FICK'S LAW SELECTS THE NEUMANN BOUNDARY CONDITION

DANIELLE HILHORST, SEUNG-MIN KANG, HO-YOUN KIM, AND YONG-JUNG KIM

ABSTRACT. We show that the Neumann boundary condition appears along the boundary of an inner domain when the diffusivity of the outer domain goes to zero. We take Fick's diffusion law with a bistable reaction function, and the diffusivity is 1 in the inner domain and $\epsilon > 0$ in the outer domain. The convergence of the solution as $\epsilon \to 0$ is shown, where the limit satisfies the Neumann boundary condition along the boundary of an inner domain. This observation says that the Neumann boundary condition is a natural choice of boundary conditions when Fick's diffusion law is taken.

keywords: Heterogeneous diffusion equation, Reaction-diffusion equation, Fick's law diffusion, Singular limit, Neumann boundary condition

1. INTRODUCTION

1.1. **Problem setup and results.**

We consider an initial value problem for a reaction-diffusion equation,

$$(P_0^\epsilon) \qquad \begin{cases} u_t = \nabla \cdot (D_\epsilon \nabla u) + f(u), & (t,x) \in (0,T) \times \Omega, \\ \mathbf{n} \cdot \nabla u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector on the boundary, and $u_0 \in C(\overline{\Omega})$ is an initial value bounded by

$$0 \le u_0(x) \le M, \qquad x \in \Omega, \quad M \ge 1. \tag{1.1}$$

The nonlinear reaction function f is continuously differentiable and bistable. More specifically, we take the following hypotheses; for $\alpha \in (0,1)$,

$$\begin{cases} f(0) = f(1) = f(\alpha) = 0, \quad f'(0) < 0,\, f'(1) < 0,\, f'(\alpha) > 0, \\ f(u) < 0 \ \text{ for } u \in (0,\alpha) \cup (1,\infty), \quad f(u) > 0 \text{ for } u \in (\alpha, 1). \end{cases} \tag{1.2}$$

A typical example of such a bistable non-linearity is $f(u) = u(u-\alpha)(1-u)$. Note that $u = 0$ and $u = 1$ are stable steady-states, $u = \alpha$ is unstable, and

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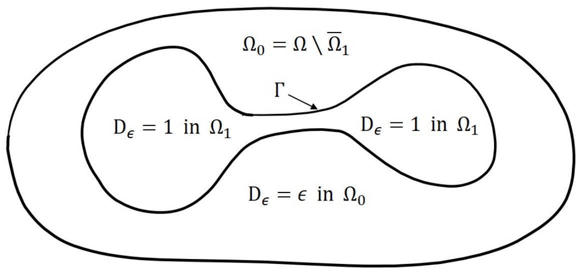

FIGURE 1. Components of domain Ω and diffusivity D_ϵ.

the upper bound M in (1.1) is taken larger than the bigger stable steady-state $u = 1$. The diffusivity D_ϵ is given as below. First, the domain Ω is divided into two parts, as in Figure 1. The inner domain Ω_1 ($\overline{\Omega}_1 \subset \Omega$) is a connected open set with a smooth boundary denoted by $\Gamma := \partial\Omega_1$. The outer domain $\Omega_0 = \Omega \setminus \overline{\Omega}_1$ is also open and connected. The diffusivity D_ϵ is given by

$$D_\epsilon(x) = \begin{cases} 1, & x \in \Omega_1 \cup \Gamma, \\ \epsilon, & x \in \Omega_0. \end{cases} \tag{1.3}$$

Let u be the limit of the solution u^ϵ of Problem (P_0^ϵ) as $\epsilon \to 0$. Then, since the diffusivity in the inner domain Ω_1 is fixed at 1, the limit will obviously satisfy

$$u_t = \Delta u + f(u), (t, x) \in (0, T) \times \Omega_1.$$

The missing part is the boundary condition. The purpose of the paper is to show that the limit u is the unique solution of

$$(P_0^0) \qquad \begin{cases} u_t = \Delta u + f(u), & (t, x) \in (0, T) \times \Omega_1, \\ \mathbf{n} \cdot \nabla u = 0, & (t, x) \in (0, T) \times \partial\Omega_1, \\ u(0, x) = u_0(x), & x \in \Omega_1, \end{cases}$$

where $\mathbf{n}$ is the outward unit normal vector on $\Gamma(= \partial\Omega_1)$. Usually, since a boundary condition is needed to complete a second-order problem, we often choose the Neumann or Dirichlet boundary conditions. In the paper, we show that the Neumann boundary condition appears naturally in the context of the Problem (P_0^ϵ). Note that the solution behavior in one of the two domains is influenced by the one in the other domain due to diffusion. This interaction eventually disappears if $\epsilon \to 0$. However, traces of the interaction remain in the boundary condition at the interface Γ. The resulting boundary condition may depend on the types of diffusion and reaction function. The diffusion law in Problem (P_0^ϵ) is Fick's diffusion law, and the main conclusion of the paper is that Fick's law selects the Neumann boundary condition. We also prove that, in the outer domain Ω_0, u^ϵ converges to the solution

$u \in C^1([0,T]; L^\infty(\Omega_0))$ of the initial value problem

$$(Q_0^0) \qquad \begin{cases} u_t = f(u), & (t,x) \in (0,T) \times \Omega_0 \\ u(0,x) = u_0(x), & x \in \Omega_0, \end{cases}$$

which is an ODE system, and a boundary condition is not needed.

The main results of the paper are in the following three theorems:

Theorem 1.1 (Existence)**.** *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary, $u_0 \in C(\overline{\Omega})$ an initial value bounded by* (1.1)*, f a continuously differentiable bistable non-linearity satisfying* (1.2)*, and D_ϵ the diffusivity given by* (1.3)*. Then, Problem (P_0^ϵ) possesses a unique weak solution u^ϵ and $\|\nabla u^\epsilon\|_{L^2((0,T)\times\Omega_1)}$ is uniformly bounded with respect to $\epsilon > 0$.*

Theorem 1.2 (Neumann condition is selected)**.** *Under the same assumptions as in Theorem 1.1, the solution u^ϵ of Problem (P_0^ϵ) converges to the unique solution of Problem (P_0^0) strongly in $L^2((0,T)\times\Omega_1)$ as $\epsilon \to 0$.*

Moreover, we also prove the following convergence result.

Theorem 1.3 (ODE solution)**.** *Under the same assumption as in Theorem 1.1, the solution u^ϵ of Problem (P_0^ϵ) converges to the unique solution of Problem (Q_0^0) weakly in $L^2((0,T)\times\Omega_0)$ as $\epsilon \to 0$.*

The organization of this paper is as follows. In Section 2, we introduce a sequence of perturbed problems (P_δ^ϵ) where the diffusion coefficient $D_{\delta,\epsilon}$ is a smooth approximation of D_ϵ. We present a priori estimates which are uniform in the parameters δ and ϵ. In Section 3, we let δ tend to zero and deduce the solution existence of Problem (P_0^ϵ). Then, the uniqueness of the problem is proved, which gives Theorem 1.1. Finally, we let ϵ tend to zero in Section 4 and present the proofs of Theorem 1.2 and Theorem 1.3.

1.2. **Heterogeneous diffusion for geometry effects.** The paper is written under the motivation of a project to reinterpret the effect of domain geometry and boundary conditions using heterogeneous diffusion. For example, let u be a solution of a reaction-diffusion equation

$$(1.4) \qquad \begin{cases} u_t = \Delta u + f(u), & (t,x) \in (0,T) \times \Omega_1, \\ \alpha u + (1-\alpha)\mathbf{n} \cdot \nabla u = 0, & (t,x) \in (0,T) \times \partial\Omega_1, \\ u(0,x) = u_0(x), & x \in \Omega_1, \end{cases}$$

where $f(u)$ is a bistable nonlinearity. The parameter $\alpha = 0$ gives the Neumann boundary condition, and $\alpha = 1$ the Dirichlet boundary condition. The solution behavior of the problem depends on the shape of the domain and the boundary condition. Consider two examples. H. Matano showed in his seminal paper that, under the Neumann boundary condition ($\alpha = 0$), nonconstant steady-state solutions of the problem are unstable if Ω_1 is convex [11, Theorem 5.1]. However, it can be stable if Ω_1 is nonconvex [11, Theorem 6.2] (see [13] and references therein for more works on dumbbell-shaped domains). On the other hand, Berestycki *et al.* [2] took the Neumann boundary condition and showed that a bistable traveling wave might propagate or be blocked depending on the exit shape of the domain, which is independent of the domain size. However, if the boundary condition is Dirichlet, the results depend on the domain size, too.

