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An Oleinik-type estimate for a convection–diffusion equation and convergence to N-waves ☆

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Abstract

In this article we propose an Oleinik-type estimate for sign-changing solutions to a convection–diffusion equation

$$u_t + (|u|^\gamma/\gamma)_x = \mu u_{xx}, \quad u(x,0) = u_0(x), \quad u, x \in \boldsymbol{R},\ 1<\gamma\leqslant 2,\ \mu, t>0.$$

Since the Oleinik entropy inequality holds for nonnegative solutions or inviscid case ($\mu = 0$) only, the theoretical progress for the case was limited. In this paper we show that its solution satisfies an Oleinik-type estimate,

$$t^{\frac{2}{\gamma}} u_x \leqslant C, \quad 1<\gamma\leqslant 2,\ t>0,$$

where $C = C(u_0, \gamma) > 0$. Using this estimate, the convergence to an N-wave is proved for sign changing solutions and the theoretical gap in asymptotic convergence of the corresponding problem is filled.

1. Introduction

We investigate the competition between the convection and the diffusion which frequently appears in many physical phenomena. Since such a co-relation plays an important role in the evolution of solutions of the corresponding mathematical

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models, a survey of their interaction in a simpler model may provide a good insight of those models. In this paper we consider a Cauchy problem of a scalar convection–diffusion equation

$$u_t + \partial_x f(u) = \mu u_{xx}, \quad u(x,0) = u_0(x), \quad \mu, t > 0, \; x, u \in \boldsymbol{R}, \tag{1}$$

where the initial value is integrable $u_0 \in L^1(\boldsymbol{R})$ and the convection is given by the convex power law

$$f(u) = \frac{1}{\gamma}|u|^\gamma, \qquad \gamma > 1. \tag{2}$$

The convexity of the convection can be easily verified, i.e., $f''(u) = (\gamma - 1)|u|^{\gamma-2} \geqslant 0$ for all $u \in R$.

Benilan and Crandall [2] have studied regularizing effects of convection and diffusion together in a single framework based on the homogeneity. However, the long time regularizing effect generated by the diffusion is different from the one by the convection and, therefore, it is required to understand the difference to obtain a better asymptotics. There are two kinds of sources to generate the competitions between them. The first one is due to the difference in the similarity structure between the diffusion equation ($f = 0$) and the convection one ($\mu = 0$). This kind of competition is now well understood thanks to recent results to be mentioned below.

A convenient way to see this phenomenon is to transform the problem using similarity variables:

$$s = \ln(t), \quad \xi = x/\sqrt[\gamma]{t}, \quad w(\xi, s) = \sqrt[\gamma]{t}u(x,t). \tag{3}$$

We can easily check that problem (1), (2) is transformed to

$$w_s + \frac{1}{\gamma}(|w|^\gamma - \xi w)_\xi = \mu e^{(\gamma-2)s/2} w_{\xi\xi}, \quad \xi, s, w \in \boldsymbol{R}, \; \mu > 0. \tag{4}$$

For $\gamma > 2$, the coefficient in the diffusion term increases exponentially as $s \to \infty$. Hence it is expected that the effect of the diffusion dominates the one of the convection in this case. In fact the asymptotic structure of the solution is same as the one of the heat equation and is obtained by a technique based on the diffusion (see [9]). If $\gamma = 2$, the equation is called the Burgers equation and is the border case. In this case the coefficient in the diffusion term is constant and the effects of the diffusion and the convection are balanced. The asymptotic structure of this case is a diffusion wave which is an intermediate stage between the heat kernel and the N-wave (see [16,17]).

Note that the N-wave of the convection equation ($\mu = 0$) under the convex power law (2) is given by

$$N_{p,q}(x,t) = \begin{cases} \mathrm{sign}(x)\sqrt[\gamma-1]{|x/t|}, & -(\frac{\gamma p}{\gamma-1})^{\frac{\gamma-1}{\gamma}} < x/t < (\frac{\gamma q}{\gamma-1})^{\frac{\gamma-1}{\gamma}}, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

where $\int_0^{\infty} N_{p,q}(x,t)\,dx = q$ and $\int_{-\infty}^{0} N_{p,q}(x,t)\,dx = -p$. If $p = 0$, the N-wave is a positive function.

For $\gamma < 2$, the coefficient in the diffusion term decreases to zero exponentially as $s \to \infty$. Hence it is natural to expect that the convection dominates the whole evolution. In fact, Escobedo et al. [7] show that a positive solution converges to a positive N-wave which is the asymptotic structure of the convection equation. This convergence is obtained by a technique based on the Oleinik estimate. This result is extended to multidimensional space and to various convection functions (see [4,6,8]). For a sign changing solution with $1 < \gamma < 2$ there is no result on the asymptotic convergence so far.

The other kind of competition between the convection and the diffusion appears across a point of a sign change, which is of our main interest in this paper. The role of the convection is usually interpreted in terms of the wave speed $f'(u) = \mathrm{sign}(u)|u|^{\gamma-1}$. Since the wave speed $f'(u)$ for the positive value $u > 0$ is opposite to the one for the negative value $u < 0$, the positive and the negative humps of the solution may collide together or get separated away from each other. On the other hand the flux generated by the diffusion depends on the slope $\partial_x u$ only, and it makes a positive and a negative humps interact as long as $\partial_x u \neq 0$ across a zero point. Note that the Oleinik estimate does not hold across a sign-change because of this kind of competition.

For a convex-concave convection such as

$$f(u) = \mathrm{sign}(u)|u|^{\gamma}/\gamma, \quad \gamma > 1,$$

the second kind of competition is not observed since the wave speed $f'(u) = |u|^{\gamma-1}$ is always positive. For this type of convection, the asymptotic convergence for sign changing solutions has been shown in [7], which is identical to the one of positive solutions. For detailed asymptotic structure we refer readers to [5,14,19] for inviscid conservation laws and [4] for convection–diffusion equations.

The only asymptotic convergence for the convection–diffusion equation of the type (1) that still remains open is of sign changing solutions under the convex power law (2) with $1 < \gamma < 2$. In [7], it is shown that positive solutions satisfy the Oleinik estimate and that, using this estimate, they converge to positive N-waves. This technique is not directly applicable to sign changing solutions since the estimate does not hold anymore (see Remark 5). To overcome this difficulty we introduce a weak form of the Oleinik estimate that is satisfied by the solution of the convection–diffusion equation with $1 < \gamma \leqslant 2$.

The generalization of the Oleinik estimate has been considered for several cases. We refer to Hoff [13] for multidimensional problems, Sinestrari [21] for conservation laws with source terms, Jenssen and Sinestrari [14] for a nonconvex convection and Bressan and LeFloch [3] for genuinely nonlinear systems.

