

# Lecture note on convection and diffusion

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( Figures are not given yet.)

## 1 Similarity & Scaling Invariance

The logic we are going to employ to survey the similarity structure of several kinds of PDEs is the following. The preliminary requirement is that the problem under consideration should be well-posed, or the uniqueness should be guaranteed at least. Let  $u(x, t)$  be a solution to such a problem and consider a rescaled function  $v$  given by

$$u(x, t) = av(bx, ct), \quad a, b, c > 0. \quad (1.1)$$

The main step is to find the relation among  $a, b, c$  that guarantees for  $v$  to be the solution of the same problem. Then the uniqueness of the problem implies  $u = v$  and hence

$$u(x, t) = au(bx, ct), \quad (1.2)$$

which indicates that the solution should have certain structure provided by the equation. Survey of such behavior is the purpose of this section.

### 1.1 Riemann Problem

The first example is the Riemann problems of conservation laws given by

$$\begin{aligned} u_t + f(u)_x &= 0, \\ \lim_{t \downarrow 0} u(x, t) &= \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \end{aligned} \quad (1.3)$$

One can easily see that the rescaled function  $v$  satisfies the initial value

$$\lim_{t \downarrow 0} v(x, t) = \begin{cases} au_l, & x < 0, \\ au_r, & x > 0. \end{cases}$$

Therefore, if  $v$  is expected to be the solution of the Riemann problem (1.3), then one should pick  $a = 1$ . Now let's find the relation between  $b$  and  $c$ . One can easily check that

$$u_t = cv_t, \quad f'(u)u_x = bf'(v)v_x \quad \text{and} \quad u_t + f(u)_x = cv_t + bf(v)_x = 0.$$

Therefore, if  $b = c$ , then  $v$  satisfies the conservation law  $v_t + f(v)_x = 0$ . The solution of the conservation law that satisfies the entropy condition is unique. For example for the convex scalar case the entropy condition implies that the solution satisfy

$$\lim_{x \uparrow x_0} u(x, t) \geq \lim_{x \downarrow x_0} u(x, t).$$

Since  $b > 0$ , such relations are not changed and  $v$  also satisfies the entropy condition. Since the solution is unique we may conclude  $v(x, t) = u(x, t)$  and, hence,

$$u(x, t) = v(bx, bt) = u(bx, bt). \quad (1.4)$$

This relation indicates that the solution  $u$  is constant along any line that emanates from the origin. (See Figure 1.) In other words  $u$  is a function of  $x/t$ . Therefore, it is natural to introduce a new variable

$$\xi = \frac{x}{t} \quad (1.5)$$

One may say that the problem (1.3) is invariant under the scaling  $u(x, t) \rightarrow u(bx, bt)$  and the variable  $\xi$  also has such a scaling invariance, i.e., even if we substitute  $bx$  in the place of  $x$  and  $bt$  in the place of  $t$ ,  $\xi$  is not changed:

$$\frac{bx}{bt} = \frac{x}{t} = \xi.$$

Therefore, one call this variable  $\xi$  a similarity variable of the Riemann problem (1.3). One can easily rewrite the equation in terms of this similarity variable. Consider

$$\xi_x = t^{-1}, \quad \xi_t = -\xi t^{-1}, \quad \frac{\partial}{\partial t} u(\xi) = u' \xi_t = -u' \xi t^{-1}, \quad \frac{\partial}{\partial x} f(u) = f(u)_{\xi} \xi_x = f'(u) u' t^{-1}.$$

Therefore, the problem (1.3) is transformed to a one dimensional boundary problem:

$$\begin{aligned} -\xi u' + f'(u) u' &= 0, \\ \lim_{t \rightarrow -\infty} u(\xi) &= u_l, \quad \lim_{\xi \rightarrow +\infty} u(\xi) = u_r. \end{aligned} \quad (1.6)$$

Notice that we use almost nothing on the flux function  $f(u)$ . It can be a system of any kind as long as it provide proper uniqueness property. Therefore, the similarity structure is for systems of conservation laws with Riemann initial data.

**HOME WORK.** Discuss the similarity structure in the following problem.

$$\begin{aligned} u_t &= u_{xx}, \\ \lim_{t \downarrow 0} u(x, t) &= \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \end{aligned}$$

## 1.2 $L^1$ solutions to convection equations

Let  $u(x, t)$  be an  $L^1$  solution to

$$\begin{aligned} u_t + f(u)_x &= 0, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned}$$

Since the total mass is preserved, one should consider a scaling that preserve the total mass. If not, there is no chance that  $v$  and  $u$  are identical. Therefore, we set  $a = b$ , i.e., the scaling considered is

$$u(x, t) = av(ax, ct), \quad a, c > 0. \quad (1.7)$$

Unlike the Riemann problem case the relation of scaling of invariance depends on the flux function  $f$  since  $a \neq 1$ . Therefore, we can not consider the flux function with the generality of the Riemann problem. Here, we consider a scalar case under a power law:

$$\begin{aligned} u_t + f(u)_x &= 0, \quad f(u) = |u|^q/q, \quad q > 1 \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \quad (1.8)$$

One can easily check that

$$u_t = acv_t, \quad u_x = a^2v_x \quad \text{and} \quad u_t + f'(u)u_x = acv_t + a^{q+1}f'(v)v_x = 0.$$

Therefore, if  $c = a^q$ , then  $v$  also satisfies the same convection equation. Now we consider similarity variables. Consider

$$\xi = xt^{-1/q}, \quad u(x, t) = t^{-1/q}\tilde{u}(\xi, t).$$

Then the quantities  $\xi$  and  $\tilde{u}$  are not changed under the rescaling  $(x, t) \rightarrow (ax, a^qt)$  and  $u \rightarrow u/a$ . One may derive an equation satisfied by these similarity variables. Using the chain rule, one may obtain

$$\xi_x = t^{-1/q}, \quad \xi_t = -\xi t^{-1}/q, \quad u_x = t^{-2/q}\tilde{u}_\xi, \quad u_t = t^{-1/q}t^{-1}\tilde{u}/q - t^{-1/q}\tilde{u}_\xi\xi t^{-1}/q + t^{-1/q}\tilde{u}_t.$$

Substituting these quantities to the equation we obtain

$$u_t + \text{sign}(u)|u|^{q-1}u_x = t^{-1/q}(\tilde{u}_t + t^{-1}(\text{sign}(\tilde{u})|\tilde{u}|^{q-1}\tilde{u}_\xi - (\xi\tilde{u}/q)_\xi)) = 0.$$

Therefore,  $\tilde{u}$  satisfies

$$\tilde{u}_t + t^{-1}(|\tilde{u}|^q/q - \xi\tilde{u}/q)_\xi = 0.$$

Since the dependent factor  $t^{-1}$  appears explicitly in the equation, people consider a new time scale  $s = \ln t$ . The full similarity variables are

$$\xi = \frac{x}{\sqrt[q]{t}}, \quad s = \ln t, \quad \tilde{u}(\xi, s) = \sqrt[q]{t}u(x, t). \quad (1.9)$$

Under the is exponentially fast time scale,  $t = e^s$ , the only difference is  $\tilde{u}_t = \tilde{u}_s t^{-1}$  and the equation is turned into

$$\tilde{u}_s + (|\tilde{u}|^q/q - \xi\tilde{u}/q)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1). \quad (1.10)$$

Notice that  $s = 0$  corresponds to  $t = 1$  and, hence we have  $\tilde{u}(\xi, 0) = u(\xi, 1)$ . It is also common use the variables  $s = \ln(t+1)$ ,  $\xi = x/\sqrt[q]{t+1}$ ,  $\tilde{u} = \sqrt[q]{t+1}u$ . In the case one have  $\tilde{u}(\xi, 0) = u(\xi, 0)$  and the computation is slightly more complicate.