To reinterpret the geometry effect, we may embed the domain Ω_1 into the whole space $\Omega = \mathbf{R}^n$ and assign diffusivity $\epsilon > 0$ to the outer domain $\Omega \setminus \Omega_1$. One might say that the solution of the whole domain problem converges to the solution of the original bounded domain problem as $\epsilon \to 0$ and gives the geometric effect of the original problem. However, Theorem 1.2 implies that if Fick's diffsuion law is taken as in (P_0^ϵ), the solutions converge to a problem with the Neumann boundary condition, not Dirichlet. We propose the problem with a general diffusion law,

$$
(1.5) \qquad \begin{cases} u_t = \nabla \cdot (D_\epsilon^{1-q} \nabla (D_\epsilon^q u)) + f(u), & (t,x) \in (0,T) \times \Omega, \\ \mathbf{n} \cdot \nabla u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}
$$

If $q = 0$, (1.5) returns to Fick's law case in (P_0^ϵ). If $q = 1$, the diffusion law is called Chapman [4], and if $q = 0.5$, Wereide [14]. Diffusion laws with a general exponent q may appear depending on the choice of reference points of the spatial heterogeneity (see [1, 8]). The resulting boundary condition depends on q. For example, the case without reaction ($f = 0$) has been studied in one space dimension [5]. The resulting boundary condition is Neumann if $q < 0.5$ and Dirichlet if $q > 0.5$. There is no specific restriction for the critical case with $q = 0.5$, which is Wereide's diffusion law. However, the critical exponent $q = q^*$ may depend on the reaction function f, the space dimension, and the regularity of the boundary, which requires further study. In this context, this paper deals with the case that $q = 0$, and the boundary condition which we obtain is the Neumann boundary condition.

2. Classical solution for a smooth diffusivity

We start by regularising the diffusivity D_ϵ in (1.3), which is discontinuous along the interface Γ. For small parameters $\epsilon, \delta > 0$, let

$$
D_{\delta,\epsilon}(x) = \begin{cases} 1, & x \in \Omega_1 \cup \Gamma \\ \epsilon, & x \in \Omega_0 \text{ and } \operatorname{dist}(\Gamma, x) > \delta, \end{cases}
$$

and then extend $D_{\delta,\epsilon}$ smoothly to the whole domain Ω so that $\epsilon \le D_{\delta,\epsilon} \le 1$. Then, it converges to the discontinuous diffusion coefficient pointwise, i.e.,

$$
\lim_{\delta \to 0} D_{\delta,\epsilon}(x) = D_\epsilon(x).
$$

We consider a regularized problem

$$(P_\delta^\epsilon) \qquad \begin{cases} u_t = \nabla \cdot (D_{\delta,\epsilon}(x)\nabla u) + f(u), & (t,x) \in (0,T)\times\Omega \\ \mathbf{n}\cdot\nabla u = 0, & (t,x)\in(0,T)\times\partial\Omega \\ u(0,x) = u_0(x), & x\in\Omega. \end{cases}$$

Problem (P_δ^ϵ) formally converges to Problem (P_0^ϵ) as $\delta \to 0$. We prove this below.

Definition 2.1. *A function $u^{\delta,\epsilon} \in C^{1,2}((0,T)\times\Omega)\cap C([0,T]\times\overline{\Omega})$ is called a classical solution of Problem (P_δ^ϵ) if $u^{\delta,\epsilon}$ and its derivatives satisfy the partial differential equation in Problem (P_δ^ϵ) pointwise as well the initial and boundary conditions.*

Lemma 2.2. *There exists a unique classical solution $u^{\delta,\epsilon}$ of Problem (P_δ^ϵ) which is bounded by*

$$0 \le u^{\delta,\epsilon} \le M \quad \text{in } (0,T)\times\Omega. \tag{2.1}$$

Proof. The inequalities in (2.1) follow from the standard maximum principle and the initial condition (1.1). Since $f(u)\le 0$ when $u = 0, M$, $u = 0$ is a lower solution, and $u = M$ is an upper solution of (P_δ^ϵ).

Now, the reaction function $f(u)$ is Lipschitz since it is a continuous function for bounded domain $0 \le u \le M$. We apply [12, Theorem 4.2] to deduce the existence of a unique classical solution, and it completes the proof. □

Next, we obtain a priori estimates for the solution $u^{\delta,\epsilon}$.

Lemma 2.3. *Let $\epsilon \in (0,1)$ and let $u^{\delta,\epsilon}$ be the classical solution of Problem (P_δ^ϵ) and define $M_f := \sup_{s\in[0,M]}|f(s)|$. Then, we have the following inequalities.*

$$\|\nabla u^{\delta,\epsilon}\|_{L^2((0,T)\times\Omega_1)} \le C_1(T,|\Omega|,M,M_f), \tag{2.2}$$

$$\|\nabla u^{\delta,\epsilon}\|_{L^2((0,T)\times\Omega)} \le \frac{1}{\sqrt{\epsilon}}C_1(T,|\Omega|,M,M_f), \tag{2.3}$$

$$\|u_t^{\delta,\epsilon}\|_{L^2(0,T;(H^1)^*(\Omega))} \le C_2(T,|\Omega|,M,M_f). \tag{2.4}$$

Proof. We first prove (2.2) and (2.3). We multiply the reaction-diffusion equation in Problem (P_δ^ϵ) by $u^{\delta,\epsilon}$ and integrate the result on Ω. This gives

$$\int_\Omega u^{\delta,\epsilon}u_t^{\delta,\epsilon}dx = \int_\Omega \nabla\cdot(D_{\delta,\epsilon}\nabla u^{\delta,\epsilon})u^{\delta,\epsilon}dx + \int_\Omega f(u^{\delta,\epsilon})u^{\delta,\epsilon}dx.$$

Integrating by parts gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^{\delta,\epsilon})^2dx = -\int_\Omega D_{\delta,\epsilon}|\nabla u^{\delta,\epsilon}|^2dx + \int_\Omega f(u^{\delta,\epsilon})u^{\delta,\epsilon}dx.$$

Integrating the above equation on $[0,T]$ yields

$$\frac{1}{2}\int_\Omega (u^{\delta,\epsilon}(T,x))^2\ dx + \int_0^T\int_\Omega D_{\delta,\epsilon}|\nabla u^{\delta,\epsilon}|^2 dxdt = \frac{1}{2}\int_\Omega (u_0)^2 dx + \int_0^T\int_\Omega f(u^{\delta,\epsilon})u^{\delta,\epsilon} dxdt.$$

Since the first term on the left-hand-side is positive, we have

$$\int_0^T\int_\Omega D_{\delta,\epsilon}|\nabla u^{\delta,\epsilon}|^2 dxdt \le \frac{1}{2}\int_\Omega (u_0)^2 dx + \int_0^T\int_\Omega f(u^{\delta,\epsilon})u^{\delta,\epsilon} dxdt.$$

The bounds $0 \le u^{\delta,\epsilon} \le M$ and $|f(u^{\delta,\epsilon})| \le M_f$ imply that

$$\int_0^T\int_\Omega D_{\delta,\epsilon}|\nabla u^{\delta,\epsilon}|^2 dxdt \le \frac{1}{2}|\Omega|M^2 + MM_f|\Omega|T. \tag{2.5}$$

Now, we define C_1 from the right side of (2.5), which is $C_1(T,|\Omega|,M,M_f) = \left(\frac{1}{2}|\Omega|M^2 + MM_f|\Omega|T\right)^{\frac{1}{2}}$. Note that $D_{\delta,\epsilon} = 1$ in Ω_1 and $\epsilon \le D_{\delta,\epsilon} \le 1$ in Ω, so that we have

$$\int_0^T\int_{\Omega_1} |\nabla u^{\delta,\epsilon}|^2 dxdt \le C_1^2,$$

and

$$\int_0^T\int_\Omega |\nabla u^{\delta,\epsilon}|^2 dxdt \le \frac{1}{\epsilon}C_1^2.$$