It is well known that the solution of the inviscid problem ($\mu = 0$) satisfies the Oleinik estimate:

$$\partial_x f'(u) (\equiv (\gamma - 1)|u|^{\gamma-2} u_x) \leqslant 1/t, \quad \gamma > 1, \quad t > 0. \tag{6}$$

In Section 3 we show that the solution of the convection–diffusion equation satisfies

$$t^{\frac{2}{\gamma}} u_x \leqslant C, \quad 1<\gamma\leqslant 2, \ t>0, \tag{7}$$

where the constant $C>0$ depends on the initial value u_0 and γ (see Theorem 7). This estimate is a weak form of the Oleinik estimate for the solution of the convection–diffusion equation. For the comparison with the original Oleinik estimate (6), we may rewrite estimate (7) as

$$C^{-1}(\sqrt[\gamma]{t})^{2-\gamma} u_x \leqslant 1/t, \qquad 1<\gamma\leqslant 2, \ t>0. \tag{8}$$

Since $\max_x |u(x,t)| = O(1/\sqrt[\gamma]{t})$ (see [15]), these two estimates, (6) and (8), are almost equivalent for a large time $t\gg 1$. On the other hand, since $(\sqrt[\gamma]{t})^{2-\gamma}\to 0$ as $t\to 0$, we may say (8) is weaker than (6) for small $t>0$. Note that, for the Burgers equation $(\gamma = 2)$ we show that these two are identical with $C = 1$. Using this Oleinik-type estimate, we show our main result:

Theorem 1. *Let $u(x,t)$ be the solution of the convection–diffusion equation* (1), (2) *with $1<\gamma<2$. Then there exists a constant $0\leqslant\bar{p}\leqslant -\inf_\xi \int_{-\infty}^{\xi} u_0(\xi)\, d\xi$ such that*

$$||u(\cdot,t) - N_{\bar{p},\bar{p}+M}(\cdot,t)||_{L^1(R)}\to 0 \quad as \ t\to\infty, \tag{9}$$

where $M = \int u_0(x)\,dx$ and $N_{\bar{p},\bar{p}+M}(x,t)$ is the N-wave given by (5).

This papers is organized as follows. In Section 2, we introduce a technique to obtain the uniform estimates of similarity solutions to the inviscid problem which is based on the Oleinik estimate. This technique is modified in Section 3 to obtain the corresponding uniform estimates of similarity solutions of the convection–diffusion equation. The weak form of the Oleinik estimate (7) is obtained as one of the results of this process. In Section 4 we develop a technique that connects the solutions in similarity variables and in original ones by introducing an artificial time variable. Finally, using this technique and the uniform estimates in Section 3, the asymptotic convergence of the sign changing solution (Theorem 1) is proved.

2. The Oleinik estimate for inviscid problems

It is well-known that, if there is no diffusion, the nonlinearity in the convection equation,

$$u_t + \partial_x f(u) = 0, \quad u(x,0) = u_0(x), \quad \mu, t>0, \ x, u\in \boldsymbol{R}, \tag{10}$$

introduces a singularity to the solution even with a smooth initial value. Therefore, weak solutions are considered in this paper with an entropy admissibility condition that is

$$u(x-,t) \geqslant u(x+,t), \quad x \in \boldsymbol{R}, \ \ t>0. \tag{11}$$

Let $u(x,t)$ be the solution to the conservation law (2), (10) satisfying the entropy condition (11). It is also well known that the nonlinearity of the convection gives extra regularizing effects to the solution, which is reflected in the Oleinik estimate,

$$\partial_x f'(u)(\equiv (\gamma-1)|u|^{\gamma-2}u_x) \leqslant 1/t, \ \ \gamma>1, \ \ t>0 \qquad (6).$$

This estimate implies that the upper bound of $f''(u)u_x$ converges to zero having order $O(1/t)$ as $t \to \infty$. On the other hand its lower bound may break down in a finite time, and this is the mechanism of the shock appearance and makes the entropy condition (11) valid.

The Oleinik estimate has played a key role in the development of the theory of shock waves. We refer to Oleinik [20] for the uniqueness theorem of entropy solutions, Glimm and Lax [12] and Liu and Pierre [19] for the large time convergence to N-waves, and Escobedo et al. [7] for convection dominant convection–diffusion equations.

The similarity profile $g: R \to R$ of the convection equation (10) is defined by the relation $f'(g(x)) = x, x \in R$. Under the power law (2), it is given as

$$g(x) = \mathrm{sign}(x) \sqrt[\gamma-1]{|x|}, \quad x \in R. \tag{12}$$

Roughly speaking, the equality of the Oleinik estimate holds for a solution given by $u(x,t) = g(x/t)$. In the asymptotic convergence the similarity profile at the zero state is important, which is

$$\lim_{x \to 0} g'(x) = \begin{cases} 0, & 1<\gamma<2, \\ 1, & \gamma=2, \\ \infty, & \gamma>2. \end{cases} \tag{13}$$

Consider the similarity variables in (3). Then the rescaled function $w(\xi,s)$ is a weak solution to the transformed inviscid problem

$$\begin{aligned} & w_s + \frac{1}{\gamma}(|w|^\gamma - \xi w)_\xi = 0, \\ & \qquad\qquad\qquad\qquad\qquad \xi, s, w \in \boldsymbol{R}, \ \ \gamma>1, \\ & w(\xi,0) = u(\xi,1), \end{aligned} \tag{14}$$

and satisfies the same entropy condition

$$w(\xi-,s) \geqslant w(\xi+,s), \quad \xi \in \boldsymbol{R}. \tag{15}$$

Note that $w(\xi,0)$ is not the original initial value $u_0(\xi)$ after the change of variables. The new time variable $s = \ln t$ has values in $\boldsymbol{R}$ and $s=0$ corresponds to $t=1$.

The steady state of the inviscid problem (14), (15) is an N-wave which is a member of the two-parameter family of single variable functions

$$N_{p,q}(\xi) = \begin{cases} g(\xi), & -(\frac{\gamma p}{\gamma-1})^{\frac{\gamma-1}{\gamma}} < \xi < (\frac{\gamma q}{\gamma-1})^{\frac{\gamma-1}{\gamma}}, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where positive parameters p and q measure the area (or mass) of the negative and the positive humps of the steady state, respectively. If the total mass $M = \int N_{p,q}(\xi)\,d\xi (\equiv q - p)$ is prescribed, there is an one-parameter family $N_{p,p+M}$ corresponding to the mass M. This N-wave in similarity variables is simply a rescaled one of the classical N-wave in (5). Furthermore, after the transformation, the Oleinik estimate is also transformed to

$$\partial_\xi f'(w) = (\text{sign}(w)|w|^{\gamma-1})_\xi \leqslant 1. \tag{17}$$

Since $f'(g(\xi)) = \xi$, we can easily see that the N-wave is the special solution that the equality in the transformed Oleinik estimate holds. We note here that the time variable has been disappeared after the change of variables. It simplifies the computations considerably and helps us to focus on the main issue of the phenomena. This is the reason we insist to use the similarity variables in the computation. We can convert the results in the similarity variables to the original ones whenever we want.

Lemma 2. *Suppose that a function $w(\xi, s)$ satisfies the Oleinik estimate* (17) *and that $w(z, s) = g(z - \xi_0)$ for a point $z \in \boldsymbol{R}$. Then,*

$$w(\xi, s) \geqslant g(\xi - \xi_0) \quad \textit{for } \xi < z,$$
$$w(\xi, s) \leqslant g(\xi - \xi_0) \quad \textit{for } \xi > z. \tag{18}$$

Furthermore, if $1 < \gamma < 2$ and $w(\xi_0, s) = 0$, then $w_\xi(\xi_0, s) \leqslant 0$.