We have checked that the function  $v(x, t)$  given by

$$u(x, t) = av(ax, a^qt)$$

satisfied the differential equation (1.8). Therefore, one may ask if  $u = v$ . To see that one should check the initial value. Under the above scaling one can easily check that

$$v(x, 0) = u(x/a, 0)/a.$$

Therefore, the only  $L^1$  function that provides the identical initial value is the dirac-measure-type  $M\delta(x)$ . Such a solution is usually called a source-type solution and we denote this source-type solution by  $\rho(x, t)$ , i.e.,  $\rho(x, t)$  is a solution having initial value

$$\lim_{t \downarrow 0} \rho(x, t) = M\delta(x). \quad (1.11)$$

Similarly set

$$\tilde{\rho}(\xi, s) = \sqrt[q]{t}\rho(x, t).$$

If positive solutions are considered, then the source-type solution is unique. However, there is no unique solution if sign-changing solutions are allowed. In that case two invariant variables are introduced, which are

$$P = -\inf_x U(x, t), \quad Q = P + \lim_{x \rightarrow \infty} U(x, t), \quad \text{where } U(x, t) = \int_{-\infty}^x u(y, t)dy. \quad (1.12)$$

It is well known that the positive quantities  $P$  and  $Q$  are constant in time if the flux is convex,  $f''(u) \geq 0$ , and  $Q - P = M$ . If these invariant constants are prescribed, then the source-type solution is unique and hence we may conclude that,

$$\text{if } P \text{ and } Q \text{ are prescribed, then } \rho(x, t) = a\rho(ax, a^q t).$$

Therefore, the value of  $\rho(x, t)$  along the curve  $x/\sqrt[q]{t} = \text{constant}$  is decided completely by the value at a single point, and the rescaled value  $\tilde{\rho}$  is constant along it. (See Figure 2.) Therefore,  $\tilde{\rho}$  is a function of  $\xi$  only if the initial value is given by the dirac-measure. This similarity solution is computed explicitly in the followings. Since  $\tilde{\rho}_s = 0$  and  $\tilde{\rho}$  is compactly supported,

$$(|\tilde{\rho}|^q - \xi \tilde{\rho})_\xi = 0 \Rightarrow |\tilde{\rho}|^q - \xi \tilde{\rho} = 0 \Rightarrow \tilde{\rho} = 0 \text{ or } \text{sign}(\tilde{\rho})|\tilde{\rho}|^{q-1} = \xi. \quad (1.13)$$

The solution that has the invariants  $P, Q$  and that satisfies the entropy condition should be given by

$$\tilde{\rho}(\xi) = \begin{cases} (f')^{-1}(\xi), & -a_P < \xi < a_Q, \\ 0, & \text{otherwise,} \end{cases} \quad (1.14)$$

where the end points of the support is given by the relation

$$-\int_{-a_P}^0 \tilde{\rho}(\xi) d\xi = P, \quad \int_0^{a_Q} \tilde{\rho}(\xi) d\xi = Q.$$

From these relations,  $a_P$  and  $a_Q$  are explicitly computed:

$$a_P = \left( \frac{qP}{q-1} \right)^{\frac{q-1}{q}}, \quad a_Q = \left( \frac{qQ}{q-1} \right)^{\frac{q-1}{q}}. \quad (1.15)$$

This self-similar solution is called an  $N$ -wave due to its shape. (See Figure 3.) If the positive solutions are considered, then  $p = 0$  and  $q = M$  and the  $N$ -wave has simply a triangle like shape.

Under the change of variable, one can easily transform the  $N$ -wave in terms of the original variables:

$$\rho(x, t) = \begin{cases} (f')^{-1}(x/t), & -a_P \sqrt[q]{t} < \xi < a_Q \sqrt[q]{t}, \\ 0, & \text{otherwise,} \end{cases} \quad (1.16)$$

One may consider a flux such as  $f(u) = |u|^2 + |u|^3$ . In this case the solution has no exact scaling invariance. However,  $\rho(x, t)$  in (1.16) is well defined for all convex flux,  $f''(u) \geq 0$ , and is the source-type solution.

Since  $\max_x \tilde{\rho}$  is of order  $O(1)$ , the source-type solution in the original variable has the order

$$\max_x \rho(x, t) = O(t^{-1/q}) \quad \text{as } t \rightarrow \infty. \quad (1.17)$$

### 1.3 $L^1$ solutions to diffusion equations

Let  $u(x, t)$  be a positive  $L^1$  solution to a nonlinear diffusion equation

$$\begin{aligned} u_t &= \phi(u)_{xx}, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}), \end{aligned}$$

where the diffusion function  $\phi$  is given by

$$\phi(u) = u^m, \quad m > 0. \quad (1.18)$$

Consider a scaling of

$$u(x, t) = av(ax, ct), \quad a, c > 0. \quad (1.19)$$

One can easily check that

$$u_t = acv_t, \quad \phi(u)_{xx} = a^{m+2}\phi(v)_{xx} \quad \text{and} \quad u_t - \phi(u)_{xx} = acv_t - a^{m+2}\phi(v)_{xx} = 0. \quad (1.20)$$

Therefore, if  $c = a^{m+1}$ , then  $v$  also satisfies the same diffusion equation. Now we consider similarity variables. Consider

$$\alpha = m + 1, \quad s = \ln t, \quad \xi = xt^{-1/\alpha}, \quad u(x, t) = t^{-1/\alpha}\tilde{u}(\xi, s).$$

Then the quantities  $\xi$  and  $\tilde{u}$  are not changed under the rescaling  $(x, t) \rightarrow (ax, a^\alpha t)$  and  $u \rightarrow u/a$ . One may derive an equation satisfied by these similarity variables. Using the chain rule, one obtain

$$\xi_x = t^{-\frac{1}{\alpha}}, \quad \xi_t = \frac{-\xi t^{-1}}{\alpha}, \quad \phi(u)_{xx} = t^{-\frac{1}{\alpha}} t^{-1} \phi(\tilde{u})_{\xi\xi}, \quad u_t = -\frac{t^{-\frac{1}{\alpha}} t^{-1}}{\alpha} \tilde{u} - \frac{t^{-\frac{1}{\alpha}} t^{-1}}{\alpha} \xi \tilde{u}_\xi + t^{-\frac{1}{\alpha}} t^{-1} \tilde{u}_s.$$

Substituting these quantities to the equation we obtain

$$u_t - \phi(u)_{xx} = t^{-1-1/\alpha} (\tilde{u}_s - (\phi(u)_\xi + \xi \tilde{u}/\alpha)_\xi) = 0.$$

Therefore,  $\tilde{u}(\xi, s)$  satisfies

$$\tilde{u}_s - (\phi(u)_\xi + \xi \tilde{u}/\alpha)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1). \quad (1.21)$$

Let  $\rho(x, t)$  be the positive source-type solution with initial value  $\lim_{t \downarrow 0} u(x, t) = M\delta(x)$ . Then, due to the similarity property its similarity quantity  $\tilde{\rho}(\xi, s)$  is a function of  $\xi$  only and satisfies

$$(\phi(\tilde{\rho})_\xi + \xi \tilde{\rho}/\alpha)_\xi = 0.$$

We may do the similar computation as the convection case:

$$(\phi(\tilde{\rho})_\xi + \xi \tilde{\rho}/\alpha)_\xi = 0 \Rightarrow \phi(\tilde{\rho})_\xi + \xi \tilde{\rho}/\alpha = 0 \Rightarrow \rho = 0 \quad \text{or} \quad \frac{\phi'(\tilde{\rho})}{\tilde{\rho}} \tilde{\rho}_\xi + \frac{\xi}{\alpha} = 0 \Rightarrow \tilde{\rho} = 0 \quad \text{or} \quad g(\tilde{\rho})_\xi = -\frac{\xi}{\alpha},$$

where

$$g'(x) = \phi'(x)/x, \quad (\text{i.e., } g(x) = \frac{m}{m-1} x^{m-1} \text{ for } \phi(x) = x^m).$$

Therefore, we may conclude that

$$\tilde{\rho}(\xi) = \max(0, g^{-1}(C - \xi^2/(2\alpha))), \quad (1.22)$$

where  $C$  is a constant that should be decided by the total mass relation  $\int \tilde{\rho}(\xi) d\xi = M$ .

