HETEROGENEOUS CONTINUOUS-TIME RANDOM WALK AND NONLOCAL DIFFUSION

YONG-JUNG KIM AND HYUN-JIN LIM

ABSTRACT. Diffusion in a heterogeneous environment is not understood well. There have been lots of discussions and debates on it when diffusivity varies in space. Nonlocal diffusion theory is also developed for homogeneous environments and its extension to a heterogeneous environment is limited. In this paper, we introduce a systematic method to construct heterogeneous nonlocal diffusion and show the convergence to a local diffusion equation as a parabolic scale limit. We introduce the spatial heterogeneity in the walk-length and departing rate.

1. INTRODUCTION

One of the commonly employed nonlocal diffusion models is an integro-differential equation,

$$u_t = \int_{\mathbb{R}^n} u(t,y)K(y\to x)dy - u(t,x)\int_{\mathbb{R}^n} K(x\to y)dy, \tag{1.1}$$

where $u(t,x)$ is the population density at position x and time t, and $K(x\to y)$ is the migration rate from x to y. The equation is called a convolution model if

$$K(x\to y) = J(x-y),$$

where the convolution kernel J is usually normalized and assumed to be symmetric, i.e.,

$$\int_{\mathbb{R}^n} J(z)dz = 1, \tag{1.2}$$

and

$$J(z) = J(-z). \tag{1.3}$$

The nonlocal diffusion equation has been used to explain various phenomena such as phase transition and speration [3, 4, 5, 6], biological mutation [1, 8, 9, 10, 21, 22], dispersal with long-distance events [16, 25, 29], etc. The fundamental properties of the problem such as uniqueness, existence, stability, interface motion, and traveling wave solutions are studied well [13, 19, 20, 32, 36]. It is shown that the nonlocal equation can approximate wide class of diffusion equations (e.g.,[11, 16, 24]).

The convolution model (1.1)–(1.3) is called homogeneous since a single kernel is used over the whole space. Theories for such a homogeneous nonlocal diffusion equation are well-established. If the environment is spatially heterogeneous, we need to include spatial heterogeneity in the model which will produce heterogeneous nonlocal diffusion equations. The spatial heterogeneity has been studied recently [7, 10, 27]. In particular, Molino and Rossi [28] proposed a heterogeneous nonlocal model using,

$$K(x \to y) = J(A(x)(x-y)) \det A(x), \tag{1.4}$$

where J is the convolution kernel and $A(x)$ is an $n \times n$ real matrix with $\det A(x) \geq a_0 > 0$. The spatial heterogeneity is included by assigning a dilation matrix $A(x)$ to each point x, and the multiplied Jacobian, $\det A(x)$, makes the mass conserved. It is shown that the diffusion limit (or parabolic scale limit) of the problem converges to the solution of Chapman's diffusion law [12]

$$u_t = \Delta(D(x)u) \tag{1.5}$$

when the resulting diffusion is isotropic and $D(x)$ is the diffusivity. One of the critical issues of heterogeneous diffusion theory is that there are infinitely many choices of diffusion laws when the diffusivity $D(x)$ is not constant, e.g.,

$$u_t = \nabla \cdot (D^{1-q}(x)\nabla(D^q(x)u)), \quad q \in \mathbb{R}. \tag{1.6}$$

Two other well-known examples are Fick's law [18] with $q = 0$ and Wereide's law [34] with $q = 0.5$. Of course, these diffusion laws are all identical if D is constant.

There have been lots of discussions and debates about what is the correct diffusion law when the diffusivity D is spatially nonconstant (see [26, 31] for discussions and experiments). However, the fact that Chapman's diffusion law (1.5) is derived from the nonlocal equation (1.1) with (1.4) does not imply (1.5) is the correct diffusion law. The other two diffusion laws, (1.6) with $q = 0$ or $q = 0.5$, also can be derived from related nonlocal diffusion equations (see [27]). It is clear that a derivation of a diffusion law does not justify it. We only need to find useful information from the mathematical derivation of diffusion laws, which is the mechanism that decides the diffusion laws. One of the main purposes of the paper is to show that it is the way to choose the reference point that decides the diffusion law.

The reference point is the position from which the spatial heterogeneity is taken. For example, when a particle jumps from a departure point x to the arrival point y, the spatial heterogeneity should be taken from somewhere nearby x or y. In the model (1.4), the heterogeneity is taken from the departure point, $A = A(x)$, i.e., the reference point is the departure point. In the paper, we consider the reference point dependency by taking the reference point, denoted by p, as

$$p_{x \to y} = ay + (1-a)x, \quad 0 \leq a \leq 1. \tag{1.7}$$

In the model (1.4), the spatial heterogeneity is included in the dilation matrix A and the reference point is the departure point x, i.e., $a = 0$. Recently, Alfaro *et al.* [2] proposed a heterogeneous nonlocal model using,

$$K(x \to y) = J(p, x - y) \ \big(= J(ay + (1-a)x, x - y) \big), \tag{1.8}$$

where $J(p,z)$ is a convolution kernel assigned to a position $p \in \mathbb{R}^n$, z is the variable for the convolution, and $J(p,z) = J(p,-z)$. Then, after taking a diffusion limit, the nonlocal diffusion equation gives a local diffusion equation,

$$u_t = \nabla \cdot (D^{2a}(x) \nabla (D^{1-2a}(x) u)). \tag{1.9}$$

If $a = 0$, the reference point is the departure point, $p = x$, and the obtained equation is Chapman's diffusion law. This is the same diffusion law obtained from the kernel (1.4), which says that if the reference point is the departure point, we obtain Chapman's diffusion law (1.5).

The two convolution models, (1.4) and (1.8), have different meanings. In particular, (1.8) fits a situation that organisms migrate among habitats and has the property of a discrete-patch model. For example, if $a = \frac{1}{2}$, the kernal (1.8) is symmetric, i.e., $K(x \to y) = K(y \to x)$, and the resulting diffusion equation is Fick's law (**??**). For a continuous-space random walk model for physical diffusion phenomena when $u(t,x)$ is the particle density, a Jacobian term such as in (1.4) should be included. However, it is too complicate to include such a term for a general situation such as (1.8), where the dilation in (1.4) is an exceptional case.

2. Model and results

The purpose of the paper is to introduce a spatially heterogeneous nonlocal diffusion model that naturally preserves the mass without adding a Jacobian like term and then show the convergence of its solution to the solution of a diffusion law after taking a parabolic scale limit. In this process, we will see the reference point dependency in the resulting diffusion equation. For simplicity, we consider a random walk system in the one space dimension. However, the method and result of the paper can be extended to multiple dimensions.

Let $\ell \in C^2(\mathbb{R})$ be the distance function of the walk-length at position $x \in \mathbb{R}$ and we assume ℓ and its derivatives are uniformly bounded by

$$0 < \ell_{min} \le \ell(x) \le \ell_{max} < \infty, \quad |\ell'(x)| + |\ell''(x)| < M. \tag{2.1}$$

The spatial heterogeneity of the model is included in this distance function $\ell(x)$. If a particle jumps from x to y, then the distance is given by

$$|y - x| = \ell(p) = \ell(ay + (1-a)x).$$

Thaking this relation as a rule, we may define $N_{\pm}(x)$, which is the two positions that a particle at x may jump to with the equal probability $\frac{1}{2}$. Then, the two positions $N_+(x) > x$ and $N_- < x$ are given implicitly by

$$\begin{cases} N_+(x) = x + \ell(aN_+(x) + (1-a)x), \\ N_-(x) = x - \ell(aN_-(x) + (1-a)x). \end{cases} \tag{2.2}$$