Proof. We can easily check that $\partial_\xi f'(\bar{w}) = 1$ for the function $\bar{w}(\xi) = g(\xi - \xi_0)$. Since $\partial_\xi f'(w) \leqslant \partial_\xi f'(\bar{w})$ and $f'(w(z,s)) = f'(\bar{w}(z,s))$, we have

$$f'(w(\xi, s)) \geqslant f'(\bar{w}(\xi, s)) \quad \text{for } \xi < z,$$
$$f'(w(\xi, s)) \leqslant f'(\bar{w}(\xi, s)) \quad \text{for } \xi > z.$$

The convexity of the convection, $f''(u)$, implies that $f'(u)$ is an increasing function and, hence, (18) is obtained. Since $g'(0) = 0$ for $1 < \gamma < 2$, the estimate (18) implies that $w_\xi(\xi_0, s) \leqslant 0$. $\square$

Remark 3. Let $w(\xi, s)$ be the solution of (14) that satisfies the entropy condition (15). Then, since the solution w satisfies the (transformed) Oleinik estimate (17), we may apply Lemma 2. If the estimates are transformed to the original variables, we may conclude that, if $u(z,t) = g((z-x_0)/t)$ for $z \in R$,

$$u(x,t) \geqslant g((x-x_0)/t) \quad \text{for } x<z,$$

$$u(x,t) \leqslant g((x-x_0)/t) \quad \text{for } x>z,$$

and that, if $1<\gamma<2$ and $u(x_0,t)=0$, $\partial_x u(x_0,t) \leqslant 0$.

Next we introduce two functions of integrals of the similarity solution $w(\xi,s)$ that are

$$W_-(\xi,s) = \int_{-\infty}^{\xi} w(\zeta,s)\,d\zeta \ \ (\equiv W(\xi,s)\ ),$$
$$W_+(\xi,s) = \int_{\xi}^{\infty} w(\zeta,s)\,d\zeta = M_0 - W(\xi,s). \tag{19}$$

Then W satisfies a Hamilton–Jacobi-type equation

$$W_s + \frac{1}{\gamma}(\operatorname{sign}(W_\xi)\,|W_\xi|^{\gamma-1} - \xi)\,W_\xi = 0, \tag{20}$$

and two quantities,

$$p = -\inf_{\xi}\ W(\xi,s) = -\inf_{x}\ \int_{-\infty}^{x} u(y,e^s)\,dy,$$
$$q = \sup_{\xi}\ W_+(\xi,s) = \sup_{x}\ \int_{x}^{\infty} u(y,e^s)\,dy, \tag{21}$$

are the invariant constants of the problem (see [18]). In the following lemma we show that the similarity solution $w(\xi,s)$ is uniformly bounded. In the proof of the lemma we introduce a technique based on the similarity profile $g(x)$ and the Oleinik estimate. This technique is developed in the next section to show the uniform estimate of the solution to (1).

Lemma 4. *Let $w(\xi,s)$ be the entropy solution of* (14) *and $W(\xi,s)$ be its integral given by* (19). *Then W and w are uniformly bounded by*

$$-A \leqslant W(\xi,s) \leqslant B, \tag{22}$$

$$|w(\xi,s)| \leqslant \sqrt[\gamma]{\frac{\gamma(A+B)}{\gamma-1}}, \tag{23}$$

where $A = -\inf_\xi W(\xi,0)$ and $B = \sup_\xi W(\xi,0) \geqslant 0$.

Proof. Note that the constant A is one of two invariant constants in (21) (i.e., $A = p$), but B is not the other one in general. Estimate (22) simply follows from the maximum principle for Hamilton–Jacobi equations. Now we show the uniform estimate for $w(\xi, s)$. Let $w_0 = w(z, s)$ for a given point $z \in \boldsymbol{R}$. To show the upper bound of $w(\xi, s)$ we consider $w_0 \geqslant 0$. Let $\xi_0 = z - w_0^{\gamma-1}$. Then $w(z, s) = g(z - \xi_0)$ and (18) implies that

$$0 \leqslant \int_{\xi_0}^{z} w(\xi, s) - g(\xi - \xi_0)\ d\xi = \int_{\xi_0}^{z} w(\xi, s)\, d\xi - \frac{\gamma - 1}{\gamma} w_0^{\gamma}.$$

So w_0 is bounded by

$$w_0 \leqslant \sqrt[\gamma]{\frac{\gamma}{\gamma - 1} \int_{\xi_0}^{z} w(\xi, s)\, d\xi}. \tag{24}$$

For any $a, b \in \boldsymbol{R}$, we have

$$\int_{a}^{b} w(\xi, s)\, d\xi = \int_{-\infty}^{b} w(\xi, s)\, d\xi - \int_{-\infty}^{a} w(\xi, s)\, d\xi \leqslant \sup_{\xi} W(\xi, s) - \inf_{\xi} W(\xi, s).$$

Since the right-hand side is $A + B$, we obtain the upper bound for $w(\xi, s)$, i.e.,

$$w(\xi, s) \leqslant \sqrt[\gamma]{\frac{\gamma(A + B)}{\gamma - 1}}.$$

Let $w_0 = w(z, s) < 0$ and $\xi_0 = z + |w_0|^{\gamma-1}$. Then $w(z, s) = g(z - \xi_0)$ and (18) implies that

$$0 \geqslant \int_{z}^{\xi_0} w(\xi, s) - g(\xi - \xi_0)\, d\xi = \int_{z}^{\xi_0} w(\xi, s)\, d\xi + \frac{\gamma - 1}{\gamma} |w_0|^{\gamma}.$$

So w_0 is uniformly bounded below by

$$w_0 \geqslant -\sqrt[\gamma]{\frac{-\gamma}{\gamma - 1} \int_{z}^{\xi_0} w(\xi, s)\, d\xi} \geqslant -\sqrt[\gamma]{\frac{\gamma(A + B)}{\gamma - 1}}.$$

Since the choice of $z \in R$ is arbitrary, the uniform estimate (23) holds. $\square$

The uniform estimate of the transformed solution $w(\xi, s)$ is transformed to the original variables

$$|u(x, t)| \leqslant \sqrt[\gamma]{\frac{\gamma(A + B)}{\gamma - 1}}\ t^{-\frac{1}{\gamma}}. \tag{25}$$

The decay rate $O(t^{-1/\gamma})$ is exact. Considering the N-wave like solutions, we can see that the coefficient part is also optimal.

The main tool to obtain the asymptotic convergence for the inviscid problem is the method of characteristics. Since this method is not applicable under the presence of diffusion, we do not pursue the convergence in this paper. We refer readers to [12,15,18] for the asymptotic convergence of inviscid cases.

3. An Oleinik-type estimate for viscous problems

Now we consider the solution $w(\xi, s; \mu)$ (or simply $w(\xi, s)$) to the transformed convection–diffusion equation,

$$\begin{aligned} w_s + \left(\frac{1}{\gamma}|w|^{\gamma} - \frac{1}{\gamma}\xi w\right)_{\xi} &= \mu e^{(\gamma-2)s/2}\, w_{\xi\xi}, \\ w(\xi, 0) &= u(\xi, 1), \end{aligned} \tag{26}$$

where $\xi, s, w \in \boldsymbol{R}$, $\mu > 0$ and $1 < \gamma \leqslant 2$.