Under the change of variable, one can easily transform the N-wave in terms of the original variables:

$$\rho(x, t) = \max(0, g^{-1}(Ct^{\frac{1-m}{m+1}} - x^2/(2\alpha t))) = \max(0, g^{-1}([Ct^{\frac{2}{m+1}} - x^2/(2\alpha)]/t)). \quad (1.23)$$

Notice that, unlike the convection case, the expression of  $\rho(x, t)$  depends on the power  $m$ . Therefore, it is not obvious to find the explicit formula for the source-type solution for a general case when the diffusion is not given by the power law. we are not sure if we may find one.

Since  $\max_x \tilde{\rho}$  is of order  $O(1)$ , the source-type solution in the original variable has the order

$$\max_x \rho(x, t) = O(t^{-1/(m+1)}) \quad \text{as } t \rightarrow \infty. \quad (1.24)$$

**HOME WORK.**

1. Show that the self-similar solution  $\rho(x, t)$  is the Gaussian for the linear heat equation case  $m = 1$ .
2. Show that the support of  $\rho(x, t)$  is compact for any given time  $t > 0$  if  $m > 1$ .
3. Show that the support of  $\rho(x, t)$  is strictly positive compact on the whole real line for any given time  $t > 0$  if  $m > 1$ .
4. Discuss the similarity structure for the multidimensional case

$$u_t = \Delta u^m, \quad m > 0.$$

#### 1.4 p-Laplacian

Let  $u(x, t)$  be a positive  $L^1$  solution to a different kind of nonlinear diffusion equation, so called the p-Laplacian,

$$\begin{aligned} u_t &= (|u_x|^{p-2} u_x)_x, \quad p > 1, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \tag{1.25}$$

One may similarly study the similarity structure of the p-Laplacian equation. From the similarity variables one can show the asymptotic magnitude of the source-type solution:

$$\max_x \rho(x, t) = O(t^{-1/(2p-2)}) \quad \text{as } t \rightarrow \infty. \tag{1.26}$$

#### HOME WORK.

1. Find the similarity variables of the solutions to (1.25).
2. Find the equation that the similarity variables satisfy.
3. Find the explicit formula that the self-similar solution satisfies.
4. Verify that the self-similar solution  $\rho(x, t)$  is the Gaussian for the linear heat equation case  $p = 2$ .
5. Find the range of the power  $p > 1$  that the self-similar solution  $\rho(x, t)$  is compactly supported.
6. Discuss the similarity structure for the multidimensional case

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 1.$$

One can easily check that

$$u(x, t) = av(ax, a^{2p-2}t)$$

is the invariance scaling of the p-Laplacian. Therefore, the similarity variables are

$$\alpha = 2p - 2, \quad \xi = xt^{1/\alpha}, \quad s = \ln t, \quad \tilde{u}(\xi, s) = t^{1/\alpha} u(x, t) \tag{1.27}$$

and they satisfy

$$\tilde{u}_s - (|\tilde{u}_\xi|^{p-2} \tilde{u}_\xi + \xi \tilde{u}/\alpha)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1).$$

Let  $\rho(x, t)$  be the source-type solution and  $\tilde{\rho}$  be the self-similar correspondence. Then,  $\tilde{\rho}$  is the solution of the ordinary differential equation

$$|\tilde{\rho}_\xi|^{p-2} \tilde{\rho}_\xi + \xi \tilde{\rho}/\alpha = 0.$$

How to solve it?

## 1.5 Co-existence of several different phenomena

Let  $u(x, t)$  be a positive  $L^1$  solution to a different kind of the following mixed problem,

$$\begin{aligned} u_t &= (u^m)_{xx} + (|u_x|^{p-2}u_x)_x - (u^q/q)_x, \quad m > 0, q > 1, p > 1, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \quad (1.28)$$

If

$$q = m + 1 = 2p - 2,$$

then the problem has a invariance scaling and hence we may consider similarity structure of the problem. If not, there is no exact similarity property. However, one may find the dominant factor of the evolution. We may conclude that

$$\max_x |u(x, t)| = O(t^{-1/\alpha}) \quad \text{as } t \rightarrow \infty, \quad \alpha = \min(q, m + 1, 2p - 2). \quad (1.29)$$

If  $\alpha = q$ , then we may say that the convection dominates the evolution. Similarly the PME type diffusion dominates the evolution if  $\alpha = m + 1$  and the p-Laplacian type diffusion decide the asymptotics if  $\alpha = 2p - 2$ .

### HOME WORK.

1. Suppose that  $\alpha = q$  in (1.29). Transform the equation (1.28) using the similarity variables for the convection case (1.9) and observe that the coefficients for other terms decays to zero.
2. Discuss what decide the similarity structure of the problems. What is the basic difference in the influence of convection and diffusion when they decide the scaling invariance.
3. When the magnitude of the solution decays faster than the order  $1/t$ ?

## 2 Characteristics for convection equations

In this section we consider convection equations. Let  $u(x, t)$  be the entropy solution to the convection equation

$$u_t + a(x, t)u_x = 0, \quad \lim_{t \downarrow 0} u(x, t) = u_0(x) \in L^1(\mathbf{R}). \quad (2.1)$$

The characteristics is a curve  $x = x(t)$  in the  $x - t$  plane that satisfies

$$x'(t) = a(x(t), t).$$

Then one can easily check that

$$\frac{d}{dt}u(x(t), t) = u_t + x'(t)u_x = u_t + a(x(t), t)u_x = 0.$$

Therefore,  $u$  is constant along the characteristic curve  $x = x(t)$ ,

## 2.1 scalar conservation laws

Consider

$$u_t + f(u)_x = u_t + f'(u)u_x = 0 \quad (2.2)$$

under the assumption on the smooth flux of

$$f(0) = f'(0) = 0, \quad f''(u) \geq 0.$$

In this case  $x'(t) = f'(u)$  and  $u$  is constant along the characteristics. Therefore, the slope of the curve is constant, i.e, characteristics are strait lines. (If the waves speed  $a(x, t)$  is discontinuous, then we consider minimal or maximal characteristics.)

If characteristics are collide to each other, then discontinuities may appear. In the case the speed of the propagation of the discontinuity, or shock speed, is given by the Rankini-Hugoniot jump condition,

$$\text{shock speed } s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{and } s = \frac{f(u)}{u} \text{ if } u_- = 0.$$

We employ the primitive of the solution as its potential:

$$U(x, t) = \int_{-\infty}^x u(y, t) dy. \quad (2.3)$$

Integrating (2.3) on  $(-\infty, x)$ , one obtains

$$U_t + f(u) = U_t + \frac{f(u)}{u} U_x = 0$$

Here, potential  $U$  is constant along the characteristic  $x'(t) = f(u)/u$ . Notice that this curve is not the same the characteristic of  $u$  and is not a strait line any more. This curve is related to the shock characteristics when one state is connected to the zero state.