Next, we take the kernel $K(x \to y) = K(x, y)$ as to a variable function for the conditional probability of Markov process, where the kernel gives the conditional probability $\int_A K(x, y)dy = P(X_t \in A|X(0) = x)$. Hence, $K(x, y)$ is the rate for a particle to move from x to y per unit time. Then, the rate for a particle to depart a position x is given by

$$\gamma(x) = \int_{\mathbb{R}} K(x, y)dy. \tag{2.3}$$

For a homogeneous case, the departing rate is constant and is normalized by one (1.2). We assume that the departing rate $\gamma(x)$ is bounded by

$$0 < \gamma_{min} \le \gamma(x) \le \gamma_{max} < \infty. \tag{2.4}$$

Notice that since K is a rate for probability, not probability itself, γ is not necessarily bounded by one. Then, a random walk system that walks to one of the two positions $N_\pm(x)$ with equal probability with departure rate $\gamma(x)$ can be obtained by taking a kernel,

$$K(x, y) = \frac{\gamma(x)}{2}\delta(y - N_-(x)) + \frac{\gamma(x)}{2}\delta(y - N_+(x)). \tag{2.5}$$

This kernel is not symmetric. For example, $\int K(x, y)dy = \gamma(x) \neq \int K(y, x)dy$. Finally, the nonlocal integration equation (1.1)with an initial value becomes

$$\begin{cases} u_t(t, x) = \displaystyle\int_{\mathbb{R}} u(t, y)K(y, x)dy - \gamma(x)u(t, x), \\ u(0, x) = u_0(x). \end{cases} \tag{2.6}$$

If the functions $N_\pm(x)$ for the next positions have inverse functions and denote them by $B_\pm(x)$, the (2.6) is written as

$$\begin{aligned} u_t(t, x) = &\frac{\gamma(B_+(x))}{2}u(B_+(x), t)\partial_x B_+(x) + \frac{\gamma(B_-(x))}{2}u(B_-(x), t)\partial_x B_-(x) \\ &- \gamma(x)u(t, x). \end{aligned}$$

The terms ${B_+}'(x)$ and ${B_-}'(x)$ play the role of the Jacobian determinant and make the mass conserved.

The main result of the paper is Theorem 6.1. If we rewrite the conclusion in n space dimensions for comparison, the diffusivity of the model is given as

$$D(x) = \frac{\ell^2(x)\gamma(x)}{2n}$$

and the diffusion limit of the solution of the nonlocal problem (1.1) converges to the solution of a diffusion equation

$$u_t = \nabla \cdot \Big(\Big(\frac{D(x)}{\gamma(x)}\Big)^a \nabla\Big(\Big(\frac{D(x)}{\gamma(x)}\Big)^{1-a}\gamma(x)u\Big)\Big). \tag{2.7}$$

If $\gamma(x) = 1$, it is written as

$$u_t = \nabla \cdot \big(D(x)^a \nabla\big(D(x)^{1-a}u\big)\big). \tag{2.8}$$

If we rewrite (2.7) in terms of $\ell = \ell(x)$ and $\gamma = \gamma(x)$, we obtain

$$u_t = \frac{1}{2n}\nabla \cdot \big(\ell^{2a}\nabla\big(\ell^{2-2a}\gamma u\big)\big).$$

Remark 2.1. *The kernel* (2.5) *is the case that a particle departing x moves to exactly $N_+(x)$ or $N_-(x)$. We may generalize the situation and allow particles to move in a neighborhood of $N_\pm(x)$ by taking*

$$K(x,y) = \frac{\gamma(x)}{2}\varphi(y - N_-(x)) + \frac{\gamma(x)}{2}\varphi(y - N_+(x)),$$

where φ be a compactly supported smooth function with $\int \varphi(x)dx = 1$.

In Section 3, we introduce a heterogeneous position jump process and formulate a nonlocal equation. In Section 4, we give formal calculation of local diffusion equation by parabolic scaling limit of the nonlocal model. In Section 5, we consider a case with $\gamma(x) = 1$ and prove the convergence of the solution of non-local equation to a local diffusion equation after taking a diffusion limit. In Section 6, we extend the result to a case when $\gamma(x)$ is not constant.

3. Random walk model as a Markov process

In this section, we develop the continuous-time random walk model (1.1) with (2.5) in terms of Markov process and see the connection with Continuous Time Markov Chain (or CTMC for brevity). Since the walk-length of the random walk is heterogeneous, we consider continuous-space random walk to handle the spatial heterogeneity properly. The obtained continuous-time random walk system is one of the simplest nonlocal diffusion equation which jumps to one of two possible positions.

3.1. **CTMC.**

A random walk system is a Markov process that the current position of a particle decides the probability distribution of the next position. In our case with a non-constant walk-length and sojourn time, the distribution is a spatial function. The process is a CTMC given by a stochastic process $\{X_t\}_{t\in T}$, where $T = [0,\infty)$ and $X_t : (t,\omega) \mapsto s \in S$ represents the position at time t and ω in filtered probability space $\big(\Omega, F, \{F_t\}_{t\in T}, P\big)$. In a CTMC, stopping time steps are sequence of moments τ_n such that the particle jumps to the next state and stays until the next jump. We may define them inductively by $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t \in [\tau_n,\infty); X_t \neq X_{\tau_n}\} = \inf\{t \in (\tau_n,\infty); X_t \neq X_{\tau_n}\}.$$

We can relate the stopping time steps to staying period of time at each position, e.g., τ_1 is the time period to stay at $X_0 = s \in S$, and is called holding time. The holding time is a random variable and takes an exponential distribution with parameter $\gamma(x) \in [0,\infty]$. In order to avoid well-known pathologies such as infinite number of transition in finite time or staying at some position forever, we assume the uniform bounds in (2.4). Then, such a

Markov chain is regular, i.e., it is almost surely right continuous and there is no infinite transition in a finite time.

A jump chain of a given CTMC is defined as $\{Y_n = X_{\tau_n}\}$, which is called an embedded discrete time Markov chain or sinply a jump chain. The Markovity of a jump chain follows from the strong Markovity of the CTMC. Assuming time homogeneity of the CTMC, one-step transition matrix Q of the jump chain Y is given by

$$Q(x,y) = P(X_{\tau_1} = y | X_0 = x), \quad x, y \in S.$$

Note that transition matrix is measurable with respect to the current state x and takes a probability mass with respect to the next state y. The jump chain and holding time are independent of given initial state.

In the paper, our main interest is the effect of the spatial heterogeneity in the diffusion equation and we take a simplest case when particles jump to either right or left from their current position on a finite interval $S = [0,1]$ with the periodic boundary condition. More specifically, we choose $Q(x,y)$ as

$$Q(x,y) = \frac{1}{2}\delta\left(y - N_+(x)\right) + \frac{1}{2}\delta\left(y - N_-(x)\right)$$

where $N_\pm(x)$ is next position when the current position is x. Finally, for a given CTMC, a generator is defined as a one-sided derivative of the transition kernel $\{P_t\}$ at 0, i.e.,

$$G = \lim_{t\to 0^+} \frac{P_t - I}{t}.$$

It is known that the generator matrix $G(x,y)$ is defined well and satisfies

$$G(x,y) = -\gamma(x) I(x,y) + \gamma(x) Q(x,y),$$

where $I(x,y)$ is identity matrix and $Q(x,y)$ is the transition matrix of the embedded DTMC. When sojourn time is constant, every state has identical distribution for holding time and one can assume $\gamma(\cdot) = 1$. Then, $G(x,y)$ is identical to $Q(x,y)$ which is the one step transition matrix of embedded DTMC if the identity part $I(x,y)$ is ignored.

Note that $\sum_y G(x,y) = 0$,

$$G(x,x) = -\sum_{y\neq x} G(x,y), \quad \text{and} \quad Q(x,y) = -\frac{G(x,y)}{G(x,x)}.$$

For a finite or a discrete states case, $G(x,y)$ is the jumping rate from x to y, and the departing rate $\gamma(x) := -G(x,x)$ is assumed to be bounded by (2.4). A CTMC gives the dynamics of population $\mu(t,x)$ in as a differential form,

$$\partial_t \mu(t,x) = \sum_y \mu(t,y) G(y,x). \tag{3.1}$$

Since $\sum_y G(x,y) = 0$, it can also be written as

$$\partial_t \mu(t,x) = \sum_y \mu(t,y) G(y,x) - \mu(t,x) \sum_y G(x,y). \tag{3.2}$$

3.2. **Model for continuous state space.** In our case, the sample space is continuous and we consider densities. Then, the population μ is replaced with the population density $u(t,x)$ and the jumping probability $G(x,y)$ is with jumping probability density $K(x,y)$. In the probability theory, the generator matrix $G(x,y)$ gives the probability to move from x to y only when $x \neq y$. If $x = y$, $G(x,x) = -\sum_{y\neq x} G(x,y)$ is the probability to depart x. However, we cannot take this convection for a continuous state case. Hence, we use the formula (3.2), not (3.1), and take an integral equation,

$$\partial_t u(t,x) = \int u(t,y)K(y,x)dy - u(t,x)\int K(x,y)dy,$$

as our continuous state model.