Remark 5. In general a sign changing solution to the convection–diffusion equation (26) with $1 < \gamma < 2$ does not satisfy the Oleinik estimate (17). Suppose that $w(\xi, s)$ is the solution of the problem with a special initial value $w(\xi, 0) = \xi$. Differentiate Eq. (26) with respect to ξ variable. Then, after setting $z(\xi, s) = w_\xi(\xi, s)$, we obtain

$$\mu e^{(\gamma-2)s/2}\, z_{\xi\xi} - \frac{1}{\gamma}(\gamma\ \mathrm{sign}(w)|w|^{\gamma-1} - \xi)z_\xi - \frac{1}{\gamma}(\gamma(\gamma-1)|w|^{\gamma-2}z - 2)z - z_s = 0$$

with its initial value $z(\xi, 0) = 1$. Clearly, there exists a zero point $\xi_0(s) \in R$ such that $w(\xi(s), s) = 0$. Then, if $w(\cdot, s)$ satisfies the Oleinik estimate, then $z(\xi_0, s) = w_\xi(\xi_0, s) \leqslant 0$ from Lemma 2, which contradicts to the maximum principle for parabolic equations (for example, see [11], Theorem 3 in Chapter 2). Hence, the Oleinik estimate does not hold for sign changing solutions of the convection–diffusion equation. Without the diffusion the solution would have the similarity profile like structure which is flat at the sign changing points. But, under the effect of the diffusion, the solution cannot be so flat.

Since the solution $w(\xi, s)$ does not satisfy the Oleinik estimate (17), we may not apply the technique in the proof of Lemma 4. In the followings we modify the technique and obtain uniform estimates of $w(\xi, s)$ and its derivative $w_\xi(\xi, s)$. As a result we obtain a weaker form of the Oleinik estimate which is satisfied by the solutions of the convection–diffusion equation. Recall that (26) arises from the similarity transformation (3) and is set on $\boldsymbol{R} \times \boldsymbol{R}$. We note that the estimates in this section are independent of μ and s.

Lemma 6. *Let $w(\xi,s)$ be the solution to the Cauchy problem* (26). *If $w(\xi,s)$ is uniformly bounded for $s<S<\infty$, i.e.,*

$$|w(\xi,s)|<M, \quad s<S, \tag{27}$$

then

$$w_\xi \leqslant \frac{2}{\gamma(\gamma-1)} M^{2-\gamma}, \quad s<S. \tag{28}$$

Proof. Differentiate Eq. (26) with respect to ξ and obtain

$$z_s + \frac{1}{\gamma}(\gamma\ \mathrm{sign}(w)|w|^{\gamma-1} - \xi)z_\xi + \frac{z}{\gamma}(\gamma(\gamma-1)|w|^{\gamma-2}z - 2) = \mu e^{(\gamma-2)s/2} z_{\xi\xi},$$

where $z = w_\xi$. If z has an interior maximum at $(\xi,s), s<S$, then $z_s = z_\xi = 0$ and $z_{\xi\xi}<0$ at the point, and, therefore,

$$z(\gamma(\gamma-1)|w|^{\gamma-2}z - 2) \leqslant 0. \tag{29}$$

Since $\gamma \leqslant 2$ and w is bounded by (27), we may conclude that (28) holds at interior maximum points. If z has its maximum at the final time level $s=S$, then $z_s \geqslant 0$, $z_\xi = 0, z_{\xi\xi} \leqslant 0$ at the point and (29) holds at the point and we obtain (28) at the maximum point again. Since

$$w_\xi(\xi, \ln t) = \sqrt[\gamma]{t^2}\, u_x(\sqrt[\gamma]{t}\,\xi, t), \tag{30}$$

$\lim_{s\to-\infty} w_\xi(\xi,s) = 0$ for a smooth initial value u_0 and (28) follows on the whole domain. If the initial value is not smooth, we may approximate it by smooth functions in a standard way and conclude (28) by a density argument. □

In the following we show the uniform boundedness of the solution w, which is assumed in Lemma 6. Consider

$$\begin{aligned} W_-(\xi,s) &= \int_{-\infty}^{\xi} w(\zeta,s)\,d\zeta \ \ (\equiv W(\xi,s)),\\ W_+(\xi,s) &= \int_{\xi}^{\infty} w(\zeta,s)\,d\zeta = M_0 - W(\xi,s). \end{aligned} \tag{31}$$

Then $W(\xi,s)$ satisfies a viscous Hamilton–Jacobi equation

$$W_s + \frac{1}{\gamma}(\mathrm{sign}(W_\xi)\,|W_\xi|^{\gamma-1} - \xi)\,W_\xi = \mu e^{(\gamma-2)s/2}\,W_{\xi\xi}. \tag{32}$$

In this case the quantities in (21) are functions of the time variable s, i.e.,

$$\begin{aligned} p(s) &= -\inf_{\xi} \; W(\xi, s), \\ q(s) &= \sup_{\xi} \; W_+(\xi, s) = M_0 + p(s) \end{aligned} \tag{33}$$

are not constant anymore.

Theorem 7. *Let $w(\xi, s)$ be the solution of* (26), *W be its integral given by* (31) *and*

$$A = -\inf_x \int_{-\infty}^{x} u_0(x)\, dx, \quad B = \sup_x \int_{-\infty}^{x} u_0(x)\, dx.$$

Then W, w and w_ξ are uniformly bounded by

$$-A \leqslant W(\xi, s) \leqslant B, \tag{34}$$

$$|w(\xi, s)| \leqslant \sqrt[\gamma]{\frac{4(A+B)}{\gamma(\gamma-1)}}, \tag{35}$$

$$w_\xi(\xi, s) \leqslant \frac{1}{2(A+B)} \left(\frac{4(A+B)}{\gamma(\gamma-1)} \right)^{\frac{2}{\gamma}}. \tag{36}$$

Proof. Estimate (34) follows from the maximum principle for Hamilton–Jacobi equations. Suppose that

$$\sqrt[\gamma]{\frac{4(A+B)}{\gamma(\gamma-1)}} < \sup_{0<s<S, \xi \in \boldsymbol{R}} w(\xi, s). \tag{37}$$

Clearly, the supremum $M \equiv \sup w(\xi, s)$ over a finite time domain $0<s<S$ is finite. For any $0 \leqslant M_1 < M$, there exist $\xi_1 \in \boldsymbol{R}$ and $s<S$ such that $w(\xi_1, s) = M_1$. Since $w_\xi \leqslant \frac{2}{\gamma(\gamma-1)} M^{2-\gamma}$ from Lemma 6, the function $w(\xi, s) - (\frac{2}{\gamma(\gamma-1)} M^{2-\gamma}(\xi - \xi_1) + M_1)$ is decreasing. Let $\xi_0 = \xi_1 - \frac{\gamma(\gamma-1)}{2} M_1 M^{\gamma-2}$. Then, we have

$$\begin{aligned} 0 &\leqslant \int_{\xi_0}^{\xi_1} \left(w(\xi, s) - \left(\frac{2}{\gamma(\gamma-1)} M^{2-\gamma}(\xi - \xi_1) + M_1 \right) \right) d\xi \\ &= \int_{\xi_0}^{\xi_1} w(\xi, s)\, d\xi - \frac{\gamma(\gamma-1) M_1^2 M^{\gamma-2}}{4}. \end{aligned}$$