## 2.2 Isentropic Gas Dynamics

In this section  $\rho(x, t)$  represents the density of the given gas,  $u(x, t)$  is the velocity,  $m$  is the momentum,  $m = \rho u$ , and  $p$  is the pressure. In the isentropic gas the pressure is assumed to be a function of density only  $p = p(\rho)$  and they satisfies the equation

$$\begin{aligned} \rho_t + m_x &= 0, \\ m_t + (m^2/\rho + p(\rho))_x &= 0. \end{aligned} \quad (2.4)$$

Set potentials as

$$R(x, t) = \int_{-\infty}^x \rho(y, t) dy, \quad M(x, t) = \int_{-\infty}^x m(y, t) dy.$$

Then after the integration of the equation (2.4), one obtain

$$\begin{aligned} R_t + uR_x &= 0, \\ M_t + (u + p(\rho)/m)M_x &= 0. \end{aligned} \quad (2.5)$$

The first relation in (2.5) implies that  $R$  is constant along the curve

$$x'_1(t) = u|_{(x_1(t), t)}.$$

That is natural since the characteristic of the first class is the trace of a moving particle and the total mass of the gas on the left hand side of it should be constant.

Under the power law  $p(\rho) = \rho^\gamma$ ,  $\gamma \geq 1$ , the characteristic curve for the momentum potential is

$$x'_2(t) = (u + \rho^{\gamma-1}/u)|_{(x_2(t),t)}.$$

This indicates that the characteristic curve on which the momentum potential is constant has higher speed than the particle trace has. In particular, if the velocity  $u$  is near zero, then the characteristic speed increases to infinity and, if the density  $\rho$  approaches to zero, two curves  $x_{1,2}(t)$  approaches to each other.

### 3 Asymptotics in the diffusion and convection

(This section is a part of independent paper work.)

#### 3.1 Introduction

Let  $u(x, t)$  be a positive solution of

$$u_t = \frac{\partial}{\partial x} \sigma(u, u_x), \quad \lim_{t \downarrow 0} u(x, t) = u_0(x) \geq 0, \quad t > 0, x \in \mathbf{R}, \quad (3.1)$$

where the limit is understood in  $L^1$  sense and the positive initial value  $u_0$  is compactly supported and integrable. We may assume without loss of generality that

$$\text{spt}(u_0) \subset [0, L] \quad \text{and} \quad \int u_0(x) dx = 1 \quad (3.2)$$

after an appropriate normalization. Consider three representative cases. The convection is the case that the operator  $A$  is a function of  $u$  only. For this case we consider a smooth flux function,

$$\sigma_c(u) := f(u), \quad f(0) = f'(0) = 0. \quad (3.3)$$

The extra conditions  $f(0) = f'(0) = 0$  can be assumed without loss of generality after appropriate translations. In many papers convex flux functions are considered under the power law  $f(u) = -u^q/q$ ,  $q > 1$ , and the case with  $q = 2$  is the well known Burgers equation. In this paper the convexity of the flux  $f(u)$  is not assumed.

For the diffusion case the nonlinear diffusion,

$$\sigma_d(u, u_x) := \phi'(u)u_x = \phi(u)_x, \quad m > 0, \quad (3.4)$$

is considered, where  $\phi$  is assumed to satisfy

$$\phi \in C[0, +\infty) \cap C^1(0, +\infty), \quad \phi(0) = 0 \quad \text{and} \quad \phi'(u) \geq 0 \quad \text{for all } u > 0. \quad (3.5)$$

In this case the diffusion speed,  $\phi'(u)$ , is a function of the solution  $u$ . In the study of porous medium equations the power law,  $\phi(u) = u^m$ , is mostly considered for  $m > 0$  and the case  $m = 1$  is the linear heat equation.

The last case considered is the one when the diffusion speed is a function of the gradient of the solution. Consider

$$\sigma_p(u_x) := \psi(u_x), \quad (3.6)$$

where  $\psi$  is an odd function,  $\psi(-v) = -\psi(v)$ , and satisfies the same condition as  $\phi$  in (3.5) for  $v := u_x > 0$ , i.e.,

$$\psi \in C[0, +\infty) \cap C^1(0, +\infty), \quad \psi(0) = 0 \text{ and } \psi'(v) \geq 0 \text{ for all } v > 0. \quad (3.7)$$

The p-Laplacian is the case under power law  $\psi(v) = \text{sign}(v)|v|^{p-1}$ ,  $p > 1$ .

In the entropy diminishing technique one of the main steps is to transform the problem using time dependent similarity variables. Then nontrivial steady solutions appear and one may find explicit canonical solutions such as N-waves or Barenblatt type solutions. Intermediate asymptotics is then obtained via diminishing of entropy difference between the solution and a steady state.

In this paper we consider the cases in

$$0 \neq \sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c, \quad c_i = 0 \text{ or } 1. \quad (3.8)$$

In these cases there exist two or more phenomena of different similarity scales and there is no similarity variables that provide any nontrivial steady solution. Let  $\rho(x, t) \geq 0$  be the source solution to (3.1), i.e.,

$$\rho_t = \frac{\partial}{\partial x} \sigma(\rho, \rho_x), \quad \lim_{t \downarrow 0} \rho(x, t) = \delta(x), \quad x \in \mathbf{R}, \quad t \in \mathbf{R}^+. \quad (3.9)$$

If the operator  $A$  is given by (3.3), (3.4) or (3.6), under the power law, the canonical solution  $\rho(x, t)$  is given explicitly due to the similarity structure of each phenomenon. However, such an explicit solution does not exist for the general case in (3.8). Therefore, we should continue the study of asymptotic decay without such an explicit solution. It is well known that the difference between a general solution  $u(x, t)$  to (3.1),(3.2) and the source-type solution  $\rho(x, t)$  decays to zero in  $L^1$  sense,

$$\|u(t) - \rho(t)\|_1 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Here and in the remainder of this paper, we use the notation  $\|\cdot\|_1$  to denote the  $L^1$ -norm and  $u(t)$  to denote the function on the real line given by  $u(t)|_x = u(x, t)$ . Since the  $L^1$ -norm of a solution is preserved, it is a natural way to measure the distance between two solutions. Note that one may obtain asymptotic orders in  $L^p$ -norm,  $p \geq 1$  via classical interpolation arguments.

The main result of this paper is the following theorem:

**Theorem 1** *Let  $\sigma_c, \sigma_d, \sigma_p$  and  $A$  be given by (3.3)–(3.8),  $u(x, t)$  be the solution to (3.1)–(3.2) and  $\rho(x, t)$  be the source-type solution (3.9). Then,*

$$\|u(t) - \rho(t)\|_1 \leq 2L \max_x \rho(x, t). \quad (3.10)$$

For the case with convection only,  $\sigma = \sigma_c$ , its asymptotics has been studied in [7] and hence we consider the other 6 possible cases in this paper. To obtain a convergence of the next order one should place the solution  $u(x, t)$  at the correct position. Then one may obtain an intermediate asymptotics in  $L^1$ -norm, which is

$$\|u_c(t) - \rho(t)\|_1 = O(1/t) \quad \text{as } t \rightarrow \infty, \quad (3.11)$$

where  $u_c(x, t) := u(x + c, t)$ . In the literature the convergence order of  $1/t$  has been also studied in various cases. We could not obtain this convergence order with the generality in Theorem 1. The convergence order corresponding to the order  $1/t$  is obtained for the following two cases:

**Theorem 2** *Let  $u(x, t)$  be the solution of (3.1)–(3.2),  $\sigma_d$  and  $\sigma_p$  be given by (3.4) and (3.6), respectively, and  $\rho(x, t)$  be a canonical solution (3.9). Then,*