Next, we prove (2.4). Let $\phi \in L^2(0,T;H^1(\Omega))$ be a test function. We multiply the partial differential equation in Problem (P_δ^ϵ) by ϕ and integrate on $(0,T)\times\Omega$ to obtain

$$\int_0^T \langle u_t^{\delta,\epsilon},\phi\rangle dt = \int_0^T\int_\Omega \nabla\cdot(D_{\delta,\epsilon}\nabla u^{\delta,\epsilon})\phi dxdt + \int_0^T\int_\Omega f(u^{\delta,\epsilon})\phi dxdt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H^1(\Omega)$ and $(H^1)^*(\Omega)$. Applying integration by parts, we obtain

$$\int_0^T \langle u_t^{\delta,\epsilon},\phi\rangle dt = -\int_0^T\int_\Omega D_{\delta,\epsilon}\nabla u^{\delta,\epsilon}\cdot\nabla\phi dxdt + \int_0^T\int_\Omega f(u^{\delta,\epsilon})\phi dxdt.$$

The terms on the right-hand side are bounded as follows. From (2.5),

$$\begin{aligned}
&\left|\int_0^T\int_\Omega D_{\delta,\epsilon}\nabla u^{\delta,\epsilon}\cdot\nabla\phi dxdt\right| \\
\le&\Big(\int_0^T\int_\Omega D_{\delta,\epsilon}|\nabla u^{\delta,\epsilon}|^2 dxdt\Big)^{\frac{1}{2}}\Big(\int_0^T\int_\Omega D_{\delta,\epsilon}|\nabla\phi|^2 dxdt\Big)^{\frac{1}{2}} \\
\le&C_1\Big(\int_0^T\int_\Omega |\nabla\phi|^2 dxdt\Big)^{\frac{1}{2}},
\end{aligned}$$

and

$$\Big|\int_0^T \int_\Omega f(u^{\delta,\epsilon})\phi dxdt\Big| \le M_f\sqrt{|\Omega|T}\Big(\int_0^T \int_\Omega |\phi|^2 dxdt\Big)^{\frac{1}{2}}.$$

We deduce that

$$\Big|\int_0^T \langle u_t^{\delta,\epsilon}, \phi\rangle dt\Big| \le \Big(C_1 + M_f\sqrt{|\Omega|T}\Big)\|\phi\|_{L^2(0,T;H^1(\Omega))},$$

which implies that

$$\|u_t^{\delta,\epsilon}\|_{L^2(0,T;(H^1)^*(\Omega))} \le C_1 + M_f\sqrt{|\Omega|T} := C_2(T,|\Omega|,M,M_f).$$

This completes the proof of (2.4). □

3. Existence and uniqueness of a weak solution of (P_0^ϵ)

In this section, we prove that the solution of Problem (P_0^ϵ) exists by taking the singular limit as $\delta \to 0$ in Problem (P_δ^ϵ). The solution of Problem (P_0^ϵ) is defined in a weak sense.

Definition 3.1. *A function $u \in C([0,T];L^2(\Omega)) \cap L^2((0,T);H^1(\Omega)) \cap L^\infty((0,T)\times\Omega)$ is called a weak solution of Problem (P_0^ϵ) if*

$$\int_0^T \int_\Omega (-u\phi_t + D_\epsilon(x)\nabla u\cdot\nabla\phi - f(u)\phi)dxdt$$
$$= \int_\Omega u_0(x)\phi(0,x) - u(T,x)\phi(T,x)dx, \tag{3.1}$$

for all test functions $\phi \in H^1((0,T)\times\Omega)$.

In order to show the existence of a weak solution of Problem (P_0^ϵ), we consider the solution $u^{\delta,\epsilon}$ of Problem (P_δ^ϵ) and apply Fréchet-Kolmogorov theorem which is introduced in [6, Proposition 2.5], [3, Theorem 4.26, p.111, Corollary 4.27, p.113]. In the following proposition, $Q_T := (0,T)\times\Omega$ and $\bar\epsilon$ and $\bar\delta$ are small constants which are not related to ϵ and δ in Problem (P_δ^ϵ).

Proposition 3.2 (Fréchet-Kolmogorov)**.** *A bounded set $B \subset L^2(Q_T)$ is precompact in $L^2(Q_T)$ if*

(1) *For any $\bar\epsilon > 0$ and any subset $Q \Subset Q_T$, there exists a $\bar\delta > 0$ such that $\bar\delta < \operatorname{dist}(Q,\partial Q_T)$ and*

$$\|u(t+\tau,x) - u(t,x)\|_{L^2(Q)} + \|u(t,x+\xi) - u(t,x)\|_{L^2(Q)} < \bar\epsilon$$

for all τ,ξ, and $u \in B$ whenever $|\tau| + |\xi| < \bar\delta$.

(2) *For any $\bar\epsilon > 0$, there exists $Q \Subset Q_T$ such that*

$$\|u\|_{L^2(Q_T\setminus Q)} < \bar\epsilon$$

for all $u \in B$.

Next, we apply the Fréchet-Kolmogorov theorem to the collection of classical solutions $\{u^{\delta,\epsilon}\}$ of (P^ϵ_δ) and show that it is precompact in $L^2(Q_T)$. We recall that the constant $\epsilon > 0$ is fixed through this section. Since the classical solutions $\{u^{\delta,\epsilon}\}$ are uniformly bounded, the second assertion of Proposition 3.2 can be easily derived. Indeed, let $Q = (0, T-\tau) \times \Omega^r$ for some small $\tau, r > 0$, where $\Omega^r = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}$. Then,

$$\|u^{\delta,\epsilon}\|^2_{L^2(Q_T\setminus Q)} \leq \int_{T-\tau}^T \int_\Omega (u^{\delta,\epsilon})^2 dxdt + \int_0^T \int_{\Omega\setminus\Omega^r} (u^{\delta,\epsilon})^2 dxdt.$$

From Lemma 2.3, we have

$$\int_{T-\tau}^T \int_\Omega (u^{\delta,\epsilon})^2 dxdt \leq \tau|\Omega|M^2, \tag{3.2}$$

and

$$\int_0^T \int_{\Omega\setminus\Omega^r} (u^{\delta,\epsilon})^2 dxdt \leq |\Omega\setminus\Omega_r|TM^2. \tag{3.3}$$

Note that the right-hand sides of the inequalities (3.2) and (3.3) tend to zero as $\tau \to 0$ and $r \to 0$. Thus for any $\bar\epsilon > 0$, we may choose $\tau > 0$ and $r > 0$ so small that

$$\|u^{\delta,\epsilon}\|^2_{L^2(Q_T\setminus Q)} \leq \tau|\Omega|M^2 + |\Omega\setminus\Omega^r|TM^2 < \bar\epsilon.$$

This completes the proof of the property (2) in the Fréchet-Kolmogorov theorem.

We now estimate the equicontinuity of time and space variables for $\{u^{\delta,\epsilon}\}$ (see [6, Lemmas 2.6 and 2.7]). In the following lemmas, r is a given constant from compact subset Q which can be taken as any small number with $r < \mathrm{dist}(Q, \partial Q_T)$. Thus, $\bar\delta$ is choosen as $\bar\delta < r$ where it gives $C_r|\xi|^2 < \bar\epsilon$ and $C_r\tau < \bar\epsilon$ for the right side of inequalities in Lemma 3.3 and Lemma 3.4, and it gives the equicontinuity of the Fréchet-Kolmogorov theorem.

Lemma 3.3. *For any small positive constant $r > 0$,*

(1) *There exists a positive constant C_ϵ which is independent of δ such that*

$$\int_0^T \int_{\Omega^r} (u^{\delta,\epsilon}(t, x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \leq C_\epsilon|\xi|^2, \tag{3.4}$$

for all real values $|\xi| \leq r$.

(2) *There exists a positive constant C which is independent of δ and ϵ such that*

$$\int_0^T \int_{\Omega_1^r} (u^{\delta,\epsilon}(t, x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \leq C|\xi|^2, \tag{3.5}$$

for all real values $|\xi| \leq r$.

Proof. We first prove (3.4).

$$\begin{aligned}
&\int_0^T \int_{\Omega^r} (u^{\delta,\epsilon}(t,x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \\
&\qquad = \int_0^T \int_{\Omega^r} \Big(\int_0^1 \nabla u^{\delta,\epsilon}(t,x+\theta\xi)\cdot \xi d\theta \Big)^2 dxdt \\
&\qquad \le |\xi|^2 \int_0^1 \int_0^T \int_{\Omega^r} |\nabla u^{\delta,\epsilon}(t,x+\theta\xi)|^2 dxdtd\theta \\
&\qquad \le |\xi|^2 \int_0^1 \int_0^T \int_{\Omega} |\nabla u^{\delta,\epsilon}(t,x)|^2 dxdtd\theta = |\xi|^2 \|\nabla u^{\delta,\epsilon}\|^2_{L^2(Q_T)}
\end{aligned}$$

Thus we deduce from (2.3) that

$$\int_0^T \int_{\Omega} (u^{\delta,\epsilon}(t,x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \le \frac{1}{\epsilon} C_1^2 |\xi|^2,$$

where C_1 is the constant in (2.3) which is independent of δ and ϵ.