Using estimate (34), we obtain

$$W(\xi_1, s) = W(\xi_0, s) + \int_{\xi_0}^{\xi_1} w(\xi, s)\, d\xi > -A + \frac{\gamma(\gamma-1)M_1^2 M^{\gamma-2}}{4} \tag{38}$$

for any $M_1 < M$. Finally, from assumption (37), we obtain

$$W(\xi_1, s) \geqslant -A + \frac{\gamma(\gamma-1)M^\gamma}{4} > B,$$

which violates (34). Since this contradiction is from assumption (37), we may conclude that $w(\xi, s) \leqslant \sqrt[\gamma]{4(A+B)/\gamma(\gamma-1)}$ for any $\xi \in \boldsymbol{R}$ and $s < S$. Furthermore, since the upper bound is independent from $S > 0$, it is the upper bound for all $s < \infty$.

If $\inf w(\xi, s) < -\sqrt[\gamma]{4(A+B)/\gamma(\gamma-1)}$ is assumed, then similar arguments lead to a same kind of contradiction and, hence, (35) holds. Estimate (36) comes from (28) and (35). □

Remark 8. Compare the uniform estimate (35) with the one for the inviscid problem (23). For the Burgers equation, $\gamma = 2$, both of the upper bounds are identical. For $1 < \gamma < 2$, our upper bound for the viscous problem is bigger than the one for the inviscid problem. This is not for the actual decay speed, but for the technical difficulty of the viscous case. In fact, the asymptotic limit in the Theorem 1 shows that the solution of the viscous problem decays faster in the sense that the asymptotic limit for the viscous problem is smaller than the one for the inviscid problem.

Remark 9. The uniform estimate (36) for the upper bound of $w_\xi(\xi, s)$ can be written as

$$t^{\frac{2}{\gamma}} u_x \leqslant C \equiv \frac{1}{2(A+B)} \left(\frac{4(A+B)}{\gamma(\gamma-1)} \right)^{\frac{2}{\gamma}}. \tag{39}$$

This estimate is the counter part of the Oleinik estimate for the solution of convection–diffusion equation (1), (2) with $1 < \gamma \leqslant 2$. If $\gamma = 2$, the equation is called the Burgers equation, and we can easily check that the constant in the estimate is $C = 1$. Hence, it is identical to the Oleinik estimate. For $\gamma < 2$, estimate (39) can be considered as a weaker form of the Oleinik estimate which holds for the solutions to the convection–diffusion equations.

4. Long time behavior of the convection–diffusion equation

In this section, we prove our main result that the solution of the convection–diffusion equation (26) with $1 < \gamma < 2$ converges to an N-wave asymptotically, which is a steady state of the inviscid problem (14). First we show the existence of a convergent subsequence and the basic structures of its limit.

Lemma 10. *Let $w(\xi, s)$ be the solution of the convection–diffusion equation* (26) *with* $1<\gamma<2$. *Then there exist a sequence s_k and a function $\bar{w}(\xi)$ such that $w(\xi, s_k) \to \bar{w}(\xi)$ as $s_k \to \infty$ for any $\xi \in \boldsymbol{R}$. Furthermore,* $\inf_\xi W(\xi, s_k) \to -\bar{p}$ *as $s_k \to \infty$ for a nonnegative constant* $\bar{p} \geqslant 0$, *and*

$$\int \bar{w}(\zeta)\, d\zeta = \lim_{s_k \to \infty} \int w(\zeta, s_k)\, d\zeta = M, \tag{40}$$

$$\inf_\xi \int_{-\infty}^{\xi} \bar{w}(\zeta)\, d\zeta = \lim_{s_k \to \infty} \inf_\xi \int_{-\infty}^{\xi} w(\zeta, s_k)\, d\zeta = -\bar{p}, \tag{41}$$

where $M = \int u_0(x)\, dx$ $(= \int w(\xi, s)\, d\xi$ *for all* $s \in \boldsymbol{R})$. *The convergence is uniform on any closed interval on which $\bar{w}$ is continuous.*

Proof. Since $w_\xi(\xi, s)$ is uniformly bounded from the above as in (36), $w(\xi, s) - C\xi$ is a decreasing function with $C = \frac{1}{2(A+B)}(\frac{4(A+B)}{\gamma(\gamma-1)})^{2/\gamma}$. Furthermore, since $|w(\xi, s) - C\xi|$ is uniformly bounded on a closed interval $[-N, N]$ for any $N \in \boldsymbol{R}^+$, the Helly's selection theorem implies that there exist a sequence s_k and a limit function $\bar{w}(\xi)$ such that $w(\xi, s_k) \to \bar{w}(\xi)$ for $\xi \in [-N, N]$ as $s_k \to \infty$. By taking a subsequence of s_k using classical diagonal arguments, if needed, we obtain a subsequence s_k and a limit function $\bar{w}(\xi)$ such that

$$w(\xi, s_k) \to \bar{w}(\xi) \quad \text{as } s_k \to \infty$$

for any $\xi \in \boldsymbol{R}$.

The maximum principle for the Hamilton–Jacobi equation (32) implies that the infimum of $W(\xi, s)$ (i.e., $\inf_\xi \int_{-\infty}^{\xi} w(\zeta, s)\, d\zeta$) increases as $s \to \infty$. Since the infimum is bounded above by the zero value, there exists $\bar{p} \geqslant 0$ such that $\inf_\xi \int_{-\infty}^{\xi} w(\xi, s)\, d\xi \to -\bar{p}$ as $s \to \infty$.

Now we show a claim which is the main part of the proof

Claim. *For any given* $\varepsilon>0$, *there exist* $k, s_0>0$ *such that*

$$-\varepsilon < \int_{\xi<-k} w(\xi, s)\, d\xi < \varepsilon, \quad s<s_0. \tag{42}$$

We show this claim constructing a solution to the heat equation as a super solution. Consider the solution $\psi(x, t)$ of the heat equation

$$\psi_t = \mu\psi_{xx}, \qquad \psi(x, 0) = u_0(x),$$

which has the same initial value as the one of problem (1). Let $h(\xi, s) = \sqrt[\gamma]{t}\psi(x, t)$ be the similarity transformation given by the variables in (3) and $H(\xi, s)$ be its integral $H(\xi, s) = \int_{-\infty}^{\xi} h(\zeta, s)\, d\zeta$. Then, from the explicit formula for the solution of the heat equation, $h(\xi, s)$ is given explicitly by

$$h(\xi, s) = \frac{1}{\sqrt{4\pi\mu}} \int_{-\infty}^{\infty} \frac{\sqrt[\gamma]{t}}{\sqrt[\gamma]{t}} e^{(\frac{\sqrt[\gamma]{t}}{\sqrt[2]{t}}(\xi-\zeta))^2/4\mu}\, \sqrt[\gamma]{t}\, u_0(\sqrt[\gamma]{t}\zeta)\, d\zeta, \quad s = \ln t.$$