(i) Let  $\sigma = \sigma_d$ . If  $c = \int x u_0(x) dx \in \mathbf{R}$  is the center of mass, then there exists  $T > 0$  such that

$$\|\rho(t) - u_c(t)\|_1 \leq 8T \max_x (\rho(x, t) \phi(\rho(x, t))). \quad (3.12)$$

(ii) Let  $\sigma = \sigma_p$  and  $u_0$  is symmetric and continuous, and satisfies

$$\text{spt}(u_0) = [-L, L], u_0(x) > 0 \text{ for } x \in (-L, L) \text{ and } \int u_0(x) dx = 1. \quad (3.13)$$

Then, there exists  $T > 0$  such that

$$\|\rho(t) - u(t)\|_1 \leq 2T \max_x (\psi(\rho_x(x, t))). \quad (3.14)$$

Similar convergence order has been shown in [7] for the case of  $\sigma = \sigma_c$  where the flux is convex: If  $c = \min(\text{spt}(u_0))$  and there exist  $\epsilon, t_0 > 0$  such that

$$\int_{-\infty}^x \rho(y, t_0) dy \leq \int_{-\infty}^x u(y + c, 0) dy, \quad 0 < x < \epsilon, \quad (3.15)$$

then there exists  $T > 0$  such that

$$\|\rho(t) - u_c(t)\|_1 \leq 2T f(\max_x \rho(x, t)). \quad (3.16)$$

One of the main goals of this article is to introduce a potential comparison technique in the simple setting of one space dimension. However, the technique has been originally developed for several space dimension for nonlinear diffusion equation [8]. This note consists as followings. In Section 2 several preliminary steps are constructed including the potential comparison principle and then Theorem 1 is proved using the potential comparison technique. Section 3 is for the porous medium type equations and Theorem 2 (i) is proved.

In Section 4 the derivative of the solution to the p-Laplacian type equation (3.32) is understood as the sign-changing solution to the porous medium equation. The technique for the porous medium equation is naturally applied to show Theorem 2 (ii). To obtain the convergence orders in Theorems explicitly we consider special cases under the power laws in Section 5. If  $f, \phi$  and  $\psi$  are given by (3.37), the convergence orders in (3.12), (3.14) and (3.16) turn out to be the order  $1/t$ .

### 3.2 $L^1$ convergence order of the magnitude of solutions

In this section we show Theorem 1. For the convection equation case,  $\sigma = \sigma_c$ , the entropy solutions are considered and the convergence order has been shown in [6], Theorem 1 (13), by trapping a solution between two rarefaction waves. In this chapter we consider the integrals of the solution  $u(x, t)$  and the canonical one  $\rho(x, t)$  as their potentials,

$$U(x, t) = \int_{-\infty}^x u(y, t) dy, \quad R(x, t) = \int_{-\infty}^x \rho(y, t) dy. \quad (3.17)$$

Integrating (3.1) over  $(-\infty, x)$  yields

$$U_t = \sigma(u, u_x) = \sigma(U_x, U_{xx}). \quad (3.18)$$

To justify this Hamilton-Jacobi type equation we should have

$$\lim_{x \rightarrow -\infty} \sigma(u(x, t), u_x(x, t)) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \int_{-\infty}^x u(y, t) dy = \int_{-\infty}^x \frac{\partial}{\partial t} u(y, t) dy,$$

which can be easily verified.

It is well known that, if  $u(x, 0) \geq \tilde{u}(x, 0)$ , then the solutions  $u, \tilde{u}$  to (3.1) satisfies  $u(x, t) \geq \tilde{u}(x, t)$  for all  $t > 0$ . However, one can not expect this relation between  $u$  and  $\rho$  even after a time and space translation since  $\rho$  and  $u$  share the same total mass. In the following proposition we show that a similar comparison principle holds between potentials.

**Proposition 3 (Comparison Principle)** *Let  $A$  be one of the 7 cases in Theorem 1,  $u$  and  $\tilde{u}$  be solutions of (3.1) and  $U$  and  $\tilde{U}$  be their potentials given by their integrals as in (3.17), respectively. Then, if  $U(x, 0) \geq \tilde{U}(x, 0)$ , then  $U(x, t) \geq \tilde{U}(x, t)$  for all  $t \geq 0$ .*

**Proof.** Let  $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$  where  $c_i = 0$  or  $1$ . One can easily check that the potential difference  $E = U - \tilde{U}$  satisfies

$$\mathcal{L}[E] := (c_1a(x, t) + c_2b(x, t))E_{xx} + (c_3c(x, t))E_x - E_t = 0, \quad E(x, 0) \geq 0,$$

where

$$a := \frac{\phi'(u)u_x - \phi'(\tilde{u})\tilde{u}_x}{u_x - \tilde{u}_x}, \quad b := \frac{\psi(u_x) - \psi(\tilde{u}_x)}{u_x - \tilde{u}_x}, \quad c := \frac{f(u) - f(\tilde{u})}{u - \tilde{u}}.$$

Notice that  $b(x, t) \geq 0$  and  $c(x, t) \geq 0$  since  $f(u)$  and  $\text{sign}(u_x)\psi(|u_x|)$  are increasing functions with respect to  $u \geq 0$  and  $u_x$ , respectively. However,  $a(x, t)$  may change its sign. In the followings we check if the maximum principle type arguments hold. First observe the uniform decay of  $E$ , i.e.,

$$|E(x, t)| \leq \int_{-\infty}^x |u(y, t) - \tilde{u}(y, t)| dy \leq \|u(x, t) - \tilde{u}(x, t)\|_1 \rightarrow 0$$

as  $t \rightarrow \infty$ . Suppose that there exists  $t \geq 0$  and  $x \in \mathbf{R}$  such that  $E(x, t) < 0$ . Then the uniform convergence and the initial condition  $E(x, 0) \geq 0$  imply that there exists a global minimum point  $(x_0, t_0)$ .

First consider the case  $c_1 = 1$ . Since the solutions  $u, \tilde{u}$  are continuous,  $E$  is differentiable and hence

$$E_x(x_0, t_0) = 0.$$

Therefore,  $u(x_0, t_0) = \tilde{u}(x_0, t_0)$  and  $a(x_0, t_0) = \phi'(u(x_0, t_0))$ . The point  $x_0$  should be an interior point of the support of  $u(\cdot, t_0)$  or  $\tilde{u}(\cdot, t_0)$  (otherwise  $E(x_0, t_0) = 0$ ) and hence  $a(x_0, t_0) > 0$  and the equation is uniformly parabolic in a small neighborhood of  $(x_0, t_0)$ . The maximum principle now easily derives a contradiction.

Now consider the case  $c_1 = 0$  and  $c_2 = 1$ . (In this case the coefficient  $b$  may diverges and hence the maximum principle is not obvious.) Since  $E(x, t_0)$  is not a constant function and  $x_0$  is a minimum point, there exists  $\delta > 0$  such that  $E(x, t_0) > E(x_0, t_0)$  for  $x \in (x_0 - \delta, x_0)$  after retaking  $x_0$  if needed. By taking smaller  $\delta > 0$  if needed, we may assume  $E_x(\cdot, t_0) < 0$  and  $E_{xx}(\cdot, t_0) > 0$  on  $(x_0 - \delta, x_0)$ . If  $b(x_0, t_0) > 0$ , the problem is uniformly parabolic in neighborhood of  $(x_0, t_0)$  again. Therefore, consider the case when  $b(x_0, t_0) = 0$ . Let  $W = E + \epsilon z$ , where

$$z(x) = 1 - e^{x_0 - x} \quad \text{and} \quad 0 < \epsilon < \frac{E(x_0, t_0) - E(x_0 - \delta, t_0)}{z(x_0 - \delta)}.$$