Next, we prove the inequality (3.5).

$$\begin{aligned}
&\int_0^T \int_{\Omega_1^r} (u^{\delta,\epsilon}(t,x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \\
&\qquad = \int_0^T \int_{\Omega_1^r} \Big(\int_0^1 \nabla u^{\delta,\epsilon}(t,x+\theta\xi)\cdot \xi d\theta \Big)^2 dxdt \\
&\qquad \le |\xi|^2 \int_0^1 \int_0^T \int_{\Omega_1^r} |\nabla u^{\delta,\epsilon}(t,x+\theta\xi)|^2 dxdtd\theta \\
&\qquad \le |\xi|^2 \int_0^1 \int_0^T \int_{\Omega_1} |\nabla u^{\delta,\epsilon}(t,x)|^2 dxdtd\theta = |\xi|^2 \|\nabla u^{\delta,\epsilon}\|^2_{L^2((0,T)\times\Omega_1)},
\end{aligned}$$

which in view of the inequality (2.2) implies that

$$\int_0^T \int_{\Omega_1^r} (u^{\delta,\epsilon}(t,x+\xi) - u^{\delta,\epsilon}(t,x))^2 dxdt \le C_1^2 |\xi|^2,$$

where C_1 is independent of δ and ϵ. □

Lemma 3.4. *For any small positive constant $r > 0$,*

(1) *There exists a positive constant C_ϵ which is independent of δ such that*

$$\int_r^{T-r} \int_{\Omega} (u^{\delta,\epsilon}(t+\tau,x) - u^{\delta,\epsilon}(t,x))^2 dxdt \le C_\epsilon \tau,$$

for all real values τ with $0 < \tau \le r < T$.

(2) *There exists a positive constant C which is independent of δ and ϵ such that*

$$\int_r^{T-r} \int_{\Omega_1^r} (u^{\delta,\epsilon}(t+\tau,x) - u^{\delta,\epsilon}(t,x))^2 dxdt \le C\tau, \tag{3.6}$$

for all real values τ *with* $0 < \tau \le r < T$.

Proof. We have

$$\begin{aligned}
&\int_r^{T-r}\int_\Omega (u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))^2 dxdt\\
=&\int_r^{T-r}\int_\Omega (u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))\Big(\int_0^\tau u_t^{\delta,\epsilon}(t+s,x)ds\Big)dxdt\\
\le&\Big|\int_0^\tau\int_r^{T-r}\langle u_t^{\delta,\epsilon}(t+s,x),u^{\delta,\epsilon}(t+\tau,x)\rangle dtds\Big|\\
&+\Big|\int_0^\tau\int_r^{T-r}\langle u_t^{\delta,\epsilon}(t+s,x),u^{\delta,\epsilon}(t,x)\rangle dtds\Big|\\
\le&\ 2\tau\|u_t^{\delta,\epsilon}\|_{L^2(0,T;(H^1)^*(\Omega))}\|u^{\delta,\epsilon}\|_{L^2(0,T;H^1(\Omega))}\\
\le&\ 2\tau C_2\sqrt{|\Omega|TM^2+C_1^2/\epsilon},
\end{aligned}$$

where C_1, C_2 are the upper bounds in Lemma 2.3, which are independent of δ and ϵ.

Next, we prove (3.6). Let $\mu(x)\in C_c^\infty(\Omega_1)$ be such that $0\le\mu\le 1$ in Ω_1 and $\mu=1$ on Ω_1^r. We extend μ by 0 on Ω_0. Then we have

$$\begin{aligned}
&\int_r^{T-r}\int_{\Omega_1^r} (u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))^2 dxdt\\
\le&\int_r^{T-r}\int_\Omega \mu(x)(u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))^2 dxdt\\
=&\int_r^{T-r}\int_\Omega \mu(x)(u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))\Big(\int_0^\tau u_t^{\delta,\epsilon}(t+s,x)ds\Big)dxdt\\
=&\Big|\int_0^\tau\int_r^{T-r}\langle u_t^{\delta,\epsilon}(t+s,x),\mu(x)u^{\delta,\epsilon}(t+\tau,x)\rangle dtds\Big|\\
&+\Big|\int_0^\tau\int_r^{T-r}\langle u_t^{\delta,\epsilon}(t+s,x),\mu(x)u^{\delta,\epsilon}(t,x)\rangle dtds\Big|\\
\le&\ 2\tau\|u_t^{\delta,\epsilon}\|_{L^2(0,T;(H^1)^*(\Omega))}\|\mu u^{\delta,\epsilon}\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

We remark that

$$\begin{aligned}
\|\mu u^{\delta,\epsilon}\|_{L^2(0,T;H^1(\Omega))} =&\Big(\int_0^T\int_\Omega \mu^2(u^{\delta,\epsilon})^2+\mu^2|\nabla u^{\delta,\epsilon}|^2+(u^{\delta,\epsilon})^2|\nabla\mu|^2dxdt\Big)^{1/2}\\
\le&\Big(\int_0^T\int_{\Omega_1}(u^{\delta,\epsilon})^2+|\nabla u^{\delta,\epsilon}|^2+M^2|\nabla\mu|^2dxdt\Big)^{1/2}\\
\le&\sqrt{TM^2(|\Omega_1|+\|\nabla\mu\|^2_{L^2(\Omega_1)})+C_1^2},
\end{aligned}$$

where C_1 is the upper bound in (2.2). Then we obtain

$$\int_r^{T-r}\int_{\Omega_1^r}(u^{\delta,\epsilon}(t+\tau,x)-u^{\delta,\epsilon}(t,x))^2dxdt$$
$$\leq 2\tau C_2\sqrt{TM^2(|\Omega_1|+\|\nabla\mu\|^2_{L^2(\Omega_1)})+C_1^2},$$

which is independent of δ and ϵ. □

Thus we conclude that $\{u^{\delta,\epsilon}\}$ is precompact in $L^2(Q_T)$ and that there exists a function $u^\epsilon \in L^2(Q_T)$ such that $u^{\delta,\epsilon}$ converges strongly in $L^2(Q_T)$ along a subsequence. We are ready to show that the function $u^\epsilon(t,x)$ is a weak solution of Problem (P_0^ϵ).

Proof of Theorem 1.1. From the Fréchet-Kolmogorov theorem, we deduce that there exists a function $u^\epsilon \in L^2(Q_T)$ and a subsequence $\{u^{\delta_i,\epsilon}\}$, which we denote again by $u^{\delta,\epsilon}$ such that

$$u^{\delta,\epsilon}\to u^\epsilon \quad \text{strongly in } L^2((0,T)\times\Omega) \text{ and a.e in } Q_T \quad \text{as } \delta\to 0.$$

Moreover, we deduce from (2.2) that

$$u^{\delta,\epsilon}\rightharpoonup u^\epsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad \text{as } \delta\to 0,$$

and it follows from (2.4) that

$$u_t^{\delta,\epsilon}\rightharpoonup u_t^\epsilon \quad \text{weakly in } L^2(0,T;(H^1)^*(\Omega)) \quad \text{as } \delta\to 0,$$

so that $u^\epsilon \in C([0,T];L^2(\Omega))$. Next, we show that

$$u^{\delta,\epsilon}(t,\cdot)\rightharpoonup u^\epsilon(t,\cdot) \text{ weakly in } (H^1)^*(\Omega), \text{ for all } 0\leq t\leq T. \tag{3.7}$$

Indeed, for any $0\leq\tau\leq T$, for all test function $\phi\in H^1(\Omega)$,

$$\int_\Omega(u^{\delta,\epsilon}(\tau,x)-u_0(x))\phi dx=\int_0^\tau\int_\Omega u_t^{\delta,\epsilon}\phi dxdt=\int_0^\tau\langle u_t^{\delta,\epsilon},\phi\rangle dt. \tag{3.8}$$

We remark that

$$\lim_{\delta\to 0}\int_0^\tau\langle u_t^{\delta,\epsilon},\phi\rangle dt=\int_0^\tau\langle u_t^\epsilon,\phi\rangle dt=\int_\Omega(u^\epsilon(\tau,x)-u_0(x))\phi dx. \tag{3.9}$$

We deduce from (3.8) and (3.9) that

$$\lim_{\delta\to 0}\int_\Omega(u^{\delta,\epsilon}(\tau,x)-u_0(x))\phi dx=\int_\Omega(u^\epsilon(\tau,x)-u_0(x))\phi dx, \tag{3.10}$$

for all $\phi\in H^1(\Omega)$.