We need an assumption on the kernel $K(x,y)$ that there exists a function γ and it satisfies

$$\int K(x,y)dy = \gamma(x). \tag{3.3}$$

Note that the kernel $K(x,y)$ is not a function of two points x and y only, but may depend on the points in a neighborhood. Hence, (3.3) is an assumption and a definition for $\gamma(x)$ at the same time, which says that the total rate that departs x is given by x alone. However, in general, $\int K(x,y)dx$ is not a function of y alone. The $\gamma(x)$ plays the same role of $\gamma(x)$ for the CTMC (3.1) and (3.2). We also need upper and lower bounds in (2.4). Notice that since K is a probability density, not probability itself, γ is not necessarily bounded by one.

In the paper, we take a kernel

$$K(x,y) = \frac{\gamma(x)}{2}\delta(y - N_-(x)) + \frac{\gamma(x)}{2}\delta(y - N_+(x)). \tag{3.4}$$

This kernel models a random walk that a particle jumps to one of two positions $N_\pm(x)$ with equal probability $\frac{1}{2}$ when its current position is x. The specific choice of $N_\pm(x)$ in this paper is given in the following section. The probability to jump from x is given by $\gamma(x)$. We can easily see that (3.3) is satisfied. However, the derivative $K(x,y)$ with respect to x is

$$\int K(x,y)dx = \frac{\gamma(B_+(y))}{2}\frac{1}{{N_+}'(B_+(y))} + \frac{\gamma(B_-(y))}{2}\frac{1}{{N_-}'(B_-(y))},$$

where $B_\pm$ are inverse functions of $N_\pm$, i.e., the function for previous position. Then, the integral form (2.6) is written as,

$$\begin{aligned} u_t(t,x) = {}& \frac{\gamma(B_+(x))}{2}u(B_+(x),t)\partial_x B_+(x) + \frac{\gamma(B_-(x))}{2}u(B_-(x),t)\partial_x B_-(x) \\ & - \gamma(x)u(t,x). \end{aligned} \tag{3.5}$$

3.3. **Kernel generalization.** In this paper, we consider (3.4), which is one of the simplest models that a particle jumps to one of two possible positions. However, the approach in the paper is flexible and one can generalize the system in various ways. For example, by taking a linear combination of delta measures, $K(x,y)$ can be given as a discrete sum,

$$K(x,y) = \sum_j p_j \delta(y - N_j(x)),$$

or as an integration form,

$$K(x,y) = \int p(\alpha)\delta\left(y - N(\alpha,x)\right) d\alpha,$$

where $N_j(\cdot)$ ane $N(\alpha,\cdot)$ are to denote possible positions in the next step, i.e., a particle placed at x jumps to one of $N_j(x)$ or $N(\alpha,x)$.

If the state x and the index α are in the same space, we can rewrite the kernel as

$$\begin{aligned} K(x,y) &= \int p(\alpha)\delta\left(y - N(\alpha,x)\right) d\alpha \\ &= \sum_{\alpha, y = N(\alpha,x)} \frac{p(\alpha)}{\partial_\alpha N(\alpha,x)}, \end{aligned}$$

where the summation is for α such that $y = N(\alpha,x)$. Then, by denoting $N^x(\alpha) = N(\alpha,x)$ as a function of α with a fixed x and taking its inverse function $B^x = (N^x)^{-1}$, we may write $\alpha = B^x(y)$. Then,

$$K(x,y) = p\left(B^x(y)\right) \partial_y B^x(y).$$

If this idea is generalized in $\mathbb{R}^n$, derivative is replaced by the Jacobian determinant.

4. Formal derivation

4.1. **Two points random walk.** We assume the walk-length function is smooth, $\ell \in C^3$, and satisfies

$$0 < \ell_{min} \le \ell(x) \le \ell_{max} < \infty, \quad |\ell'(x)| + |\ell''(x)| \le M < \infty. \tag{4.1}$$

The function $N_+^\epsilon(x) > x$ is for the position at the next step when a particle jumps to right from x. Similarly, $N_-^\epsilon(x) < x$ is the position when the particle jumps to left. These are implicitly given by

$$N_\pm^\epsilon(x) = x \pm \epsilon\ell(aN_\pm^\epsilon(x) + (1-a)x), \tag{4.2}$$

which decides the functions uniquely under the assumptions in (4.1) and the smallness of $\epsilon > 0$. Furthermore, $N_\pm^\epsilon$ are invertible for small $\epsilon > 0$, and we denote their inverse functions as $B_\pm^\epsilon$, respectively. Then,

$$B_\pm^\epsilon(x) = x \mp \epsilon\ell(ax + (1-a)B_\pm^\epsilon(x)). \tag{4.3}$$

The diffusion kernel is defined by

$$K^\epsilon(x,y) = \frac{1}{2}\delta(y - N_-^\epsilon(x)) + \frac{1}{2}\delta(y - N_+^\epsilon(x)). \tag{4.4}$$

Then,

$$\gamma(x) = \int K^\epsilon(x,y)dy = 1, \tag{4.5}$$

and (2.6) is written as

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{\epsilon^2}\displaystyle\int u(t,y)K^\epsilon(y,x)dy - u(t,x), \\ u(0,x) \ = u_0(x). \end{cases} \tag{4.6}$$

The small parameter $\epsilon > 0$ is to take the diffusion limit. The coefficient $\frac{1}{\epsilon^2}$ is from the macroscopic time scale and ϵ in (4.2) is for the space scale. The solution u of (4.6) depends on ϵ and we may write it as u^ϵ if we want to denote the dependency explicitly.

In this section, we formally show that, after taking a diffusion limit, the solution of (4.6) converges to a solution of

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x, \\ u(0,x) \ = u_0(x). \end{cases} \tag{4.7}$$

The proof is completed rigorously in the following section. We start with the case $a = 1$ when the reference point is the arrival point and then extend it to a general case. If $a = 1$, the reference point is the arrival point $p = x$ and $B_\pm^\epsilon$ are explicitly given by

$$B_+^\epsilon(x) = x - \epsilon\ell(x), \qquad B_-^\epsilon(x) = x + \epsilon\ell(x). \tag{4.8}$$

Then, by the Taylor expansion, (4.6) is written as

$$\begin{aligned}
\partial_t u(x) &= \frac{1}{2\epsilon^2}\big(u(x+\epsilon\ell(x))(1+\epsilon\ell'(x)) + u(x-\epsilon\ell(x))(1-\epsilon\ell'(x)) - 2u(x)\big) \\
&= \frac{1}{2\epsilon^2}\big((u + \epsilon\ell u' + \frac{1}{2}\epsilon^2\ell^2 u'')\cdot(1+\epsilon\ell') \\
&\qquad\qquad + (u - \epsilon\ell u' + \frac{1}{2}\epsilon^2\ell^2 u'')\cdot(1-\epsilon\ell') - 2u + O(\epsilon^3)\big) \\
&= \frac{1}{2\epsilon^2}\big(u + \epsilon u\ell' + \epsilon\ell u'\cdot(1+\epsilon\ell') + \frac{1}{2}\epsilon^2\ell^2 u'' + u - \epsilon u\ell' \\
&\qquad\qquad - \epsilon\ell u'\cdot(1-\epsilon\ell') + \frac{1}{2}\epsilon^2\ell^2 u'' - 2u\big) + O(\epsilon^3) \\
&= \ell\ell' u' + \frac{1}{2}\ell^2 u'' + O(\epsilon) = \frac{1}{2}(\ell^2 u_x)_x + O(\epsilon).
\end{aligned}$$

Therefore, after taking $\epsilon \to 0$, we obtain the equation (4.7) with $a = 1$, which is Fick's law, i.e.,

$$\partial_t u(x) = (Du_x)_x, \quad D = \ell^2/2.$$

Next, we consider the general case with $0 \le a \le 1$. In the following Lemma, we first show that $N^\epsilon_\pm(x)$ is invertible if ϵ is small enough and find an expansion of the inverse $B^\epsilon_\pm(x)$.