Then, since  $z(x_0) = 0$  and  $z_x(x_0) = 1$ , we get  $W(x_0, t_0) = E(x_0, t_0)$ ,  $W_x(x_0, t_0) = \epsilon > 0$  and  $W(x_0 - \delta, t_0) > E(x_0, t_0)$ . Therefore,  $W(\cdot, t_0)$  has an interior minimum point in  $(x_0 - \delta, x_0)$ . Let  $(y_0, t_0)$  be the minimum point. Since  $E(x_0, t_0)$  is the global minimum of  $E$  and the auxiliary function  $z$  depends on  $x$  variable only, this minimum point is also a minimum of  $W$  on  $Q := (x_0 - \delta, x_0) \times (0, T)$  for any  $T > 0$ . Notice that, since  $E_x(\cdot, t_0) \neq 0 \neq E_{xx}(\cdot, t_0)$  in the interval

$(x_0 - \delta, x_0)$ , both of  $b(x, t)$  and  $c(x, t)$  are strictly positive. Therefore,  $\mathcal{L}$  is uniformly parabolic in a neighborhood of  $(y_0, t_0)$  and

$$\mathcal{L}[W] = \mathcal{L}[E] + \epsilon \mathcal{L}[z] = \epsilon(bz_{xx} - c_3 cz_x) = -\epsilon(b + c_3 c) \leq 0$$

on it. The assumption that  $(y_0, t_0)$  is a interior minimum point of  $W$  contradicts to the maximum principle. The case  $c_1 = c_2 = 0$  and  $c_3 = 1$  is done in [7] and hence omitted.

**End of Proof.**

Notice that this comparison principle holds for potentials given by integrals of solutions (3.17) and that it seems to hold under a more general operator  $\sigma = \sigma(u, u_x)$ . The next step is to place the potential  $U$  between two potentials of canonical solutions.

**Corollary 4 (Trapped!)** *Let  $u$  be a solution of (3.1)-(3.2) and  $\rho$  be the canonical solution of (3.9). Then their potentials  $U, R$  given by (3.17) satisfies*

$$R(x - L, t) \leq U(x, t) \leq R(x, t) \quad \text{for all } t > 0, x \in \mathbf{R}. \quad (3.19)$$

**Proof.** Since  $\rho(x, 0) = \delta(x)$ ,  $R(x, 0)$  is the heavy side function

$$R(x, 0) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Therefore the restriction (3.2) on the initial value  $u_0$  implies that

$$R(x - L, 0) \leq U(x, 0) \leq R(x, 0)$$

(see Figure 1). Proposition 3 implies that (3.19) holds for all  $t > 0$ .

**End of Proof.**

The next step is to compute the convergence rate of potentials using (3.19) which comes easily:

**Lemma 5 (Potential Convergence Rate)** *Under the same conditions as in Corollary 4,*

$$\|U(x, t) - R(x, t)\|_\infty \leq L \max_{x \in \mathbf{R}} \rho(x, t). \quad (3.20)$$

**Proof.** Using the comparison inequality (3.19) and the Mean Value Theorem, we obtain

$$\begin{aligned} |U(x, t) - R(x, t)| &\leq |R(x - L, t) - R(x, t)| \\ &= LR_x(c, t) = L\rho(c, t) \leq L \max_{x \in \mathbf{R}} \rho(x, t), \end{aligned}$$

where  $c \in [x - L, x]$ . Since the estimate is independent of the point  $x$ , the estimate is uniform as in (3.20).

**End of Proof.**

Now the last step is to transfer this potential convergence rate to the desired solution convergence rate. Following the approach in [8], one may employ the classical regularity theory in [10] to derive the estimate of  $U_x - R_x (\equiv u - \tilde{u})$  from  $U - R$ . It seems that there is a simpler approach under a conjecture that  $\rho$  is steeper than any other solution  $u$ .

**Proposition 6** *Let  $u(x, t)$  and  $\rho(x, t)$  be solutions of (3.1)-(3.2) and (3.9), respectively. Then,*

$$\|\rho(t) - u(t)\|_1 = 2\|R(t) - U(t)\|_\infty \leq 2L \max_x \rho(x, t). \quad (3.21)$$

**Proof.** Let  $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$  where  $c_i = 0$  or  $1$ . One can easily check that the potential difference  $E = R - U$  satisfies

$$\mathcal{L}[E] := (c_1a(x, t) + c_2b(x, t))E_{xx} + (c_3c(x, t))E_x - E_t = 0, \quad E(x, 0) \geq 0,$$

where

$$\begin{aligned} a &:= \frac{\phi'(\rho)\rho_x - \phi'(u)u_x}{\rho_x - u_x}, \\ b &:= \frac{\text{sign}(\rho_x)\psi(|\rho_x|) - \text{sign}(u_x)\psi(|u_x|)}{\rho_x - u_x}, \\ c &:= \frac{f(\rho) - f(u)}{\rho - u}. \end{aligned}$$

Since  $\text{sign}(x)\psi(x)$  is an odd function with  $\psi'(x) \geq 0$  for  $x > 0$ , the coefficient  $b$  is non-negative. Therefore, if  $c_1 = 0$ , we may apply the property of the nonincrease of the lap number [14]. If  $c_1 = 1$ , then the coefficient of the leading order term  $a + c_2b$  can be negative. However, the lap number changes at the points  $\rho = u$  and  $a = \phi'(u) > 0$  near such a point. Therefore, we may apply the argument similarly and show the nonincrease of the lap number. We may conclude that the number of sign changes is at most once since  $\lim_{t \downarrow 0} \rho(x, t) = \delta(x)$ , the Dirac-delta measure. Let  $x(t)$  be the sign-changing point of  $e = \rho - u$ . Then, clearly,

$$\begin{aligned} \|u(t) - \rho(t)\|_1 &= \int |u(x, t) - \rho(x, t)| dx = 2 \int_{-\infty}^x [\rho(x, t) - u(x, t)] dx \\ &= 2\|U(t) - R(t)\|_\infty. \end{aligned}$$

Furthermore, using the comparison inequality (3.19), we obtain

$$|U(x, t) - R(x, t)| \leq |R(x - L, t) - R(x, t)| = \int_{x-L}^x \rho(y, t) dy \leq L \max_x \rho(x, t).$$

Since the estimate is independent of the point  $x$ , the estimate is uniform and the inequality in (3.21) is obtained.

**End of Proof.**

### 3.3 Porous medium type nonlinear diffusion

In this section we show Theorem 2(i). For the nonlinear diffusion case,  $\sigma = \sigma_d$ , the equation (3.1) is written as

$$u_t = \phi(u)_{xx}, \quad \lim_{t \downarrow 0} u(x, t) = u_0(x) \geq 0, \quad x \in \mathbf{R}, \quad t > 0. \quad (3.22)$$

The source solution  $\rho$  satisfies

$$\rho_t = \phi(\rho)_{xx}, \quad \lim_{t \downarrow 0} \rho(x, t) = \delta(x), \quad x \in \mathbf{R}, \quad t > 0. \quad (3.23)$$

Under the power law,  $\phi(u) = u^m$ ,  $m > 0$ , the canonical solution  $\rho(x, t)$  is explicitly given by

$$\rho^{m-1}(x, t) = \max \left\{ 0, At^{\frac{1-m}{m+1}} + \frac{1-m}{2m(m+1)} |x|^2 t^{-1} \right\}, \quad (3.24)$$

where the constant  $A$  is decided by the relation  $\int \rho(x, t) dx = 1$ . One can easily check that  $\rho(x, t)$  has its maximum at  $x = 0$  and

$$\max_{x \in \mathbf{R}} \rho(x, t) = A^{\frac{1}{m-1}} t^{\frac{-1}{m+1}} = O(t^{\frac{-1}{m+1}}). \quad (3.25)$$

For the fast diffusion range,  $0 < m < 1$ , the convergence order  $1/t$  has been shown in [8].