Next, we show that u^ϵ is a weak solution of Problem (P_0^ϵ). Since $u^{\delta,\epsilon}$ is a classical solution of Problem (P_δ^ϵ), it is also a weak solution of this problem. Thus it satisfies

$$\int_0^T\int_\Omega(-u^{\delta,\epsilon}\phi_t+D_{\delta,\epsilon}(x)\nabla u^{\delta,\epsilon}\cdot\nabla\phi-f(u^{\delta,\epsilon})\phi)dxdt$$
$$=\int_\Omega u_0(x)\phi dx-\int_\Omega u^{\delta,\epsilon}(T,x)\phi(T,x)dx, \tag{3.11}$$

for all test functions $\phi \in C^1([0,T]\times\overline{\Omega})$.

Since $u^{\delta,\epsilon} \to u^\epsilon$ in $L^2((0,T)\times\Omega)$ as $\delta \to 0$, we deduce that

$$\int_0^T \int_\Omega u^{\delta,\epsilon}\phi_t \to \int_0^T \int_\Omega u^\epsilon \phi_t \quad \text{as } \delta \to 0.$$

Moreover, we remark that

$$D_{\delta,\epsilon} \to D_\epsilon \quad \text{strongly in } L^2(\Omega) \quad \text{as } \delta \to 0$$

so that $D_{\delta,\epsilon}\nabla\phi \to D_\epsilon\nabla\phi$ strongly in $L^2((0,T)\times\Omega)$. This, combined with the fact that as $\delta \to 0$,

$$\nabla u^{\delta,\epsilon} \rightharpoonup \nabla u^\epsilon \quad \text{weakly in } L^2((0,T)\times\Omega)$$

implies that

$$\int_0^T \int_\Omega D_{\delta,\epsilon}\nabla u^{\delta,\epsilon}\cdot\nabla\phi \to \int_0^T \int_\Omega D_\epsilon \nabla u^\epsilon \cdot \nabla\phi$$

as $\delta \to 0$. Since $u^{\delta,\epsilon}$ converges to u^ϵ a.e. in Q_T as $\delta \to 0$, it follows from the continuity of f that $f(u^{\delta,\epsilon})$ converges to $f(u^\epsilon)$ a.e. in Q_T as $\delta \to 0$. Moreover, since that $|f(u^{\delta,\epsilon})| \le M_f$, we deduce from the dominated convergence theorem that

$$\int_0^T \int_\Omega f(u^{\delta,\epsilon})\phi \to \int_0^T \int_\Omega f(u^\epsilon)\phi$$

as $\delta \to 0$.

Since $u^{\delta,\epsilon}(T,\cdot) \rightharpoonup u^\epsilon(T,\cdot)$ weakly in $(H^1)^*(\Omega)$ by (3.7),

$$\int_\Omega u^{\delta,\epsilon}(T,x)\phi(T,x)dx \to \int_\Omega u^\epsilon(T,x)\phi(T,x)dx \quad \text{as } \delta \to 0.$$

Therefore the passing to the limit $\delta \to 0$ in (3.11), we conclude that u^ϵ satisfies the integral equality for all $\phi \in C^1([0,T]\times\overline{\Omega})$

$$\int_0^T \int_\Omega (-u^\epsilon\phi_t + D_\epsilon(x)\nabla u^\epsilon\cdot\nabla\phi - f(u^\epsilon)\phi)dxdt = \int_\Omega u_0(x)\phi dx - \int_\Omega u^\epsilon(T,x)\phi(T,x)dx. \tag{3.12}$$

Since $C^1([0,T]\times\overline{\Omega})$ is dense in $H^1((0,T)\times\Omega)$, we deduce that the identity (3.12) also holds for all $\phi \in H^1((0,T)\times\Omega)$.

Next, we prove the uniqueness of the weak solution. First, we remark

$$u_t^\epsilon = \nabla\cdot(D_\epsilon(x)\nabla u^\epsilon) + f(u^\epsilon) \quad \text{in } L^2(0,T;(H^1)^*(\Omega)).$$

Suppose that there exist two solutions u_1^ϵ and u_2^ϵ of Problem (P_δ^ϵ). Setting $w := u_2^\epsilon - u_1^\epsilon$, we have that

$$w_t = \nabla\cdot(D_\epsilon(x)\nabla w) + f(u_2^\epsilon) - f(u_1^\epsilon) \quad \text{in } L^2(0,T;(H^1)^*(\Omega)),$$

which implies

$$\int_0^\tau \langle w_t, w\rangle dt = \int_0^\tau \langle \nabla\cdot(D_\epsilon(x)\nabla w), w\rangle + \langle f(u_2^\epsilon) - f(u_1^\epsilon), w\rangle dt,$$

for all $0 \le \tau \le T$, that is

$$\begin{aligned}\frac{1}{2}\int_\Omega w^2(\tau) + \int_0^\tau\int_\Omega D_\epsilon(x)|\nabla w|^2 dxdt &= \int_0^\tau\int_\Omega w\left(\int_{u_1^\epsilon}^{u_2^\epsilon} f'(s)ds\right)dxdt\\ &= \int_0^\tau\int_\Omega w(u_2^\epsilon - u_1^\epsilon)\left(\int_0^1 f'(\theta u_1^\epsilon + (1-\theta)u_2^\epsilon)d\theta\right)dxdt\\ &\le \tilde{M}_f\int_0^\tau\int_\Omega w^2 dxdt,\end{aligned}$$

where $\tilde{M}_f := \sup_{s\in[0,M]} f'(s)$, for all $0 \le \tau \le T$. Applying Gronwall's lemma, we deduce that $w = 0$ a.e. in Q_T. □

4. Singular limit $\epsilon \to 0$

In this section, we prove that the solution of Problem (P_0^ϵ) converges strongly in $L^2((0,T)\times\Omega_1)$ to the solution of (P_0^0)

$$(P_0^0)\qquad \begin{cases} u_t = \Delta u + f(u), & (t,x)\in(0,T)\times\Omega_1\\ \dfrac{\partial u}{\partial\nu} = 0, & (t,x)\in(0,T)\times\partial\Omega_1\\ u(0,x) = u_0(x), & x\in\Omega_1,\end{cases}$$

and that the solution of Problem (P_0^ϵ) converges weakly in $L^2((0,T)\times\Omega_0)$ to the solution of Problem (Q_0^0)

$$(Q_0^0)\qquad \begin{cases} u_t = f(u), & (t,x)\in(0,T)\times\Omega_0\\ u(0,x) = u_0(x), & x\in\Omega_0.\end{cases}$$

The solution of (P_0^0) is defined in a weak sense.

Definition 4.1. *A function $u \in C([0,T]; L^2(0,T)) \cap L^2((0,T); H^1(\Omega_1)) \cap L^\infty((0,T)\times\Omega_1)$ is called a weak solution of (P_0^0) if*

$$\begin{aligned}&\int_0^T\int_{\Omega_1}(-u\phi_t + \nabla u\cdot\nabla\phi - f(u)\phi)dxdt\\ &= \int_{\Omega_1} u_0(x)\phi(0,x)dx - \int_{\Omega_1} u(T,x)\phi(T,x)dx,\end{aligned}\tag{4.1}$$

for any test function $\phi \in H^1((0,T)\times\Omega_1)$.

It is standard that Problem (P_0^0) possesses a unique weak solution. Furthermore, this weak solution is actually a classical solution $u \in C^{1,2}((0,T)\times\Omega_1)\cap C([0,T]\times\overline{\Omega}_1)$. Therefore, to complete the proof of Theorem 1.2, we need to show that the limit of u^ϵ converges strongly in $L^2((0,T)\times\Omega_1)$ and satisfies (4.1).