We can easily check that, for $\gamma<2$, $h(\xi, s) \to M\delta(\xi)$ as $s \to \infty$, $h(\xi, 0) = w(\xi, 0)$, $H(\xi, s)$ converges to the heavy side function with the weight M and that $H(\xi, s)$ is a solution of

$$\mu e^{(\gamma-2)s/2}\, H_{\xi\xi} + \frac{1}{\gamma}\xi H_{\xi} - H_s = 0, \quad H(\xi, 0) = \int_{-\infty}^{\xi} h(\xi, 0)\, d\xi. \tag{43}$$

Subtracting (43) from (32), we may check that $U(\xi, s) \equiv W(\xi, s) - H(\xi, s)$ satisfies

$$\mu e^{(\gamma-2)s/2}\, U_{\xi\xi} + \frac{1}{\gamma}\xi U_{\xi} - U_s = \frac{1}{\gamma}|W_{\xi}|^{\gamma} \geqslant 0.$$

Since $U(\xi, 0) = 0$, the maximum principle implies that $W(\xi, s) \leqslant H(\xi, s)$. The upper bound of (42) is now clear since $H(\xi, s) \to 0$ as $s \to \infty$ for any $\xi<0$.

Let $c = (\frac{4(A+B)}{\gamma(\gamma-1)})^{\frac{\gamma-1}{\gamma}}$. Then the uniform estimate for $w(\xi, s)$ in Theorem 7 implies that $|W_{\xi}|^{\gamma-1} \leqslant c$. The Hamilton–Jacobi equation (32) for $W(\xi, s)$ is rewritten as

$$\begin{aligned} W_s(\xi - c, s) &+ \frac{1}{\gamma}(\mathrm{sign}(W_{\xi}(\xi - c, s))|W_{\xi}(\xi - c, s)|^{\gamma-1} \\ &- (\xi - c))W_{\xi}(\xi - c, s) = \mu e^{(\gamma-2)s/2} W_{\xi\xi}(\xi - c, s). \end{aligned}$$

Consider a translation $W_c(\xi, s) = W(\xi - c, s)$ and a domain $D = \{(\xi, s) : w(\xi, s)<0, s>0\}$. Then W_c satisfies

$$\begin{aligned} &\mu e^{(\gamma-2)s/2}(W_c)_{\xi\xi} + \frac{1}{\gamma}\xi(W_c)_{\xi} - (W_c)_s \\ &\quad = \frac{1}{\gamma}(\mathrm{sign}(W_{\xi}(\xi - c, s)))|W_{\xi}(\xi - c, s)|^{\gamma-1} + c)W_{\xi}(\xi - c, s) \leqslant 0 \end{aligned}$$

for all $(\xi - c, s) \in D$. So $U_c = W_c - H$ satisfies

$$\mu e^{(\gamma-2)s/2}(U_c)_{\xi\xi} + \frac{1}{\gamma}\xi(U_c)_{\xi} - (U_c)_s \leqslant 0.$$

The maximum principle implies that $W_c(\xi, s) \geqslant H(\xi, s)$ for all $(\xi - c, s) \in D$. For any given $\xi_0>0$ and $\varepsilon>0$, there exists $s_0>0$ such that $H(\xi, s)> -\varepsilon$ for all $s>s_0, \xi< -$

ξ_0. So we have $W(\xi - c, s) \geqslant -\varepsilon$ for all $s > s_0, \xi < -\xi_0$ and $(\xi - c, s) \in D$. Since $W(\xi, s)$ has its infimum in the domain $w(\xi, s) \leqslant 0$, we may conclude that $W(\xi, s) \geqslant -\varepsilon$ for all $s > s_0, \xi < -(\xi_0 + c)$. Therefore, the lower bound in (42) holds with $k = c + \xi_0$. The proof for the claim is now complete.

In a similar way we may show $|\int_{\xi > k} w(\xi, s) d\xi| < \varepsilon$ and, hence,

$$\left| \int_{|\xi| > k} w(\xi, s)\, d\xi \right| < 2\varepsilon, \quad \text{for all } s > s_0.$$

Applying the Lebesgue's convergence theorem and the Fatou's lemma, we obtain

$$\int_{|\xi| < k} \bar{w}(\xi)\, d\xi = \lim_{s_k \to \infty} \int_{|\xi| < k} w(\xi, s_k)\, d\xi = M - \lim_{s_k \to \infty} \int_{|\xi| > k} w(\xi, s_k)\, d\xi,$$

$$\left| \int_{|\xi| > k} \bar{w}(\xi)\, d\xi \right| \leqslant \lim_{s_k \to \infty} \left| \int_{|\xi| > k} w(\xi, s_k)\, d\xi \right| \leqslant 2\varepsilon.$$

From these two relations and the trivial one $\int \bar{w}(\xi)\, d\xi = \int_{|\xi| < k} \bar{w}(\xi)\, d\xi + \int_{|\xi| > k} \bar{w}(\xi)\, d\xi$, we obtain

$$\left| \int \bar{w}(\xi)\, d\xi - M \right| < 4\varepsilon \quad \text{for any } \varepsilon > 0.$$

Hence the equality in (40) holds.

Eq. (40) implies that the limiting process and the integration are inter-changeable and (41) is clear from it. Note that the Helly's selection theorem also implies that the convergence of the subsequence is uniform on any closed interval if the limit $\bar{w}$ is continuous on it. □

The next step is to show that $\bar{w}(\xi)$ is an N-wave. Then (40) and (41) decide the limit $\bar{w}(\xi)$ independently from the choice of the subsequence s_k, and this implies the asymptotic convergence of the solution $w(\cdot, s)$ to the N-wave. To complete this mission we need the regularity of $w_s(\xi, s_k) \to 0$ as $s_k \to \infty$. Obtaining such a regularity is one of main issues in various asymptotic analysis. In the follows we introduce an extra variable and consider an one parameter family of equations. Then we show the corresponding limits are connected via the new variable satisfying certain relation. The basic idea of this technique has been introduced in [7], and we present it using similarity variables.