Lemma 4.1. *Let $\ell \in C^3(\mathbb{R})$ be uniformly bounded as in (4.1). Let $N^\epsilon_\pm(x)$ be implicitly given by (4.2) for $0 \le a \le 1$. Then, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $N^\epsilon_\pm$ are invertible and the inverse functions $B^\epsilon_\pm$ satisfy*

$$B^\epsilon_\pm(x) = x \mp \epsilon\ell(x) + (1-a)\epsilon^2\ell(x)\ell'(x) + O(\epsilon^3). \tag{4.9}$$

Proof. First, we show that $N^\epsilon_\pm$ are invertible if $\epsilon > 0$ is small enough. Let $y = N^\epsilon_\pm(x)$. Then, by a differentiation,

$$\frac{dy}{dx} = 1 \pm \epsilon\ell'(ay + (1-a)x)\left(a\frac{dy}{dx} + 1 - a\right).$$

Then,

$$\frac{dy}{dx}(1 \mp \epsilon\ell'(a^2y + (1-a)ax)) = 1 \pm \epsilon\ell'(ay + (1-a)x)(1-a).$$

Since $|\ell'|$ is bounded, there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$,

$$(1 \mp \epsilon\ell'(a^2y + (1-a)ax)) > 0, \quad 1 \pm \epsilon\ell'(ay + (1-a)x)(1-a) > 0.$$

Therefore, $N^\epsilon_\pm(x)$ are monotone and invertiable. The inverse functions are denoted by $B^\epsilon_\pm(x)$.

The inverse functions $B^\epsilon_\pm$ are given explicitly as in (4.8) when the reference point is the final point, $a = 1$. In general, the inverse functions are not given explicitly. However, in order to find the parabolic scaling limit, we only need the Taylor expansion of $B^\epsilon_\pm$. Since $|y - x| = \epsilon|\ell(ay + (1-a)x)| \le \epsilon\ell_{max}$,

$$\begin{aligned} y - x &= \pm\epsilon\ell(ay + (1-a)x) \\ &= \pm\epsilon\left(\ell(y) + (1-a)\ell'(y)(x-y) + O(\epsilon^2)\right). \end{aligned}$$

If we rewrite it for $y - x$, we obtain

$$y - x = \pm\frac{\epsilon\ell + O(\epsilon^3)}{1 \pm \epsilon(1-a)\ell'} = \pm\epsilon\ell(y) - (1-a)\epsilon^2\ell(y)\ell'(y) + O(\epsilon^3).$$

Finally, we obtain

$$x = B^\epsilon_\pm(y) = y \mp \epsilon\ell(y) + (1-a)\epsilon^2\ell(y)\ell'(y) + O(\epsilon^3).$$

Therefore, $B^\epsilon_\pm(x)$ satisfies the expansion (4.9). $\square$

Using the Taylor expansion of $B^\epsilon_\pm(x)$ in (4.9), we may do the same calculation as the previous case $a = 1$ to obtain the second order term in expansion with respect to ϵ at zero. In below, we introduce another approach. Fix x_0 in the domain and define $U(x)$ and $B(x,\epsilon)$ as

$$U(x) = \int_{x_0}^x u(y)dy, \qquad B(x, \pm\epsilon) = B^\epsilon_\pm(x)$$

for the $\epsilon > 0$. Then, $U(x)$ means population (not density) within the region $[x_0, x]$ assuming $x_0 < x$. Extend $B(x,s)$ smoothly for $s \in [-\epsilon, \epsilon]$ such that $B(x,0) = x$.

Expand $U(B(x,\epsilon))$ for small $\epsilon > 0$ with a fixed x and obtain

$$\begin{aligned}U(B(x,\epsilon)) &= U(x) + \epsilon u\left(B(x,\epsilon)\right)\partial_\epsilon B|_{\epsilon=0} \\ &\quad + \frac{\epsilon^2}{2}\left(u_x\left(B(x,\epsilon\right)\cdot(\partial_\epsilon B)^2 + u\left(B(x,\epsilon)\right)\partial_{\epsilon\epsilon}B\right)|_{\epsilon=0} + O(\epsilon^3) \\ &= U(x) - \epsilon u(x)\ell(x) + \frac{\epsilon^2}{2}\left(u_x(x)\ell(x)^2 + 2(1-a)u(x)\ell(x)\ell'(x)\right) + O(\epsilon^3).\end{aligned}$$

Then, by differentiating it with respect to x, we obtain

$$\begin{aligned}u\left(B(x,\epsilon)\right)B_x(x,\epsilon) &= u(x) + \epsilon\partial_x\left(u\left(B(x,\epsilon)\right)\partial_\epsilon B\right)|_{\epsilon=0} \\ &\quad + \frac{\epsilon^2}{2}\partial_x\left(u_x\left(B(x,\epsilon\right)\cdot(\partial_\epsilon B)^2 + u\left(B(x,\epsilon)\right)\partial_{\epsilon\epsilon}B\right)|_{\epsilon=0} + O(\epsilon^3) \\ &= u(x) - \epsilon(u(x)\ell(x))_x + \frac{1}{2}\epsilon^2\left(u_x(x)\ell(x)^2 + 2(1-a)u(x)\ell(x)\ell'(x)\right)_x + O(\epsilon^3).\end{aligned}$$

Do the same computation for $u\left(B(x,-\epsilon)\right)B_x(x,-\epsilon)$ and add the two, which give

$$\begin{aligned}u\left(B(x,\epsilon)\right)&B_x(x,\epsilon) + u\left(B(x,-\epsilon)\right)B_x(x,-\epsilon) - 2u(x) \\ &= \epsilon^2\left(u_x(x)\ell(x)^2 + 2(1-a)u(x)\ell(x)\ell'(x)\right)_x + O(\epsilon^3) \\ &= \epsilon^2\left(\ell^2 u_x + \ell^{2a}(\ell^{2-2a})_x u\right)_x + O(\epsilon^3) \\ &= \epsilon^2\left(\ell^{2a}(\ell^{2-2a}u)_x\right)_x + O(\epsilon^3).\end{aligned}$$

Hence, the solution of (4.6) satisfies

$$\partial_t u = \frac{1}{2}\left(\ell^{2a}(\ell^{2-2a}u)_x\right)_x + O(\epsilon)$$

for small $\epsilon > 0$. After taking $\epsilon \to 0$ limit, we obtain the desired diffusion equation (4.7).