Proof of Theorem 1.2. Since $0 \le u^\epsilon \le M$ and (2.4), there exists a function $\bar{u} \in L^\infty((0,T)\times\Omega)$ and a subsequence $\{u^{\epsilon_n}\}$ such that

$$u^{\epsilon_n} \rightharpoonup \bar{u} \quad \text{weakly in } L^2((0,T)\times\Omega) \text{ as } \epsilon_n \to 0,$$

$$u_t^{\epsilon_n} \rightharpoonup \bar{u}_t \quad \text{weakly in } L^2(0,T;(H^1)^*(\Omega)) \text{ as } \epsilon_n \to 0.$$

Next we show that $u^{\epsilon_n} \to \bar{u}$ strongly in $L^2((0,T)\times\Omega_1)$. Indeed, the sufficient estimates on differences of time and space translates follow from (3.5) and (3.6). Moreover, since u^ϵ is bounded in $L^\infty((0,T)\times\Omega)$, the property (2) in the Fréchet-Kolmogorov theorem is satisfied. Thus, we can apply the Fréchet-Kolmogorov theorem. We conclude that there exists a subsequence of $\{u^{\epsilon_n}\}$ which we denote again by $\{u^{\epsilon_n}\}$ such that

$$u^{\epsilon_n} \to \bar{u} \quad \text{strongly in } L^2((0,T)\times\Omega_1) \text{ and a.e. in } (0,T)\times\Omega_1 \text{ as } \epsilon_n \to 0.$$

We will show below that $\bar{u}$ is the unique solution of Problem (P_0^0).

Lemma 4.2. *For each $\phi \in H^1((0,T)\times\Omega)$,*

$$\int_0^T \int_{\Omega_0} D^\epsilon(x)\nabla u^\epsilon \cdot \nabla\phi dxdt \to 0 \ \text{as } \epsilon \to 0.$$

Proof. From (2.3), by the Cauchy-Schwarz inequality,

$$\begin{aligned}\Big| \int_0^T \int_{\Omega_0} D_\epsilon(x)\nabla u^\epsilon \cdot \nabla\phi dxdt \Big| &\le \epsilon \|\nabla u^\epsilon\|_{L^2((0,T)\times\Omega_0)} \|\nabla\phi\|_{L^2((0,T)\times\Omega_0)} \\ &\le C\epsilon \|\nabla u^\epsilon\|_{L^2((0,T)\times\Omega)} \\ &\le \tilde{C}\sqrt{\epsilon},\end{aligned}$$

which converges to 0 as $\epsilon \to 0$. □

It follows from (2.2) that

$$\nabla u^\epsilon \rightharpoonup \nabla\bar{u} \quad \text{weakly in } L^2((0,T)\times\Omega_1),$$

and since $|f(u^{\epsilon_n})| \le M_f$, there exists a function

$$\chi \in L^\infty((0,T)\times\Omega)$$

and a subsequence of $\{u^{\epsilon_n}\}$ which we denote again by $\{u^{\epsilon_n}\}$ such that

$$f(u^{\epsilon_n}) \rightharpoonup \chi \quad \text{weakly in } L^2((0,T)\times\Omega) \text{ as } \epsilon_n \to 0.$$

Moreover, since $u^{\epsilon_n} \to \bar{u}$ a.e. in $(0,T)\times\Omega_1$,

$$f(u^{\epsilon_n}) \to f(\bar{u}) \quad \text{a.e. in } (0,T)\times\Omega_1$$

and by Lebesgue's dominated convergence theorem,

$$f(u^{\epsilon_n}) \to f(\bar{u}) \quad \text{strongly in } L^1((0,T)\times\Omega_1) \text{ as } \epsilon_n \to 0.$$

Thus

$$\chi = f(\bar{u}) \quad \text{a.e in } (0,T)\times\Omega_1. \tag{4.2}$$

We rewrite (3.12) in the form

$$\begin{aligned}\int_0^T \int_{\Omega_1} &\{-u^{\epsilon_n}\phi_t + \nabla u^{\epsilon_n}\cdot\nabla\phi - f(u^{\epsilon_n})\phi\}\,dxdt\\ &+ \int_0^T\int_{\Omega_0}\{-u^{\epsilon_n}\phi_t + \epsilon\nabla u^{\epsilon_n}\cdot\nabla\phi - f(u^{\epsilon_n})\phi\}\,dxdt\\ = \int_{\Omega_1} &u_0(x)\phi(0,x)dx + \int_{\Omega_0} u_0(x)\phi(0,x)dx - \int_\Omega u^{\epsilon_n}(T,x)\phi(T,x)dx\end{aligned}$$

for all $\phi \in H^1((0,T)\times\Omega)$, in which we let $\epsilon_n \to 0$ to obtain

$$\begin{aligned}\int_0^T\int_{\Omega_1}(-\bar u\phi_t + \nabla\bar u\cdot\nabla\phi - f(\bar u)\phi)dxdt + \int_0^T\int_{\Omega_0}(-\bar u\phi_t - \chi\phi)dxdt\\ = \int_{\Omega_1}u_0(x)\phi(0,x)dx + \int_{\Omega_0}u_0(x)\phi(0,x)dx - \int_\Omega \bar u(T,x)\phi(T,x)dx,\end{aligned} \tag{4.3}$$

for all $\phi \in H^1((0,T)\times\Omega)$, since $u^{\epsilon_n}(T,x) \rightharpoonup \bar u(T,x)$ weakly in $(H^1)^*(\Omega)$. Take ϕ arbitrary in $H^1((0,T)\times\Omega)$ such that $\phi(T,x) = 0$ for $x \in \Omega_0$ and $\phi = 0$ in $(0,T)\times\Omega_1$. Then, (4.3) becomes

$$\int_0^T\int_{\Omega_0}(-\bar u\phi_t - \chi\phi)dxdt = \int_{\Omega_0}u_0(x)\phi(0,x)dx,$$

that is

$$-\int_0^T\langle\bar u_t,\phi\rangle dt - \int_0^T\int_{\Omega_0}\chi\phi dxdt + \int_{\Omega_0}(\bar u(0,x) - u_0(x))\phi(0,x)dx = 0. \tag{4.4}$$

Taking ϕ arbitrary in $C_c^\infty((0,T)\times\Omega_0)$, we deduce that

$$\bar u_t = \chi \quad \text{for a.e. } (t,x)\in(0,T)\times\Omega_0. \tag{4.5}$$

Now taking ϕ arbitrary in $H^1((0,T)\times\Omega)$ such that $\phi(T,x) = 0$ for $x \in \Omega_0$ and $\phi = 0$ in $(0,T)\times\Omega_1$, we deduce that

$$\bar u(0,x) = u_0(x) \quad \text{a.e. for } x \in \Omega_0. \tag{4.6}$$

Furthermore, if we take $\phi \in C_c^1([0,T)\times\overline{\Omega})$ arbitrary in (4.3), we deduce from (4.5) and (4.6) that

$$\int_0^T\int_{\Omega_1}(-\bar u\phi_t + \nabla\bar u\cdot\nabla\phi - f(\bar u)\phi)dxdt = \int_{\Omega_1}u_0(x)\phi(0,x)dx, \tag{4.7}$$

which also holds for arbitrary $\phi \in C_c^1([0,T)\times\overline{\Omega}_1)$.

Lemma 4.3. *There holds $\chi = f(\bar u)$ in $L^\infty((0,T)\times\Omega)$.*

Proof. We already proved that this property holds in $(0,T)\times\Omega_1$, but not yet in the whole domain $(0,T)\times\Omega$. We define

$$c := \sup_{s\in[0,M]} f'(s),$$

and set

$$v^\epsilon := e^{-ct}u^\epsilon, \quad \bar{v} := e^{-ct}\bar{u}$$

and

$$g(t,v) := e^{-ct}f(e^{ct}v) - cv, \quad \tilde{\chi} := e^{-ct}(\chi - c\bar{u}), \quad \psi := e^{ct}\phi.$$