To obtain a convergence order  $1/t$  we have trapped the initial potential  $U(x, 0)$  between  $R(x, t)$  and  $R(x, t + T)$  in the convection case. However, the primitive of a solution does not decouple the equation enough. Therefore, we introduce Newtonian potentials:

$$\mathcal{U}(x, t) = \int_{-\infty}^x U(y, t) dy, \quad \mathcal{R}(x, t) = \int_{-\infty}^x R(y, t) dy. \quad (3.26)$$

Let  $\varphi(x) = \max\{0, x\}$ . Then, one may check that  $\Delta\varphi(x) = \delta(x)$  and  $\mathcal{U} = \varphi * u$ . Therefore, it is natural to call  $\mathcal{U}$  a Newtonian potential. A typical way is to take  $\varphi(x) = |x|/2$  as a kernel. Here, to show the connection to Section 3.2 more closely, the Newtonian potential is given by (3.26).

**Proposition 7** *Let  $u$  be the solution of (3.22) and  $\mathcal{U}$  be its Newtonian potential. Then*

$$\mathcal{U}_t(x, t) = \phi(u(x, t)) > 0 \quad \text{for each } x \in \mathbf{R}, t > 0. \quad (3.27)$$

**Proof.** Basically, we need to check if

$$\lim_{h \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^x \frac{u(y, t+h) - u(y, t)}{h} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^x \lim_{h \rightarrow \infty} \frac{u(y, t+h) - u(y, t)}{h} dy dx.$$

Under the power law  $\phi(u) = u^m$ ,  $m > 0$ , it is clear for  $m > 1$  since the integrands are compactly supported. For the fast diffusion case  $0 < m \leq 1$  we simply refer [8]. Therefore, we need to check for the case of non-power law.

**End of Proof.**

**Proposition 8 (Potential comparison)** *Let  $\mathcal{U}(x, t)$  and  $\tilde{\mathcal{U}}(x, t)$  be the Newtonian potentials of two bounded solutions  $u$  and  $\tilde{u}$  of (3.22), respectively. Then  $\mathcal{U}, \tilde{\mathcal{U}}$  are continuous on the closure of  $\mathbf{R} \times ]0, \infty[$ . If  $\mathcal{U}(x, 0) \leq \tilde{\mathcal{U}}(x, 0)$  for all  $x \in \mathbf{R}$ , then  $\mathcal{U}(x, t) \leq \tilde{\mathcal{U}}(x, t)$  for all  $t > 0$ .*

**Proof.** Let  $\mathcal{V} = \mathcal{U} - \tilde{\mathcal{U}}$ . Then  $\mathcal{V}$  satisfies

$$\mathcal{V}_t = \mathcal{U}_t - \tilde{\mathcal{U}}_t = \phi(u) - \phi(\tilde{u}) = \frac{\phi(u) - \phi(\tilde{u})}{u - \tilde{u}} \mathcal{V}_{xx} (\equiv a(x, t) \mathcal{V}_{xx}). \quad (3.28)$$

The coefficient  $a(x, t)$  is non-negative since  $\phi$  is an increasing function and hence the maximum principle implies that  $\mathcal{V}(x, t)$  is nonnegative if the initial value  $\mathcal{V}(x, 0)$  is nonnegative.

**End of Proof.**

**Lemma 9** *Let  $c$  be the center of mass  $c = \int x u_0(x) dx$ . Then, there exists  $T > 0$  such that*

$$\mathcal{R}(x, t) \leq \mathcal{U}(x + c, t) \leq \mathcal{R}(x, t + T), \quad t > 0. \quad (3.29)$$

**Proof.** First we can easily check that  $\partial_x \mathcal{U}(x, t) = U(x, t) \leq 1$ ,  $\partial_x \mathcal{U}(x, 0) = 1$  for  $x > L$  and

$$\mathcal{U}(L, 0) = \int_0^L \int_0^x u_0(y) dy dx = \left[ x \int_0^x u_0(y) dy \right]_{-\infty}^L - \int_0^L x u_0(x) dx = L - c.$$

Therefore,  $\mathcal{U}(x, 0) = x - c$  for all  $x \geq L$  and  $\mathcal{U}(x, 0) \geq x - c$ . Clearly  $\mathcal{U}(x, 0) \geq 0$  and  $\mathcal{U}(x, 0) = 0$  for all  $x < 0$ .  $\mathcal{R}(x, 0)$  is easy to compute that we have

$$\mathcal{U}(x + c, 0) \geq \mathcal{R}(x, 0) = \begin{cases} 0, & x < 0, \\ x, & x > 0. \end{cases}$$

Considering the explicit formula (3.24) for the canonical solution  $\rho(x, t)$  we can easily check that  $\mathcal{R}(x, t) \rightarrow \infty$  as  $t \rightarrow \infty$  uniformly on any compact interval. Therefore, there exists  $T > 0$  such that

$$\mathcal{R}(x, 0) \leq \mathcal{U}(x + c, 0) \leq \mathcal{R}(x, T)$$

(see **Figure 3**). Now the comparison principle, Proposition 8, completes the proof.

**End of Proof.**

The convergence order of potentials is easily computed like followings:

**Lemma 10 (Potential convergence order)** *Let  $\mathcal{U}, \mathcal{R}$  be Newtonian potentials of  $u, \rho$ , respectively, and  $c = \int xu_0(x)dx$ . Then, there exists  $T > 0$  depending on the initial value  $u_0$  such that*

$$\|\mathcal{R}(t) - \mathcal{U}_c(t)\|_\infty = T \max_x (\phi(\rho(x, t))). \quad (3.30)$$

**Proof.** Let  $T > 0$  be the constant that satisfies (3.29). Then,

$$\begin{aligned} |\mathcal{R}(x, t) - \mathcal{U}(x + c, t)| &\leq |\mathcal{R}(x, t) - \mathcal{R}(x, t + T)| \\ &\leq T \|\mathcal{R}_t\|_\infty = T \max_x (\phi(\rho(x, t))). \end{aligned}$$

Since the estimate is independent of  $x$ , it is a uniform estimate.

**End of Proof.**

Now we transfer this convergence order to solutions. Since the coefficient  $a$  in (3.28) is non-negative, the lap number decreases and hence there is at most two sign-changing points and hence  $U(x, t) - R(x, t)$  may change its sign once. Therefore, by the same arguments in the proof of Proposition 6, we have

$$\|R(t) - U_c(t)\|_1 = 2\|\mathcal{R}(t) - \mathcal{U}_c(t)\|_\infty \leq 2T \max_x (\phi(\rho(x, t))).$$

We need to convert this  $L^1$  convergence rate to a uniform one to use the arguments again. This comes from the  $L^1 - L^\infty$  interpolation and

$$\|R(t) - U_c(t)\|_\infty \leq 2T \max_x (\phi(\rho(x, t))) \max_x (\rho(x, t)). \quad (3.31)$$

Since  $\rho - u$  may changes its sign at most twice, the estimate corresponding to the first step is

$$\|\rho(t) - u_c(t)\|_1 \leq 4\|R(t) - U_c(t)\|_\infty \leq 8T \max_x (\phi(\rho(x, t))) \max_x (\rho(x, t)).$$

This completes the proof of Theorem 2(i).