4.2. **Other random walks.** A random walk system is called revertible if a particle returns to the same position after two walks when the second walk is in the opposite direction of the first one. The random walk system with (4.2) is revertible only when $a = 1/2$. In the case, the diffusion equation (4.7) becomes Wereide's law

$$u_t = (\sqrt{D}(\sqrt{D}u)_x)_x, \quad D = \ell^2/2.$$

Suppose that f is an increasing function and f^{-1} is its inverse. Then, we may construct a revertible random walk by taking

$$K(x,y) = \frac{1}{2}\delta(y - f(x)) + \frac{1}{2}\delta(y - f^{-1}(x)).$$

In this model, $f(x)$ is the position after a walk to right and $f^{-1}(x)$ is the position after a walk to left when x is the departure point. Then, it is clearly a revertible system since $f^{-1}(f(x)) = x$. If the walk-length is of order $\epsilon > 0$, i.e., $|f(x)-x| = O(\epsilon)$, we may write $f(x) = x + \epsilon\ell(x)$ for some $\ell(x)$. If f is differentiable, the parabolic scale perturbed problem becomes

$$\begin{aligned}\epsilon^2\partial_t u(x) &= \int u(y)K(y,x) - u(x)K(x,y)dy \\ &= \frac{1}{2}\left(u(f^{-1}(x))(f^{-1})'(x) + u(f(x))f'(x)\right).\end{aligned}$$

Then,

$$\begin{aligned}\epsilon^2\partial_t u(x) &= \frac{1}{2}u(x+\epsilon\ell)(1+\epsilon\ell') \\ &\quad + \frac{1}{2}u(x-\epsilon\ell+\epsilon^2\ell\ell'+O(\epsilon^3))(1-\epsilon\ell'+\epsilon^2(\ell\ell')'+O(\epsilon^3)) - u(x) \\ &= \frac{1}{2}\Big\{u+\epsilon\ell u' + u\epsilon\ell' + \frac{1}{2}\epsilon^2\ell^2u'' + \frac{1}{2}\epsilon^2\ell\ell'u' \\ &\quad + \Big(u - \epsilon\ell u' + \epsilon^2\ell\ell'u' + \frac{1}{2}\epsilon^2\ell^2u''\Big)\left(1-\epsilon\ell'+\epsilon^2(\ell\ell')'\right)\Big\} - u + O(\epsilon^3) \\ &= \frac{1}{2}\left(\ell^2u'' + 3\ell\ell'u' + (\ell\ell')'u\right) \\ &= \frac{1}{2}\left((\ell^2u')' + (\ell\ell'u)'\right) = \frac{1}{2}\left(\ell(\ell u)'\right)',\end{aligned}$$

which is the same diffusion equation as (4.7) with $a = 1/2$, i.e., Wereide's diffsion law,

$$\partial_t u = \frac{1}{2}\left(\ell(\ell u)_x\right)_x \tag{4.10}$$

So far, we have derived diffusion equations after constructing the kernel explicitly. Next, we see which components of the random walk decide the diffusion equation when $B^\epsilon_\pm(x)$ are given with the small parameter $\epsilon > 0$. Consider a singular expansion,

$$B^\epsilon_+(\epsilon,x) = x + \epsilon f_1 + \frac{1}{2}\epsilon^2 f_2 + O(\epsilon^3).$$

Similarly, we may consider a singular expansion for $B^\epsilon_-(x)$, i.e.,

$$B^\epsilon_-(\epsilon,x) = x - \epsilon f_1 + \frac{1}{2}\epsilon^2 f_3 + O(\epsilon^3).$$

Note that the first order term is not free, but should be $-\epsilon f_1$ to have a diffusion limit. However, the second order term can be chosen arbitrarily. Then, after substituting the expansion to (3.5), we obtain

$$\begin{aligned}\epsilon^2\partial_t u &= \frac{1}{2}u(x+\epsilon f_1 + \frac{1}{2}\epsilon^2 f_2)(1+\epsilon f_1' + \frac{1}{2}\epsilon^2 f_2') \\ &\quad + \frac{1}{2}u(x-\epsilon f_1 + \frac{1}{2}\epsilon^2 f_3)(1-\epsilon f_1' + \frac{1}{2}\epsilon^2 f_3') - u + O(\epsilon^3) \\ &= \frac{1}{2}\epsilon^2\left(\Big(\frac{f_2+f_3}{2}u\Big)' + (f_1^2u')'\right) + O(\epsilon^3).\end{aligned}$$

Therefore, after taking $\epsilon \to 0$ limit, we obtain a diffusion equation

$$\partial_t u = \frac{1}{2}\left(\frac{f_2 + f_3}{2}u + f_1^2 u_x\right)_x . \tag{4.11}$$

To compare with the previous revertible case, choose $f(x) = B_+^\epsilon(x) = x + \epsilon f_1(x) + \frac{1}{2}\epsilon^2 f_2(x) + O(\epsilon^3)$ and assume $f(x)$ is monotone. The expansion of its inverse $f^{-1}(x)$ can be computed as follows. Let $y = f(x)$. Then,

$$\begin{aligned} y - x &= \epsilon f_1(x) + \frac{1}{2}\epsilon^2 f_2(x) + O(\epsilon^3) \\ &= \epsilon\Big(f_1(y) + f_1'(y)(x-y) + \frac{1}{2}\epsilon f(y) + O(\epsilon^2)\Big), \end{aligned}$$

and

$$\begin{aligned} y - x &= \frac{\epsilon f_1 + \frac{1}{2}\epsilon^2 f_2 + O(\epsilon^3)}{1 + \epsilon f_1'} \\ &= \epsilon f_1 - \epsilon^2 f_1 f_1' - \frac{1}{2}\epsilon^2 f_2 + O(\epsilon^3). \end{aligned}$$

We have omitted the argument y for convenience and finally obtain

$$f^{-1}(x) = x - \epsilon f_1(x) + \frac{1}{2}\epsilon^2\left(2f_1(x)f_1'(x) - f_2(x)\right) + O(\epsilon^3).$$

After substituting $f_3 = 2f_1(x)f_1'(x) - f_2(x)$ in the equation (4.11), we obtain the same equation as (4.10), i.e.,

$$\partial_t u = \frac{1}{2}\left(f_1(f_1 u)_x\right)_x . \tag{4.12}$$

Note that the second order term f_2 does not appear in (4.12).

5. Convergence with constant departing rate

In this section we show the uniform convergence of solutions of the non-local equation (4.6) to the solution of a diffusion equation (4.7). So far, we have considered the problem in the whole real line $\mathbb{R}$. However, for the proof of uniform convergence, we take a compact domain in time and space. In particular, we take a closed interval $X \subset \mathbb{R}$ with the periodic boundary condition, which is basically a circle without boundary. Hence, we may forget the boundary condition. The main theorem is as follows.

Theorem 5.1 (Uniform convergence with heterogeneous walk-length)**.** *Let X be a closed interval with the periodic boundary condition. Let the initial condition $u_0(x)$ and the walk-length $\ell(x)$ be smooth, bounded, and positive defined on X. Let u^ϵ be the solution of* (4.6) *when $K^\epsilon(x,y)$ is given by* (4.4). *Then for a fixed $T > 0$, u^ϵ converges to the solution u of* (4.7) *uniformly, i.e.,*

$$\|u^\epsilon - u\|_{L^\infty(X\times[0,T])} \to 0, \quad \text{as} \quad \epsilon \to 0.$$

Here we discuss the existence and the uniqueness of solutions of (4.6) and obtain the comparison principle. The time scale is ϵ^2 for $\epsilon > 0$ is small enough and consider following form with source function $f(t,x) \in C([0,T];L^1(X))$.

$$
\begin{cases}
\partial_t u^\epsilon(t,x) = \frac{1}{2}u^\epsilon(B_+^\epsilon(x),t)\partial_x B_+^\epsilon(x) + \frac{1}{2}u^\epsilon(\partial_x B_-^\epsilon(x),t)\partial_x B_-^\epsilon(x), \\
\qquad\qquad -u^\epsilon(t,x) + f(t,x) \\
u^\epsilon(0,x) = u_0(x).
\end{cases}
\tag{5.1}
$$

The previous positions $B_\pm^\epsilon(x)$ are smooth diffeomorphisms under the assumptions of smoothness, positivity, and uniform boundedness of walk-length, and smallness of $\epsilon > 0$. Note that the time scaling of order ϵ^2 is irrelevant with the uniqueness and the existence of a solution and hence the nonlocal problem without scaling has well definedness.

We have following property from the substitution.

