{max}}\ell_{\min}\ell'_{\min}.$$

Note that the sequence $\{x_0, x_1, \cdots, x_N = y_N, \cdots, y_1, y_0\}$ is a sample path taking the starting point as a reference point and walking N times to the positive direction and then walking N times to the negative direction. We see that the final point is different from the starting point and the difference is of order $O(1)$. $\square$

6. Heterogeneous sojourn time

We assume the sojourn time $\tau(x)$ and its second order derivative are uniformly bounded by

$$0 < \tau_{\min} \le \tau(x) \le \tau_{\max} < \infty, \quad |\tau''(x)| < M. \tag{6.1}$$

We assume that a particle stay for the period of sojourn time $\tau(x)$ after arriving at a point x then jumps to the next position instantaneously. We can make an individual based model for a heterogeneous sojourn time. However, both discrete-time and continuous-time model cannot handle nonconstant sojourn time except special cases (see [3]). To handle a general nonconstant sojourn time, we introduce departing rate. For a given sojourn time $\tau(x)$, we first take a time step $\tau_0 > 0$ which is smaller than or equal to $\tau_{\min}$ in (6.1). Then, we set the *departing rate* as

$$\gamma(x) = \frac{\tau_0}{\tau(x)} \le 1.$$

This is the rate of the population at x that depart the position during the fixed time period $\tau_0 > 0$. If the departing rate is included to the system, (2.4) becomes

$$\begin{cases} u^\epsilon_{n+1} = \frac{1}{2}\gamma(B_+)u^\epsilon_n(B_+)B'_+ + \frac{1}{2}\gamma(B_-)u^\epsilon_n(B_-)B'_- + (1-\gamma)u^\epsilon_n, \\ u^\epsilon_{n=0} = u_0. \end{cases} \tag{6.2}$$

Note that the departing rate is given as a function of the departure point.

Denote

$$\begin{aligned} \tilde{K}_\epsilon u &:= \frac{1}{2}\gamma(B_+)u(B_+)B'_+ + \frac{1}{2}\gamma(B_-)u(B_-)B'_- + (1-\gamma)u, \\ \tilde{L}_\epsilon u &:= \frac{\tilde{K}_\epsilon u - u}{\epsilon^2\tau_0}. \end{aligned}$$

Then,

$$\tilde{L}_\epsilon u = \frac{\tilde{K}_\epsilon u - u}{\epsilon^2\tau_0} = \frac{K_\epsilon \gamma u - \gamma u}{\epsilon^2\tau_0} = L_\epsilon(\gamma u),$$

where K_ϵ and L_ϵ are the operators given in (4.4) and (4.7). Since we assume $\gamma(\cdot)$ has the same regularity that $\ell(\cdot)$ has, Lemmas 4.6– 4.8 still hold after the operators are replaced with $\tilde{K}_\epsilon$ and $\tilde{L}_\epsilon$. Then, Lemma 4.7 is written as

$$\frac{\tilde{K}_\epsilon u - u}{\epsilon^2\tau_0} = \tilde{L}_\epsilon u = L_\epsilon(\gamma u) = L(\gamma u) + O(\epsilon) \quad \text{as } \epsilon \to 0,$$

where $Lu = \frac{1}{2\tau_0}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x$. The estimate in Lemma 4.8 is written as

$$\begin{aligned}|\tilde{L}_\epsilon(1)| &\leq |L(\gamma)| + |(L(\gamma))_x| \cdot ||\ell||\epsilon \\ &= |\frac{1}{2\tau_0}\left(\ell^{2a}(\ell^{2-2a}\gamma)_x\right)_x| + |\frac{1}{2\tau_0}\left(\ell^{2a}(\ell^{2-2a}\gamma)_x\right)_{xx}| \cdot ||\ell||\epsilon.\end{aligned}$$

Theorem 6.1 (Uniform convergence)**.** *Let $u_0(x)$, $\ell(x)$, and $\gamma(x)$ be smooth, bounded, and positive. Let u^ϵ be the linear interpolation* (4.1) *of the solution u_n^ϵ of* (6.2)*. Then, for a fixed $T > 0$, u^ϵ converges uniformly to the solution u of*

$$u_t = \frac{1}{2\tau_0}(\ell^{2a}(\ell^{2-2a}\gamma u)_x)_x. \tag{6.3}$$

Proof. We already know that $\tilde{K}_\epsilon$ is a positive operator. Let $w^\epsilon = u^\epsilon - u$ and consider a discrete time sample,

$$w_n^\epsilon(x) = w^\epsilon(n\epsilon^2\tau_0, x) = u_n^\epsilon(x) - u(n\epsilon^2\tau_0, x).$$