Let $u(x, t)$ be the solution of the convection–diffusion equation (1). Consider a transformed function

$$v(\xi, s, \tau) = e^{s/\gamma} u(e^{s/\gamma} \xi, e^s \tau), \quad s \in \boldsymbol{R}, \ \tau \in \boldsymbol{R}^+. \tag{44}$$

We can easily check that

$$v_\tau = e^{s+s/\gamma} u_t,$$

$$v_s = \frac{1}{\gamma}(\xi v)_\xi + \tau e^{s+s/\gamma} u_t,$$

$$v_{\xi\xi} = e^{3s/\gamma} u_{xx},$$

$$\partial_\xi(|v|^\gamma) = e^{s+s/\gamma}\partial_x(|u|^\gamma).$$

Now we may rewrite the convection–diffusion equation in two different ways. For a fixed $s \in \boldsymbol{R}$, (1) is written as

$$v_\tau + \frac{1}{\gamma}(|v|^\gamma)_\xi = \mu e^{(\gamma-2)s/\gamma} v_{\xi\xi}, \quad v(\xi, 0) = e^{s/\gamma} u_0(e^{s/\gamma}\xi, 0), \tag{45}$$

and, for a fixed $\tau > 0$, it is written as

$$v_s + \frac{1}{\gamma}(\tau|v|^\gamma - \xi v)_\xi = \tau\mu e^{(\gamma-2)s/\gamma} v_{\xi\xi}, \quad v(\xi, 0) = u(\xi, \tau). \tag{46}$$

We may easily check that $w(\xi, s) = v(\xi, s, 1)$, and similarity Equations (26) and (46) are identical for $\tau = 1$. It is clear that the only possible steady state for (45) is the trivial solution $v = 0$. On the other hand, steady states for the transformed problem (46) are N-waves given by

$$N_{p,q}(\xi,\tau) = \begin{cases} g(\xi/\tau), & -(\frac{\gamma p}{\gamma-1})^{\frac{(\gamma-1)}{\gamma}} < \xi/\tau < (\frac{\gamma q}{\gamma-1})^{\frac{(\gamma-1)}{\gamma}}, \\ 0 & \text{otherwise}, \end{cases} \tag{47}$$

where the similarity profile $g(x)$ is given by (12). The key observation is that $N_{p,q}(\xi,\tau)$ is a solution of the inviscid ($\mu = 0$) problem corresponding to (45) with $N_{p,q}(\xi,\tau) \to (q-p)\delta(\xi)$ as $\tau \to 0$. In fact it is identical to the original N-wave (5) of the inviscid problem.

The solution $v(\xi, s, \tau)$ satisfies the corresponding Oleinik-type estimate. Setting $z = v_\xi(\xi, s, \tau)$ with a fixed $\tau > 0$, we obtain

$$z(\gamma(\gamma-1)\tau|v|^{\gamma-2}z - 2) \leqslant 0,$$

which corresponds to (29). So, under the assumption $|v| \leqslant M$, we obtain $v_\xi \leqslant \frac{2}{\gamma(\gamma-1)\tau} M^{2-\gamma}$. If it is assumed that $v(\xi, s, \tau) > \sqrt[\gamma]{\frac{4(A+B)}{\gamma(\gamma-1)\tau}}$, then we may derive a similar contradiction as the one in the proof of Theorem 7. In the following lemma we write down the estimates.

Lemma 11. *Let $u(x,t)$ be the solution of the convection–diffusion equation* (1) *with $1 < \gamma \leqslant 2$ and $v(\xi, s, \tau)$ be its transformation given by* (44), *which is the solution of* (45) *and* (46). *Then* $v(\xi, s, \tau)$ *and* $v_\xi(\xi, s, \tau)$ *are bounded by*

$$|v(\xi, s, \tau)| \leqslant \sqrt[\gamma]{\frac{4(A+B)}{\gamma(\gamma-1)\tau}}, \tag{48}$$

$$v_\xi(\xi, s, \tau) \leqslant \frac{1}{2(A+B)}\left(\frac{4(A+B)}{\gamma(\gamma-1)\tau}\right)^{\frac{2}{\gamma}}. \tag{49}$$

For a fixed $\tau > 0$, the estimates in Lemma 11 are uniform with respect to variables ξ and s. Since $w(\xi, s) = v(\xi, s, 1)$, we are interested in the domain of $\tau \leqslant 1$ and the lemma gives the uniform estimate over $\tau \in [\tau_0, \tau_1]$ for $\tau_0 > 0$, which is used in the following lemma.

Lemma 12. *Let* $u(x, t)$ *be the solution of* (1) *with* $1 < \gamma \leqslant 2$ *and* $v(\xi, s, \tau)$ *be its transformation given by* (44). *Then there exist a sequence* s_k *and a function* $\bar{v}(\xi, \tau)$ *such that* $v(\xi, s_k, \tau) \to \bar{v}(\xi, \tau)$ *as* $s_k \to \infty$ *for any* $\xi \in \boldsymbol{R}$ *and* $\tau > 0$. *Furthermore,* $\bar{v}(\xi, \tau)$ *is the entropy solution of the inviscid problem* (10) *with its initial value* $\bar{v}(\xi, 0) = M\delta(\xi)$, $M = \int u_0(x)\, dx$.

Proof. Since $v(\xi, s, \tau)$ is uniformly bounded by $\sqrt[\gamma]{\frac{4(A+B)}{\gamma(\gamma-1)\tau_0}}$ and $v_\xi(\xi, s, \tau)$ is uniformly bounded from the above by $C = \frac{1}{2(A+B)}(\frac{4(A+B)}{\gamma(\gamma-1)\tau_0})^{\frac{2}{\gamma}}$ for $\tau \geqslant \tau_0 > 0$, $w(\xi, s) - C\xi$ is a decreasing function with respect to the ξ variable and uniformly bounded over any bounded interval $[-N, N]$. Hence, the Helly's selection theorem implies that we may take a sequence s_k and a limit function $\bar{v}(\xi, \tau)$ such that $v(\xi, s_k, \tau) \to \bar{v}(\xi, \tau)$ for any $(\xi, \tau) \in [-N, N] \times [\tau_0, \tau_1]$ for given $\tau_0 > 0, N > 0$. By taking a subsequence of s_k using classical diagonal arguments, if needed, we may assume that $v(\xi, s_k, \tau) \to \bar{v}(\xi, \tau)$ as $s \to \infty$ for any $\xi \in \boldsymbol{R}$ and $\tau > 0$.

Multiply a uniformly bounded test function $\phi(\xi)$ to (45) and integrate it over $\boldsymbol{R} \times (\tau_0, \tau_1)$ to obtain

$$\begin{aligned} &\int (v(\xi, s, \tau_1) - v(\xi, s, \tau_0))\phi(\xi)\, d\xi - \int \int_{\tau_0}^{\tau_1} \frac{1}{\gamma}(|v|^\gamma)\phi'(\xi)\, d\tau\, d\xi \\ &\quad = \int \int_{\tau_0}^{\tau_1} \mu e^{(\gamma-2)s/\gamma} v\phi''(\xi)\, d\tau\, d\xi. \end{aligned} \tag{50}$$

Taking $s \to \infty$ limit through the subsequence s_k, we obtain

$$\int (\bar{v}(\xi, \tau_1) - \bar{v}(\xi, \tau_0))\phi(\xi)\, d\xi - \int \int_{\tau_0}^{\tau_1} \frac{1}{\gamma}(|\bar{v}(\xi, \tau)|^\gamma)\phi'(\xi)\, d\tau\, d\xi = 0. \tag{51}$$

So $\bar{v}$ is a weak solution of the inviscid problem (10). The uniform estimate (49) for any fixed $\tau > 0$ implies that the limit $\bar{v}(\xi, \tau)$ satisfies the entropy condition.