Lemma 5.2 (Population Conservation). *Let $D \subset X$ be an arbitrary subdomain of X. Then, we have*

$$
\int_D u(B_+^\epsilon(x),\cdot)\partial_x B_+^\epsilon(x)dx = \int_{B_+^\epsilon(D)} u(x,\cdot)dx,
$$

$$
\int_D u(B_-^\epsilon(x),\cdot)\partial_x B_-^\epsilon(x)dx = \int_{B_-^\epsilon(D)} u(x,\cdot)dx.
$$

Now we show the uniqueness of a solution of the nonlocal equation. For a fixed $T > 0$, let

$$
X_T = C([0,T];L^1(X))
$$

be the Banach space with the uniform norm

$$
|||w||| := \max_{t\in[0,T]} \|w(\cdot,t)\|_{L^1(X)}.
$$

For given functions u_0 and f, define an operator $T_{u_0,f}$ on the space X_T as

$$
T_{u_0,f}(w)(t,x) = \int_0^t \Big(\frac{1}{2}w(B_+^\epsilon(x),t)\partial_x B_+^\epsilon(x) + \frac{1}{2}w(\partial_x B_-^\epsilon(x),t)\partial_x B_-^\epsilon(x) - w(t,x)\Big)ds
$$
$$
+ \int_0^t f(x,s)ds + u_0(x).
$$

We show the uniqueness of the solution using a fixed point theory of the operator.

Theorem 5.3 (Banach fixed point). *Let $u_0 \in C([0,T];L^1(X))$. There exists a unique solution of (5.1) if $f \in C([0,T];L^1(X))$.*

Proof. It is enough to find a fixed point in the Banach space X_T for the operator. It is easy to see that the operator $T_{u_0,f}(w)$ on the space is linear in triple $(u_{0,}f,w)$. It is well-defined since for each t,

$$
\begin{aligned}
&\|T_{u_0,f}(w)(\cdot,t)\|_{L^1(X)}\\
&\quad\le \Big\| \int_0^t \Big(\frac{1}{2}w(B_+^\epsilon(\cdot),t)\partial_x B_+^\epsilon(\cdot) + \frac{1}{2}w(\partial_x B_-^\epsilon(\cdot),t)\partial_x B_-^\epsilon(\cdot) - w(\cdot,t)\Big)\, ds\Big\|_{L^1(X)}\\
&\qquad\qquad + \int_0^t \|f(\cdot,s)\|_{L^1(X)} ds + \|u_0\|_{L^1(X)}\\
&\quad\le \|u_0\|_{L^1(X)} + 2t|||w||| + t|||f|||
\end{aligned}
$$

so L^1 bounded and continuous along time as

$$
\begin{aligned}
&\|T_{u_0,f}(w)(x,t_2) - T_{u_0,f}(w)(x,t_1)\|_{L^1(X)}\\
&\quad= \int_X \int_{t_1}^{t_2} \left|\frac{1}{2}w(B_+^\epsilon(x),t)\partial_x B_+^\epsilon(x) + \frac{1}{2}w(\partial_x B_-^\epsilon(x),t)\partial_x B_-^\epsilon(x) - w(t,x)\right| ds\\
&\qquad\qquad + \int_X \int_{t_1}^{t_2} |f(x,s)| ds dx\\
&\quad\le 2|t_2 - t_1| \cdot |||w||| + |t_2 - t_1| \cdot |||f|||
\end{aligned}
$$

using conservation property for $0 \le t_1 \le t_2 \le T$. Finally the operator is Lipschitz as

$$
\begin{aligned}
|||T_{u_0,f}(w) - T_{v_0,g}(z)||| &= |||T_{u_0-v_0,f-g}(w-z)|||\\
&\le \|u_0 - v_0\| + 2T|||w-z||| + T|||f-g|||
\end{aligned}
$$

and by taking T sufficiently small, by Banach fixed point theorem, a unique solution exists on interval $[0,T]$ and iterate to extend to define on $[0,\infty)$.

Note that by the definition of the operator, the unique solution also belongs to $C^1\big([0,T];L^1(X)\big)$. $\square$

Definition 5.4 (Super- and sub-solutions). *Let $T > 0$. We call $v(t,x) \in C^1\big([0,T];L^1(X)\big)$ a super-solution of (5.1) if*

$$
\partial_t v(x,\cdot) \ge \frac{1}{2}v(B_+^\epsilon(x),\cdot)\partial_x B_+^\epsilon(x) + \frac{1}{2}v(\partial_x B_-^\epsilon(x),\cdot)\partial_x B_-^\epsilon(x) - v(x,\cdot) + f(x,\cdot).
$$

The function $v(t,x)$ is called a sub-solution if the inequalities are reversed.

Theorem 5.5 (Comparison property). *If v is a super-solution of* (5.1), *$f \ge 0$, and $v(\cdot,0) \ge 0$, then $v \ge 0$.*

Proof. Put $v^- = -\min(v,0) = \max(-v,0)$. Then, $\partial_t v^- = 0$ for $v > 0$. If $v \le 0$,

$$
\begin{aligned}
\partial_t v^- &= -\frac{1}{2}v(B_+^\epsilon(x),\cdot)\partial_x B_+^\epsilon(x) - \frac{1}{2}v(\partial_x B_-^\epsilon(x),\cdot)\partial_x B_-^\epsilon(x) + v(x,\cdot) - f(x,\cdot),\\
&\le -\Big(\frac{1}{2}v(B_+^\epsilon(x),\cdot)\partial_x B_+^\epsilon(x) + \frac{1}{2}v(\partial_x B_-^\epsilon(x),\cdot)\partial_x B_-^\epsilon(x)\Big)\\
&\le \frac{1}{2}v^-(B_+^\epsilon(x),\cdot)\partial_x B_+^\epsilon(x) + \frac{1}{2}v^-(\partial_x B_-^\epsilon(x),\cdot)\partial_x B_-^\epsilon(x).
\end{aligned}
$$

Therefore,

$$\int \partial_t v^- \leq \int v^-,$$

and, by Gronwall's lemma,

$$\int v^- \leq 0$$

for all time $t \geq 0$. By definition, $v^- = 0$ and hence v is nonnegative. □

Corollary 5.6. *Suppose u and v are respectively super- and sub-solutions of* (5.1) *with initial condition u_0 and v_0 and source terms f and g. If $u_0 \geq v_0$ and $f \geq g$, then $u \geq v$.*

Proof. $u - v$ is supersolution of the nonlocal equation with initial condition $u_0 - v_0$ and source term $f - g$ by linearity and is nonnegative followed by previous comparison property. □

We can prove the following lemma using the formal calculation of the previous section.

Lemma 5.7 (Formal convergence)**.** *Let $a \in [0, 1]$, $B^\epsilon_\pm$ be given by* (4.3)*, and*

$$L_\epsilon u := \frac{1}{\epsilon^2}\Big(\frac{1}{2}u\left(B^\epsilon_+(x), t\right)\partial_x B^\epsilon_+(x) + \frac{1}{2}u\left(B^\epsilon_-(x), t\right)\partial_x B^\epsilon_-(x) - u(x)\Big).$$

Then, for $Lu := \frac{1}{2}\left(\ell^{2a}\left(\ell^{2-2a}u\right)_x\right)_x$,

$$L_\epsilon u = Lu + O(\epsilon) \quad \textit{as} \quad \epsilon \to 0.$$

Proof. As shown in the formal calculation, for each choice of reference $a \in [0, 1]$,

$$L_\epsilon u = \frac{1}{\epsilon^2}\Big(\frac{1}{2}u(B^\epsilon_+(x), t)\partial_x B^\epsilon_+(x) + \frac{1}{2}u(B^\epsilon_-(x), t)\partial_x B^\epsilon_-(x) - u(t, x)\Big)$$

is smooth well-defined function of ϵ when it is sufficiently small and takes Taylor expansion as

$$L_\epsilon u = Lu + O(\epsilon)$$

with the difference given as

$$|L_\epsilon u(x) - Lu(x)| = |Lu(\tilde{x}) - Lu(x)| = |(Lu)_x(\tilde{\tilde{x}})| \cdot |x - \tilde{x}|$$

for some $\tilde{x}, \tilde{\tilde{x}} \in (B^\epsilon_+(x), B^\epsilon_-(x))$ and the distance is in the order of $\epsilon \cdot \|d\|$. Once we assume smoothness and boundedness of u, ℓ and their derivatives, the difference is bounded by $C\epsilon$ with C is constant determined by u, ℓ and choice of reference. □