We will apply a classical monotonicity argument, which can be found, for instance, in [10]. We recall that by Definition 3.1, we have

$$\int_0^T \int_\Omega (-u^\epsilon \phi_t + D_\epsilon(x)\nabla u^\epsilon \cdot \nabla\phi - f(u^\epsilon)\phi)dxdt$$
$$= \int_\Omega u_0(x)\phi(0,x)dx - \int_\Omega u^\epsilon(T,x)\phi(T,x)dx,$$

for all $\phi \in H^1((0,T)\times\Omega)$, which implies that

$$\int_0^T \int_\Omega (-v^\epsilon \psi_t + D_\epsilon(x)\nabla v^\epsilon \cdot \nabla\psi - g(t,v^\epsilon)\psi)dxdt$$
$$= \int_\Omega u_0(x)\psi(0,x)dx - \int_\Omega v^\epsilon(T,x)\psi(T,x)dx,$$

for all $\psi \in H^1((0,T)\times\Omega)$, so that also

$$\int_0^T \langle v_t^\epsilon, \psi\rangle dt + \int_0^T \int_\Omega (D_\epsilon(x)\nabla v^\epsilon \cdot \nabla\psi - g(t,v^\epsilon)\psi)dxdt = 0, \tag{4.8}$$

for all $\psi \in L^2(0,T;H^1(\Omega))$. We have that

$$g_v(t,v^\epsilon) = e^{-ct}e^{ct}f'(e^{ct}v^\epsilon) - c = f'(e^{ct}v^\epsilon) - c \le 0,$$

where we have used the fact that

$$0 \le u^\epsilon = e^{ct}v^\epsilon \le M.$$

Since g is decreasing in v we deduce that

$$\int_0^T \int_\Omega (g(t,v^\epsilon) - g(t,\psi))(v^\epsilon - \psi)dxdt \le 0,$$

for $\psi \in L^2(0,T;H^1(\Omega))$. Thus,

$$\begin{aligned} 0 &\ge \liminf_{\epsilon\to 0} \int_0^T \int_\Omega (g(t,v^\epsilon) - g(t,\psi))(v^\epsilon - \psi)dxdt \\ &= \liminf_{\epsilon\to 0} (\int_0^T \int_\Omega g(t,v^\epsilon)v^\epsilon dxdt \\ &- \int_0^T \int_\Omega g(t,\psi)v^\epsilon dxdt - \int_0^T \int_\Omega (g(t,v^\epsilon) - g(t,\psi))\psi dxdt) \end{aligned} \tag{4.9}$$

Substituting $\psi = v^\epsilon$ in (4.8), we obtain

$$\int_0^T \langle v_t^\epsilon, v^\epsilon\rangle dt + \int_0^T \int_\Omega (D_\epsilon(x)\nabla v^\epsilon \cdot \nabla v^\epsilon - g(t,v^\epsilon)v^\epsilon)dxdt = 0. \tag{4.10}$$

Combining (4.9) and (4.10) yields

$$\begin{aligned}0 \geq &\liminf_{\epsilon\to 0}\Big(\int_0^T \langle v_t^\epsilon, v^\epsilon\rangle dt + \int_0^T\int_\Omega D_\epsilon(x)|\nabla v^\epsilon|^2 dxdt \\ &- \int_0^T\int_\Omega g(t,\psi)v^\epsilon dxdt - \int_0^T\int_\Omega (g(t,v^\epsilon)-g(t,\psi))\psi dxdt\Big) \\ = &\liminf_{\epsilon\to 0}\Big(\frac{1}{2}\int_\Omega |v^\epsilon(T)|^2dx - \frac{1}{2}\int_\Omega |u_0|^2 dx + \int_0^T\int_\Omega D_\epsilon(x)|\nabla v^\epsilon|^2 dxdt \\ &- \int_0^T\int_\Omega g(t,\psi)v^\epsilon dxdt - \int_0^T\int_\Omega (g(t,v^\epsilon)-g(t,\psi))\psi dxdt\Big).\end{aligned}$$

Thus letting ϵ tends to 0, we deduce that

$$\begin{aligned}0 \geq &\frac{1}{2}\int_\Omega |\bar v(T)|^2dx - \frac{1}{2}\int_\Omega |u_0|^2dx + \int_0^T\int_{\Omega_1}|\nabla \bar v|^2 dxdt \\ &- \int_0^T\int_\Omega g(t,\psi)\bar v dxdt - \int_0^T\int_\Omega(\tilde\chi - g(t,\psi))\psi dxdt\end{aligned} \tag{4.11}$$

We remark that (4.2) and (4.3) imply

$$\int_0^T \langle \bar u_t, \phi\rangle dt + \int_0^T\int_{\Omega_1}(\nabla \bar u\cdot\nabla\phi - \chi\phi)dxdt - \int_0^T\int_{\Omega_0}\chi\phi dxdt = 0,$$

for all $\phi \in L^2(0,T;H^1(\Omega))$, which in turn implies that

$$\int_0^T \langle \bar v_t + c\bar v, \psi\rangle dt + \int_0^T\int_{\Omega_1}(\nabla\bar v\cdot\nabla\psi)dxdt - \int_0^T\int_\Omega e^{-ct}\chi\psi dxdt = 0,$$

$$\int_0^T\langle \bar v_t,\psi\rangle dt + \int_0^T\int_{\Omega_1}(\nabla\bar v\cdot\nabla\psi)dxdt - \int_0^T\int_\Omega \tilde\chi\psi dxdt = 0, \tag{4.12}$$

for all $\psi \in L^2(0,T;H^1(\Omega))$. Setting $\psi = \bar v$ in (4.12) we deduce that

$$\frac{1}{2}\int_\Omega|\bar v(T)|^2dx - \frac{1}{2}\int_\Omega|u_0|^2dx + \int_0^T\int_{\Omega_1}|\nabla\bar v|^2dxdt - \int_0^T\int_\Omega\tilde\chi\bar v dxdt = 0. \tag{4.13}$$

Substituting (4.13) into (4.11) yields

$$\int_0^T\int_\Omega(\tilde\chi - g(t,\psi))(\bar v - \psi)dxdt \leq 0,$$

for all $\psi \in L^2(0,T;H^1(\Omega))$.

Taking $\psi = \bar v - \lambda w$, $\lambda > 0$, $w \in L^2(0,T;H^1(\Omega))$, we deduce that

$$\int_0^T\int_\Omega(\tilde\chi - g(t,\bar v - \lambda w))w \leq 0, \tag{4.14}$$

for all $w \in L^2(0,T;H^1(\Omega))$. Letting $\lambda \to 0$ in (4.14), we obtain

$$\int_0^T \int_\Omega (\tilde{\chi} - g(t,\bar{v}))w \leq 0, \tag{4.15}$$

for all $w \in L^2(0,T;H^1(\Omega))$. Setting $w = -w$ in (4.15), we deduce that

$$\int_0^T \int_\Omega (\tilde{\chi} - g(t,\bar{v}))w = 0,$$

which yields that

$$\tilde{\chi} = g(t,\bar{v}) \quad \text{a.e,}$$

or else

$$\chi = f(\bar{u}) \quad \text{a.e.} \tag{4.16}$$

□

Taking $\phi \in C_c^1([0,T) \times \Omega_1)$ arbitrary in (4.7) yields

$$\int_0^T \int_{\Omega_1} (-\bar{u}\phi_t + \nabla\bar{u} \cdot \nabla\phi - f(\bar{u})\phi)dxdt = \int_{\Omega_1} u_0(x)\phi(0,x)dx,$$

for all $\phi \in C_c^1([0,T) \times \Omega_1)$. Then taking ϕ arbitrary in $C_c^\infty((0,T) \times \Omega_1)$, we deduce that $\bar{u}$ satisfies the partial differential equation

$$\bar{u}_t = \Delta\bar{u} + f(\bar{u}), \qquad \text{in the sense of distributions in } (0,T) \times \Omega_1.$$

Next we take ϕ arbitrary in $C_c^\infty([0,T) \times \Omega_1)$, to deduce that

$$\bar{u}(0,x) = u_0(x), \quad x \in \Omega_1.$$

Finally, taking ϕ arbitrary in $C_c^\infty((0,T) \times \overline{\Omega}_1)$ in (4.7), we deduce that

$$\frac{\partial \bar{u}}{\partial \nu} = 0, \qquad \text{in the sense of distributions on } (0,T) \times \Gamma.$$

It then follows from standard arguments that $\bar{u}$ coincides with the unique classical solution u of Problem (P_0^0). □

Proof of Theorem 1.3. The result of Theorem 1.3 follows from (4.5) and (4.16). □

5. Numerical simulation

In this section, we observe the evolution of the solution of (P_0^ϵ) numerically and test the appearance of the Neumann boundary condition along the interior boundary $\partial\Omega_1$ when $\epsilon \to 0$. For the test, we consider a one-dimensional problem with $\Omega = (0,4)$ and its inner domain $\Omega_1 = (1,3)$. Then, the equation (P_0^ϵ) is written as

$$\begin{cases} u_t = (D_\epsilon u_x)_x + f(u), & t > 0,\ 0 < x < 4, \\ u_x(t,0) = u_x(t,4) = 0, & t > 0, \\ u(0,x) = u_0(x), & 0 < x < 4. \end{cases} \tag{5.1}$$

We take the initial value, reaction function, and diffusivity respectively as

$$u_0(x) = \sin(\pi x/4), \quad f(u) = -u(u-1/3)(u-1), \tag{5.2}$$

and

$$D_\epsilon(x) = \begin{cases} 1, & 1 \le x \le 3, \\ \epsilon, & \text{otherwise.} \end{cases} \tag{5.3}$$

The snapshots of the numerical solutions of (5.1)–(5.3) are given in Figure 2 at the moment $t = 0.1$ with four cases of ϵ values. For this computation, the MATLAB function 'pdepe' is used with a mesh size $\triangle x = 0.001$. The behavior of the solution at the interface $x = 1$ is magnified. We can observe a development of discontinuity of the gradient $|\nabla u|$ and the solution itself u at the interface $x = 1$. This is due to the continuity relation of the flux at the interface, which is $\epsilon|\nabla u(1-)| = |\nabla u(1+)|$.