Now we verify the initial value of $\bar{v}$ and complete the proof. The transformation (44) shows that $\int v(\xi,s,0)\phi(\xi)\,d\xi \to M\phi(0)$ as $s\to\infty$. Using this relation, we may repeat the same procedure as the above one with $\tau_0 = 0$ and obtain

$$\left|\int \bar{v}(\xi,\tau_1)\phi(\xi)\,d\xi - M\phi(0)\right| = \int\int_0^{\tau_1} \frac{1}{\gamma}(|\bar{v}|^\gamma)\phi'(\xi)\,d\tau\,d\xi$$
$$\leqslant \frac{1}{\gamma}||\phi'(\xi)||_\infty \int_0^{\tau_1} ||\bar{v}(\cdot,\tau)||_1\,||\bar{v}(\cdot,\tau)||_\infty^{\gamma-1}\,d\tau. \tag{52}$$

Since $||\bar{v}(\cdot,\tau)||_1 \leqslant ||u_0||_1$ and $||\bar{v}(\cdot,\tau)||_\infty$ is uniformly bounded by (48), we obtain

$$\left|\int \bar{v}(\xi,\tau_1)\phi(\xi)\,d\xi - M\phi(0)\right| \leqslant C\int_0^{\tau_1} \tau^{\frac{1-\gamma}{\gamma}}\,d\tau = \gamma C\tau_1^{\frac{1}{\gamma}} \tag{53}$$

for a constant $C>0$. Since the right-hand side of (53) converges to zero as $\tau_1\to 0$, we may conclude that the corresponding initial value for the solution $\bar{v}(\xi,\tau)$ is $M\delta(\xi)$. □

Now we prove our main result together with the pointwise convergence in similarity variables as a corollary of previous results.

Theorem 1. *Let $u(x,t)$ be the solution of the convection–diffusion equation* (1), (2) *with $1<\gamma<2$ and $w(\xi,s)$ be its similarity transformation given by* (3). *Then there exists a constant $0\leqslant\bar{p}\leqslant -\inf_\xi \int_{-\infty}^{\xi} u_0(\xi)\,d\xi$ such that*

$$w(\xi,s)\to N_{\bar{p},\bar{p}+M}(\xi) \quad \text{as } s\to\infty, \tag{54}$$

$$||u(\cdot,t) - N_{\bar{p},\bar{p}+M}(\cdot,t)||_{L^1} \to 0 \quad \text{as } t\to\infty, \tag{55}$$

where $M=\int u_0(x)\,dx$, and $N_{\bar{p},\bar{p}+M}(x,t)$ and $N_{\bar{p},\bar{p}+M}(\xi)$ are N-waves given by (5) and (16), *respectively.*

Proof. We may take the sequence s_k in Lemma 12 as a subsequence of the one in Lemma 10. The uniqueness of the entropy solution to the inviscid problem under relations (40) and (41) implies that $\bar{v}(\xi,\tau) = N_{\bar{p},\bar{p}+M}(\xi,\tau)$ (see [19] for the uniqueness). Since the limit $\bar{w}(\xi)$ in Lemma 10 is given independently from the choice of the sequence as $\bar{w}(\xi) = \bar{v}(\xi,1) = N_{\bar{p},\bar{p}+M}(\xi,1) = N_{\bar{p},\bar{p}+M}(\xi)$, $w(\xi,s)$ converges to the N-wave pointwise as $s\to\infty$.

Using the estimates in the proof of Lemma 10, we obtain

$$\lim_{s\to\infty}\int |w(\xi,s) - N_{\bar{p},\bar{p}+M}(\xi)|\,d\xi = \int \lim_{s\to\infty} |w(\xi,s) - N_{\bar{p},\bar{p}+M}(\xi)|\,d\xi = 0.$$

(The main part of the proof of Lemma 10 is to show that the limit process and the integration is interchangeable.) Since the L^1 norm is invariant under the change of variable $f(x) \to af(ax), a > 0$, we have

$$\lim_{t\to\infty} \int |u(x,t) - N_{\bar{p},\bar{p}+M}(x,t)|\, d\xi = \lim_{s\to\infty} \int |w(\xi,s) - N_{\bar{p},\bar{p}+M}(\xi)|\, d\xi = 0,$$

i.e., the L^1 convergence in (55) is obtained. □

Remark 13. Consider a solution that emanates from an N-wave like initial value, i.e., $w(\xi,0) \leqslant 0$ for $\xi < \xi_0$ and $w(\xi,0) \geqslant 0$ for $\xi > \xi_0$ with $w_\xi(\xi_0,0) \neq 0$. Let $\xi = g(s)$ be the zero-curve that emanates from the point $g(0) = \xi_0$, i.e., $w(g(s),s) = 0$. From the implicit function theorem, the curve $g(s)$ is defined on a maximal interval $[0,S), S > 0$. Roughly speaking, the number of zeroes is non-increasing and, for the N-wave like initial value as above, it cannot happen that w and w_ξ vanish at the same point $(\bar{\xi},\bar{s})$ (see Angenent [1], Theorem B for the precise statement). This implies that either $S = \infty$ or (if S is finite) $g(s) \to \pm\infty$ as $s \to S$. In either case the solution retains its N-wave like form in the interval $[0,S)$.

The infimum $p(s)$ of (33) is given by

$$p(s) = -W(g(s),s). \tag{56}$$

Let $p_0 = p(0), M = \int w(\zeta,0)\, d\zeta$. In this case we can easily check that the constants in Theorem 7 are $A = p_0, B = M$ and, therefore, the Oleinik-type estimate (36) implies that $w_\xi \leqslant C$ with $C = \frac{1}{2(p_0+M)}(\frac{4(p_0+M)}{\gamma(\gamma-1)})^{2/\gamma}$. From the fact that $w(g(s),s) = 0$, the derivative of $p(s)$ is estimated by

$$-p'(s) = W_s(g(s),s) = \mu e^{(\gamma-2)s/\gamma} w_\xi(g(s),s) \leqslant \mu e^{(\gamma-2)s/\gamma} C. \tag{57}$$

Since $\int_0^\infty \mu e^{(\gamma-2)s/\gamma} C\, ds = \mu C \frac{\gamma}{\gamma-2}$, we may estimate $\bar{p}$ in Theorem 1 by

$$\bar{p} > p_0 - \mu \frac{1}{2(p_0+M)} \left(\frac{4(p_0+M)}{\gamma(\gamma-1)}\right)^{\frac{2}{\gamma}} \frac{\gamma}{\gamma-2}. \tag{58}$$

So for small $\mu > 0$ we always have $\bar{p} > 0$. Since the slope of an N-wave at the sign-changing point is zero, i.e., $N'_{p,q}(0) = 0$, it is clear that $w_\xi(g(s),s) \to 0$ as $s \to \infty$. From these reasons the estimate in (57) may not be an optimal one. It seems like that $\bar{p} > 0$ for any $\mu > 0, p_0 > 0$.

A related structure of the solution has been already observed for the case of zero mass solutions in [10]. It is not still clear that

$$\lim_{t\to\infty} \int u^+(x,t)\, dx = \lim_{t\to-\infty} \int u^-(x,t)\, dx > 0, \quad u^\pm = \max(\pm u, 0)$$

for any nontrivial initial value. However, it is shown that, at least, there exists such an initial value (see [10], Theorem 1.1).

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Further reading

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