The diffusion equation corresponding to the nonlocal equation is

$$\partial_t u(t, x) = Lu(t, x)$$
$$u(0, x) = u_0(x)$$

Lemma 5.8. *For each choice of reference, i.e.* $a \in [0,1]$*,*

$$L_\epsilon(1)(x) = \frac{1}{2\epsilon^2}((B_+^\epsilon)'(x) + (B_-^\epsilon)'(x) - 2)$$

is bounded and thus bounded uniformly for $x \in X$ *and* $\epsilon \in [0, \epsilon_0]$ *for sufficiently small* ϵ_0*.*

Proof. We defined $N_\pm^\epsilon$ for small enough ϵ. By previous lemma, for sufficiently small ϵ,

$$\begin{aligned} |L_\epsilon(1)| &\leq |L(1)| + |(L(1))_x| \cdot \epsilon \cdot \|d\| \\ &= |\frac{1}{2}\left(\ell^{2a}(\ell^{2-2a})_x\right)_x| + |\frac{1}{2}\left(\ell^{2a}(\ell^{2-2a})_x\right)_{xx}|\epsilon \cdot \|d\| \end{aligned}$$

which is bounded for fixed ϵ by boundedness of ℓ and its derivatives. Then by continuity along ϵ, uniformly bounded for ϵ sufficiently small.

Note that for $a = 1$, that is for arrival point reference,

$$L_\epsilon(1) = \frac{1}{2}\left(1 + \epsilon\ell' + 1 - \epsilon\ell' - 2\right) = 0.$$

□

Now we prove the convergence theorem.

Proof of Theorem 5.1. Set $w^\epsilon = u^\epsilon - u$ then

$$\partial_t w^\epsilon - L_\epsilon w^\epsilon = L_\epsilon(u^\epsilon - w^\epsilon) - Lu = L_\epsilon u - Lu.$$

Let $F_\epsilon(u; x, t) = L_\epsilon u - Lu$. From Lemma 5.7,

$$|F_\epsilon(u; x, t)| \leq \theta(\epsilon) = O(\epsilon),$$

i.e., $\theta(\epsilon)$ is function of ϵ only and is independent of the time variable. This is possible by choosing $\theta(\epsilon) = \|d\|\epsilon \cdot \max_{t\in[0,T]} \|(Lu)_x\|_\infty$.

In order to apply the comparison property, we find a super-solution $\bar{w}(t, x)$ of (5.1) with source term $\theta(\epsilon)$ for each fixed ϵ. Put $\eta(\epsilon)$ as

$$\eta(\epsilon) = \max_{x\in X} |L_\epsilon(1)| = \max_{x\in X} \frac{1}{2\epsilon^2}|(B_+^\epsilon)'(x) + (B_-^\epsilon)'(x) - 2|,$$

which we know bounded from lemma. Choose

$$\bar{w}^\epsilon(t, x) = \begin{cases} \frac{\theta(\epsilon)}{\eta(\epsilon)}\left(e^{\eta(\epsilon)t} - 1\right), & \eta(\epsilon) \neq 0, \\ \theta(\epsilon)t, & \eta(\epsilon) = 0, \end{cases}$$

where $\eta(\epsilon) = 0$ case is extended by continuity. This $\bar{w}^\epsilon$ is a super-solution of

$$\begin{aligned} \partial_t v &= L_\epsilon v + F_\epsilon(u; x, t) \\ v(\cdot, 0) &= 0 \end{aligned}$$

since $\bar{w}^\epsilon(\cdot, 0) = 0$ and

$$\partial_t \bar{w}^\epsilon = \eta(\epsilon)\bar{w}^\epsilon + \theta(\epsilon) \geq L_\epsilon \bar{w}^\epsilon + \theta(\epsilon).$$

By the comparison property,

$$-\bar{w}^\epsilon \leq w^\epsilon \leq \bar{w}^\epsilon$$

Since $\bar{w}^\epsilon(t,x)$ is constant in space at each time t and is increasing, its value is uniformly bounded by $\bar{w}^\epsilon(\cdot, t=T)$ and by taking $\epsilon \to 0$, we have $\bar{w}^\epsilon \to 0$ uniformly on $X \times [0,T]$ so finally we have the uniform convergence

$$\|u^\epsilon - u\|_{L^\infty(X\times[0,T])} \to 0$$

as $\epsilon \to 0$. $\square$

6. Convergence with nonconstant departing rate

In this section, we consider the final case when the departing rate

$$\gamma(x) = \int K(x,y)dy \tag{6.1}$$

is not constant. In the previous sections, we have considered a case that the departing rate is constant $\gamma(x) = 1$ and the heterogeneity of the random walk system is in the non-constant walk-length $\ell = \ell(x)$. A classical position jump consists of walk-length Δx and the sojourn time (or jumping time) Δt. The spatial heterogeneity in the walk-length has been handled in terms of the walk-lenth function $\ell(x)$. However, the heterogeneity in Δt cause a different level of difficulty that the resulting random walk system is not Markov process anymore. The fate of a current situation cannot be decided by the previous step since Δt is not fixed.

In the accompanying paper [30, Sections 5 and 6], the relations among sojourn time, sojourn rate, and departing rate are discussed. In the discussion, the sojourn time is reversely proportional to the departing rate and proportional to the sojourn rate. The main difference is that the heterogeneity in the sojourn time is independent of the choice of reference point, but the one in the departing rate is not. However, if the reference point of departing rate is taken as the departure point, we may obtain the equivalent effect. Hence, the meaning of (6.1) is twofold in that the departing rate $\gamma(x)$ is given by the relation and chosen from the departure point x. In particular, in the formation of the kernel K (3.4), γ is taken from the departure point x.

To consider the case with a nonconstant departing rate in (6.1), the kernel in (4.4) is replaced by

$$K^\epsilon(x,y) = \frac{\gamma(x)}{2}\delta(y - N^\epsilon_-(x)) + \frac{\gamma(x)}{2}\delta(y - N^\epsilon_+(x)). \tag{6.2}$$

In the following theorem, we show that the solution u^ϵ of the nonlocal problem (4.6) uniformly converges to the solution of a local problem,

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\left(\ell^{2a}\left(\ell^{2-2a}\gamma u\right)_x\right)_x, \\ \;u(0,x) \;= u_0(x). \end{cases} \tag{6.3}$$

Note that the departing rate γ is placed inside of the inside derivative and the steady state of the problem is supposed to be reversely proportional to it.

Theorem 6.1 (Uniform convergence with heterogeneous walk-length and jumping time)**.** *Let X be a closed interval with the periodic boundary condition (i.e., a circle). Let the initial condition $u_0(x)$, the walk-length $\ell(x)$, and the departing rate $\gamma(x)$ be smooth, bounded, and positive defined on X. Let u^ϵ be the solution of* (4.6) *when $K^\epsilon(x,y)$ is given by* (6.2)*. Then for a fixed $T>0$, u^ϵ converges to the solution u of* (6.3) *uniformly, i.e.,*

$$\|u^\epsilon - u\|_{L^\infty(X\times[0,T])} \to 0, \quad as \quad \epsilon \to 0.$$

The convergence has been proved for the case when $\gamma(x)=1$ in the previous section. For a case with non-constant $\gamma(x)$, the convergence is directly obtained from the constant departing rate case. Let

$$\tilde{K}^\epsilon(x,y) := \frac{K^\epsilon(x,y)}{\gamma(x)} = \frac{1}{2}\delta(y-N_-^\epsilon(x)) + \frac{1}{2}\delta(y-N_+^\epsilon(x)).$$