Using the recursive relation gives

$$\begin{aligned}w_{n+1}^\epsilon(x) &= u_{n+1}^\epsilon(x) - u((n+1)\epsilon^2\tau_0, x) \\ &= \tilde{K}_\epsilon u_n^\epsilon(x) - u(n\epsilon^2\tau_0, x) - \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} L(\gamma u)(s,x)ds \\ &= \tilde{K}_\epsilon w_n^\epsilon(x) + \tilde{K}_\epsilon u(n\epsilon^2\tau_0, x) - u(n\epsilon^2\tau_0, x) - \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} L(\gamma u)(s,x)ds \\ &= \tilde{K}_\epsilon w_n^\epsilon(x) + \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} \left(L(\gamma u)(t,x) - L(\gamma u)(s,x) + O(\epsilon)\right) ds.\end{aligned}$$

Let $F_{\epsilon,n}(u;x) = \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} \left(L(\gamma u)(t,x) - L(\gamma u)(s,x) + O(\epsilon)\right) ds$. Then,

$$\begin{aligned}|F_{\epsilon,n}(u;x)| &\leq \int_{n\epsilon^2\tau_0}^{(n+1)\epsilon^2\tau_0} (s-t)||L(\gamma L(\gamma u))||_\infty + O(\epsilon)ds \\ &\leq ||L(\gamma L(\gamma u))||_\infty(\tau_0)^2\epsilon^4 + O(\epsilon^3) \leq \epsilon^2\theta(\epsilon) = \epsilon^2 O(\epsilon)\end{aligned}$$

for some $\theta(\epsilon)$ which is function of ϵ only. Then $\{w_n^\epsilon(x)\}$ is the solution of recursive relation

$$\begin{aligned}v_{m+1}(x) &= \tilde{K}_\epsilon v_m(x) + F_{\epsilon,n}(u;x) \\ v_{n=0}(x) &= 0\end{aligned} \tag{6.4}$$

Then we already know that $\eta(\epsilon)$ as

$$\eta(\epsilon) = \max_{x\in\mathbb{X}} \frac{1}{\epsilon^2\tau_0}|\tilde{K}_\epsilon(1) - 1|$$

is bounded similarly as in the lemma so with the same argument as in the previous section, we have uniform convergence as well.

Similar to previous proof,

$$\bar{w}_n(\epsilon) = \begin{cases} \frac{\theta(\epsilon)}{\eta(\epsilon)}\left(\left(1+\epsilon^2\eta(\epsilon)\right)^N - 1\right) & ,\eta(\epsilon)\neq 0 \\ \theta(\epsilon)n\epsilon^2 & ,\eta(\epsilon) = 0 \end{cases}$$

is super-solution of the recursive relation (6.4) by comparison principle and the lemma as

$$\begin{aligned} \bar{w}_{n+1}(\epsilon) &= (1+\epsilon^2\eta(\epsilon))\bar{w}_n(\epsilon) + \epsilon^2\theta(\epsilon) \\ &\geq \tilde{K}_\epsilon \bar{w}_n(\epsilon) + \epsilon^2\theta(\epsilon) \\ &\geq \tilde{K}_\epsilon \bar{w}_n(\epsilon) + F_{\epsilon,n}(u;x) \end{aligned}$$

and

$$\bar{w}_{n=0}(\epsilon) = 0$$

Thus by comparison,

$$-\bar{w}_n \leq w_n^\epsilon \leq \bar{w}_n$$

and with similar linear interpolation argument we achieve uniform convergence

$$||u^\epsilon - u||_{L^\infty(\mathbb{X}\times[0,T])} \to 0$$

as $\epsilon \to 0$. □

Since the sojourn time τ and γ are connected by the relation $\gamma(x) = \frac{\tau_0}{\tau(x)}$, (6.3) is written as

$$u_t = \left(\frac{\ell^{2a}}{2}\left(\frac{\ell^{2-2a}}{\tau}u\right)_x\right)_x.$$

If $a = 0.5$, the random walk becomes reversible, and the corresponding diffusion equation is

$$u_t = \frac{1}{2}\left(\ell\left(\frac{\ell}{\tau}u\right)_x\right)_x.$$

Acknowledgements. This work was supported by National Research Foundation of Korea (NRF-2017R1A2B2010398).

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