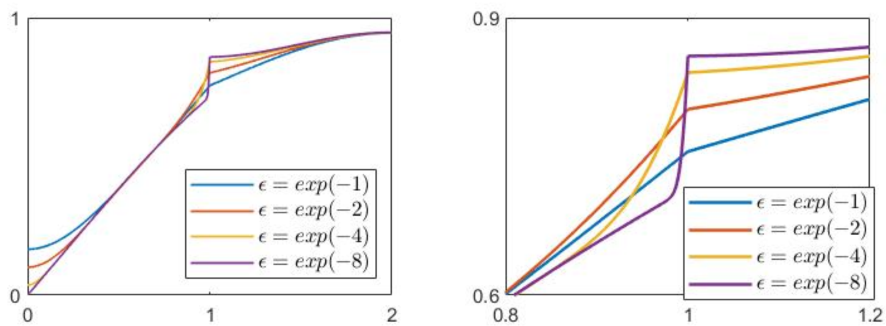

FIGURE 2. Solution snapshots at $t = 0.1$ to see the early development of interface. An interface develops as $\epsilon \to 0$.

Theorem 1.2 says that the solution of (P_0^ϵ) converges to the solution of the Neumann boundary problem (P_0^0) in the inner domain. The Neumann problem corresponds to (5.1) is

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0,\ 1 < x < 3, \\ u_x(t,1) = u_x(t,3) = 0, & t > 0, \\ u(0,x) = u_0(x), & 1 < x < 3. \end{cases} \tag{5.4}$$

The same initial value in (5.2) is taken from the inner domain $(1, 3)$. The snapshot of the numerical solution of the Neumann problem (5.4) is given in Figure 3 together with the snapshots in Figure 2 in the whole inner domain $\Omega_1 = (1, 3)$. We can observe that the solutions of (5.1) converge to the solution of the Neumann problem (5.4) as $\epsilon \to 0$.

In Figure 4, the gradient values $|\nabla u(t, 1+)|$ at the interface are given reducing ϵ with $\epsilon = e^{-j}$ for $j = 0, \cdots, 8$. They are given at three moments, $t = 0.01, 0.03$ and 0.05. We can see that these gradient values decay to zero

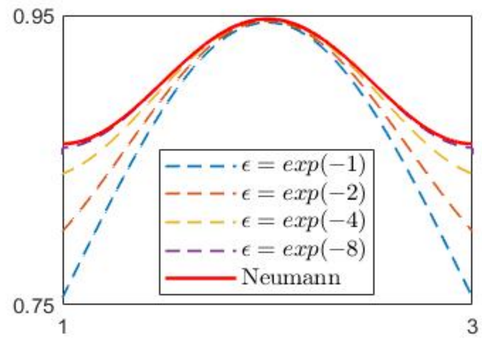

FIGURE 3. The solution of (5.1) converges to the solution of Neumann problem (5.4) as $\epsilon \to 0$. Snapshots are in the inner domain at time $t = 0.1$.

as $\epsilon \to 0$. The second figure is displayed in the log-log scale. We can see that $\ln(|\nabla u|)$ and $\ln(\epsilon)$ satisfy a linear relation,

$$\ln |\nabla u| = a \ln(\epsilon) + b,$$

which is equivalently written as

$$|\nabla u| = e^b \epsilon^a.$$

Then, the three cases with $t = 0.01, 0.04$ and 0.09 gives convergence order of

$$|\nabla u(t, 1+)| \cong \begin{cases} e^{-0.5941}\epsilon^{0.4951}, & t = 0.01 \\ e^{-0.6112}\epsilon^{0.5172}, & t = 0.04 \\ e^{-0.6650}\epsilon^{0.5333}, & t = 0.09. \end{cases}$$

We can see that the gradient of the solution at the interface converges to zero as $\epsilon \to 0$ approximately with order $O(\sqrt{\epsilon})$, and the convergence order increases as the time variable t increases. This order is not surprising. For a fixed time $t > 0$, the gradient $|\nabla u|$ in the region with diffusivity $D = 1$ is supposed to be bounded. In the region with $D = \epsilon$, the gradient $|\nabla u|$ is of order $(1/\sqrt{\epsilon})$, which is from parabolic rescaling of the space variable. Therefore, the interface condition

$$|\nabla u(t, 1+)| = \epsilon |\nabla u(t, 1-)| = O(\sqrt{\epsilon}) \quad \text{as} \quad \epsilon \to 0.$$

In Figure 5, five snapshots of numerical solutions are given, which show the long-time behavior of the solution when ϵ is small. We took $\epsilon = e^{-16}$ for this simulation. We can see that a discontinuity is developing at the interface $x = 1$ very quickly ($t = 0.1$). However, the long-time dynamics take it over and eventually converge to a step function with the discontinuity at $x = 0.4327$ decided by the initial value and the reaction function. This behavior is expected since the solution behavior in the region with $D = \epsilon$ follows the ODE solution $\dot{u} = f(u)$ as $\epsilon \to 0$. Since the unstable steady

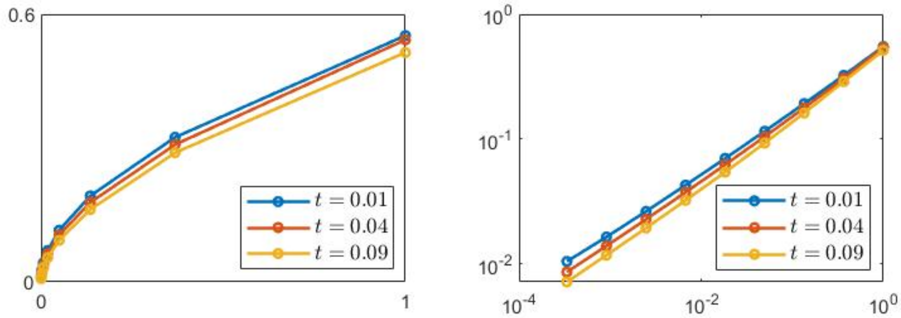

FIGURE 4. x-axis is $\epsilon > 0$. The graph is for $|\nabla u(t, 1^+)|$. The solution satisfies zero Neumann boundary condition as $\epsilon \to 0$.

state is $u = 1/3$ and the initial value takes the value at $x = 0.4327$, i.e., $u_0(0.4327) \cong 1/3$, the discontinuity develops at the point $x = 0.4327$.

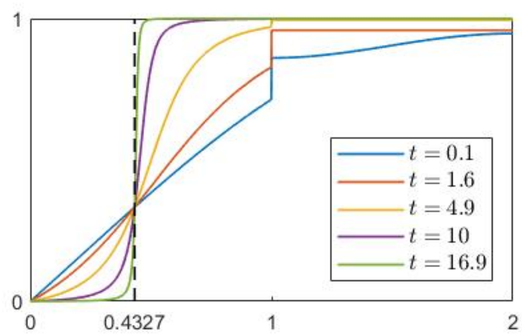

FIGURE 5. Long-time behavior with $\epsilon = e^{-16}$. For small $t > 0$, a discontinuity develops at $x = 1$. Solutions converge to a step function with a discontinuity at $x = 0.4327$ as t increases.

Acknowledgements

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(Danielle Hilhorst) CNRSandLaboratoire de Mathématiques, University Paris-Saclay, 91405 Orsay Cedex, France
Email address: danielle.hilhorst@universite-paris-saclay.fr

(Seung-Min Kang) Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, Republic of Korea
Email address: rkdtmdalsx@kaist.ac.kr

(Ho-Youn Kim) Computer, Electrical and Mathematical Sciences and Engineering Division, 4700 King Abdullah University of Science and Technology, Thuwal, 23955-6900, Kingdom of Saudi Arabia
Email address: hoyoun.kim@kaust.edu.sa

(Yong-Jung Kim) Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, Republic of Korea
Email address: yongkim@kaist.edu