Then, the integral equation with the kernel $\tilde{K}^\epsilon$ is

$$\partial_t u(t,x) = P(u) = \int u(t,y)\tilde{K}^\epsilon(y,x)dy - u(t,x),$$

where $P(\cdot)$ is generator of the nonlocal functional equation. The corresponding differential diffusion equation is

$$u_t = \frac{1}{2}(\ell^{2a}(\ell^{2-2a}u)_x)_x.$$

The integral equation with the kernel K^ϵ is written as

$$\begin{aligned}\partial_t u(t,x) &= \int u(t,y)K^\epsilon(y,x) - u(t,x)K^\epsilon(x,y)dy\\ &= \int \gamma(y)u(y)\tilde{K}^\epsilon(y,x) - \gamma(x)u(x)\tilde{K}^\epsilon(x,y)dy = P(\gamma u),\end{aligned}$$

which is the same linear equation except that u is replaced by γu. Hence, by substitution, we obtain its diffusion limit for free which is

$$u_t = \frac{1}{2}(\ell^{2a}(\ell^{2-2a}\gamma u)_x)_x. \tag{6.4}$$

This final diffusion equation fits in diverse situations. If the heterogeneity in the walk-length is forgotten, we obtain

$$u_t = \Delta(\gamma u) = \nabla\cdot(\gamma\nabla u + u\nabla\gamma),$$

where an advection term appears. This diffusion has been taken in many places. In particular, the advection term $u\nabla\gamma$ provides chemotactic phenomenon and the model is widely used in chemotaxis modeling [33][14][35][17]. The case with $a=1/2$ gives a revertible random walk and the departing rate is reversely proportional to the sojourn time, i.e., $\gamma \cong \frac{1}{\Delta t}$ after rescaling. Then, (6.4) is written as

$$u_t = \Big(\frac{\Delta x}{2}\Big(\frac{\Delta x}{\Delta t}u\Big)_x\Big)_x.$$

The turning frequency μ is reversely proportional to $\Delta t/2$. Hence, if we denote the velocity by $v = \frac{\Delta x}{\Delta t}$, the above is written as

$$u_t = \Big(\frac{v}{\mu}(vu)_x\Big)_x,$$

which is the same equation obtained from a discrete kinetic equation [23].

Acknowledgments. This research was supported in part by National Research Foundation of Korea (NRF-2022R1H1A2092302).

References

[1] M. Alfaro, P. Gabriel, O. Kavian, Confining integro-differential equations originating from evolutionary biology: Ground states and long time dynamics. Discrete and Continuous Dynamical Systems - B. doi: 10.3934/dcdsb.2022120
[2] M. Alfaro, T. Giletti, Y.-J. Kim, G.Peltier, and H. Seo, On the modelling of spatially heterogeneous nonlocal diffusion: deciding factors and preferential position of individuals. Journal of Mathematical Biology 84 (2022), 1–35.
[3] P. Bates and A. Chmaj. An integrodifferential model for phase transitions: stationary solutions in higher dimensions. J. Statistical Phys. 95 (1999) 1119–1139.
[4] P. Bates, P- Fife, X. Ren and X. Wang, Travelling waves in a convolution model for phase transitions. Arch. Rat. Mech. Anal. 138 (1997) 105–136.
[5] P. Bates and J. Han. The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation. J. Math. Anal. Appl. 311 (2005) 289–312.
[6] P. Bates and J. Han. The Neumann boundary problem for a nonlocal Cahn-Hilliard equation. J. Differential Equations 212 (2005) 235–277.
[7] G. Beltritti – J. D. Rossi. Nonlinear evolution equations that are nonlocal in space and time. Journal of Mathematical Analysis and Applications 455 (2017) 1470–1504.
[8] R. Bürger (1986) On the maintenance of genetic variation: global analysis of Kimura's continuum-of-alleles model. J Math Biol 24:341–351
[9] R. Bürger (1988) Perturbations of positive semigroups and applications to population genetics. Math Z 197:259–272
[10] C. Carrillo and P. Fife. Spatial effects in discrete generation population models. J. Math. Biol. 50(2), 161–188, (2005).
[11] E. Chasseigne - M. Chaves - J. D. Rossi. Asymptotic behavior for nonlocal diffusion equations. Journal de Mathematiques Pures et Appliquees. Vol. 86, 271-291, (2AA19).
[12] S. Chapman. On the Brownian displacements and thermal diffusion of grains suspended in a non-uniform fluid, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 119 (1928), 34–54
[13] X. Chen, Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations, Adv. Differential Equations 2, 125–160 (1997).
[14] E.Cho and Y.-J. Kim, Starvation driven diffusion as a survival strategy of biological organisms, Bull. Math. Biol. 75(5) (2013) 845-870
[15] J.S. Clark (1998) Why trees migrate so fast: confronting theory with dispersal biology and the paleorecord. Am Natl 152:204–224
[16] C. Cortazar - M. Elgueta - J. D. Rossi - N. Wolanski. How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems. Archive for Rational Mechanics and Analysis. Vol. 187(1), 137-156, (20022).
[17] L. Desvillettes, Y.-J. Kim, A. Trescases, and C. Yoon, A logarithmic chemotaxis model featuring global existence and aggregation. Nonlinear Anal. Real World Appl. 50 (20024), 562-582.
[18] A.V. Fick On liquid diffusion. Lond Edinb Dublin Philos Mag J Sci 10 (1855):30–39

[19] P. Fife. Some nonclassical trends in parabolic and parabolic-like evolutions. Trends in nonlinear analysis, 153–191, Springer, Berlin, 2AA5.

[20] P. Fife and X. Wang. A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions. Adv. Differential Equations 3(1), 85–110, (1998).

[21] W.H. Fleming (1979) Equilibrium distributions of continuous polygenic traits. SIAM J Appl Math 36:148–168,

[22] M.E. Gil, F. Hamel, G. Martin, L. Roques Dynamics of fitness distributions in the presence of a phenotypic optimum: an integro-differential approach. Nonlinearity 32 (), 3485–3522

[23] H.-Y. Kim, Y.-J. Kim, and H.-J. Lim, Heterogeneous discrete kinetic model and its diffusion limit, Kinetic and Related Models 14(5) (), 749–765

[24] J. M. Mazon - J. D. Rossi - J. Toledo. On Nonlinear Nonlocal Diffusion Problems. International Journal of Biomathematics and Biostatistics. 1 (2AA11) 181-192.

[25] J. Medlock and M. Kot (2003) Spreading disease: integro-differential equations old and new. Math Biosci 184:201–222

[26] B.P. van Milligen, P.D. Bons, B.A. Carreras, and R. Sanchez, On the applicability of Fick's law to diffusion in inhomogeneous systems. Eur. J. Phys. 26 (2005), 913–925.

[27] A. Molino - J. D. Rossi. Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence. Zeitschrift fur Angewandte Mathematik und Physik. 67 (2AA28) 1–14.

[28] A. Molino – J. D. Rossi. Nonlocal approximations to Fokker-Planck equations. Funkcialaj Ekvacioj, 62(20024) 35–60.

[29] J. Murray (2003) Mathematical biology II: spatial models and biomedical applications. Springer, New York

[30] Y.-J. Kim and H.-J. Lim, Heterogeneous discrete-time random walk and reference point dependency, preprint (2022)

[31] F. Sattin, Fick's law and Fokker–Planck equation in inhomogeneous environments, Physics Letters A 372 (2008), 3941–3945

[32] X. Wang. Metastability and stability of patterns in a convolution model for phase transitions. J. Differential Equations, 183, 434–461, (2002).

[33] M. Winkler, Can simultaneous density-determined enhancement of diffusion and cross-diffusion foster boundedness in Keller–Segel type systems involving signal-dependent motilities?, Nonlinearity 33 (2020), 6590

[34] M.T. Wereide La diffusion d'une solution dont la concentration et la température sont variables. In Annales de Physique, 9 (1914) 67–83.

[35] C. Yoon and Y.-J. Kim, Global existence with pattern formation in cell aggregation model, Acta. Appl. Math. 149 (2017) 101-123

[36] L. Zhang. Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks. J. Differential Equations 197(1), 162–196, (2004).

(Yong-Jung Kim) Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, Republic of Korea
Email address: yongkim@kaist.edu

(Hyun-Jin Lim) Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, Republic of Korea
Email address: skylerhj.lim@gmail.com