### 3.4 p-Laplacian type nonlinear diffusion

In this section we show Theorem 2(ii). Consider the case  $\sigma = \sigma_p$ , i.e.,  $u$  is the solution of

$$u_t = \psi(u_x)_x, \quad \lim_{t \downarrow 0} u(x, t) = u_0(x), \quad x \in \mathbf{R}, \quad t > 0, \quad (3.32)$$

where  $\psi$  is an odd function satisfying (3.7) and the initial value  $u_0$  is symmetric and continuous, and satisfies

$$\text{spt}(u_0) = [-L, L], \quad u_0(x) > 0 \text{ for } x \in (-L, L) \quad \text{and} \quad \int u_0(x)dx = 1. \quad (3.33)$$

The source solution  $\rho$  satisfies

$$\rho_t = \psi(\rho_x)_x, \quad \lim_{t \downarrow 0} \rho(x, t) = \delta(x), \quad x \in \mathbf{R}, \quad t > 0. \quad (3.34)$$

Let

$$v = u_x, \quad \varrho = \rho_x.$$

Then after differentiating (3.32) one can easily check that  $v$  and  $\varrho$  are solutions to the porous medium type equations

$$v_t = (\psi(v))_{xx}, \quad \varrho_t = (\psi(\varrho))_{xx}. \quad (3.35)$$

The difference from the previous case is that  $v, \varrho$  are sign-changing solution with zero total mass, i.e.,

$$\int_{-\infty}^{\infty} v(x, t) dx = \int_{-\infty}^{\infty} \varrho(x, t) dx = 0.$$

Let  $u, \tilde{u}$  be solutions to (3.32) and  $v, \tilde{v}$  be their derivatives, respectively. We take the integrals  $U, \tilde{U}$  of  $u, \tilde{u}$  as their potentials. Then,  $U, \tilde{U}$  are Newtonian potentials of  $v, \tilde{v}$ . We already saw in Proposition 3 that  $E := U - \tilde{U} \geq 0$  for all  $t \geq 0$  if  $E(x, 0) \geq 0$ . In the proof we basically used the maximum principle to the relation

$$E_t = b(x, t)E_{xx}, \quad b(x, t) := \frac{\psi(v) - \psi(\tilde{v})}{v - \tilde{v}} \geq 0.$$

Now we construct an estimate related to the time translation. Since  $\rho(x, t) \geq 0$  decays to zero as  $t \rightarrow \infty$ , preserves its initial mass, and is symmetric with respect to  $x = 0$ , we have, for any  $t \geq 0$ ,

$$\begin{aligned} R(x, t) &\rightarrow 1 \text{ as } x \rightarrow \infty, & R(x, t) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \\ R_x(x, t) &\rightarrow 0 \text{ as } t \rightarrow \infty, & R(0, t) &= 1/2 \text{ for } t \geq 0. \end{aligned}$$

Similarly, since  $u(x, t)$  is symmetric, the potential  $U$  of the solution  $u$  satisfies

$$\begin{aligned} U(x, t) &\rightarrow 1 \text{ as } x \rightarrow \infty, & U(x, t) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \\ U_x(x, t) &\rightarrow 0 \text{ as } t \rightarrow \infty, & U(0, t) &= 1/2 \text{ for } t \geq 0. \end{aligned}$$

The initial potentials  $U(x, 0), R(x, 0)$  satisfy

$$\begin{aligned} U(x, 0) &= 1 \text{ for all } x > L, & U(x, 0) &= 0 \text{ for all } x < -L, \\ R(x, 0) &= 1 \text{ for all } x > 0, & R(x, 0) &= 0 \text{ for all } x < 0. \end{aligned}$$

Under the extra assumption that  $U_x(x_0, 0) \neq 0$  at the point such that  $U(x_0, 0) = 1/2$ , one can easily see that there exist  $T > 0, c \in \mathbf{R}$  such that

$$\begin{aligned} R(x - c, T) &\leq U(x, 0) \leq R(x - c, 0), & x &> 1/2a. \\ R(x - c, 0) &\leq U(x, 0) \leq R(x - c, T), & x &< 1/2a. \end{aligned}$$

It is clear that such an estimates as (3.29) is not possible since  $R(x, t)$  and  $U(x, t)$  cross to each other at the point  $x = 0$ , i.e.,  $U(0, t) = R(0, t) = 1/2$  for all  $t > 0$ . However we may apply the maximum principle arguments by restricting the domain to  $x \geq 0$ .

**Lemma 11** *There exists  $T > 0$  such that*

$$R(x, t + T) \leq U(x, t) \leq R(x, t), \quad t > 0, \quad x \geq 0. \quad (3.36)$$

**Proof.** The second inequality in (3.36) is already obtained and we show the first one. Since the initial value is continuous and  $u_0(0) > 0$ , there exists  $\epsilon > 0, \delta > 0$  such that

$$u_0(x) > \epsilon \quad \text{for} \quad 0 < x < \delta.$$

Since  $\rho(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ , there exists  $T > 0$  such that  $\max_x \rho(x, T) < \epsilon$ . Therefore,  $R(x, T) \leq U(x, 0)$  for  $0 < x < \delta$ . Since  $R(x, t) \rightarrow 0$  uniformly as  $t \rightarrow \infty$  on the domain  $[\delta, L]$  and  $U(x, 0) > \epsilon\delta$  on  $[\delta, L]$ , we have  $R(x, T) \leq U(x, 0)$ ,  $0 < x < L$ , after taking larger  $T > 0$  if needed. Since  $U(x, 0) = 1$  for  $x \geq L$  and  $R(x, t) \leq 1$  for  $x \geq L$ , we have

$$R(x, T) \leq U(x, 0), \quad t > 0, x \geq 0.$$

Now we may apply the maximum principle on the domain  $\Omega := \{(x, t) : 0 \leq x < \infty, 0 \leq t < \infty\}$ . Since (3.36) hold on the boundary  $\partial\Omega$ , we may conclude that (3.36) holds on  $\Omega$ .

**End of Proof.**

Now the convergence order is similarly estimated as

$$\|u - \varrho\|_1 \leq 2\|R(t) - R(T+t)\|_\infty \leq 2T\|R_t\|_\infty \leq 2T \max_x \psi(\varrho(x, t))$$

and the proof for Theorem 2(ii) is complete.

In the diffusion case we obtained extra convergence order of similarity scale since the solution is the second order derivative of its potential. Here we could not do that. However, since  $\varrho$  has zero total mass, we may expect that it may decay faster than  $\rho$ .

### 3.5 Explicit computation of convergence orders

In this section we compute the convergence order in Theorems 1 and 2 explicitly under power laws:

$$f(u) = u^q/q, \quad \phi(u) = u^m, \quad \psi(v) = \text{sign}(v)|v|^{p-1}, \quad m > 0, q > 1, p > 1. \quad (3.37)$$

If  $\sigma = f(u)$ ,  $\phi(u)_x$  or  $\psi(u_x)$ , the the source-type solution  $\rho$  is given explicitly and it is well known that

$$\max_{x \in \mathbf{R}} \rho(x, t) = O(t^{-\alpha}), \quad \alpha = \begin{cases} 1/q & \text{if } \sigma = f(u), \\ 1/(m+1) & \text{if } \sigma = \phi(u)_x, \\ 1/(2p-2) & \text{if } \sigma = \psi(u_x), \end{cases} \quad (3.38)$$

as  $t \rightarrow \infty$ . If  $A$  is a sum of two or all of these three terms, the decay power  $\alpha$  is simply the largest one among these three. For the general case  $0 \neq \sigma = c_1\sigma_c + c_2\sigma_d + c_3\sigma_p$ , the decay rate of the solution  $\rho$  is

$$\max_{x \in \mathbf{R}} \rho(x, t) = O(t^{-\alpha}), \quad \alpha = \max\left(\frac{c_1}{q}, \frac{c_2}{m+1}, \frac{c_3}{2p-2}\right) \quad (3.39)$$

as  $t \rightarrow \infty$ . (This decay order has been shown for the convection-diffusion case in [5], i.e.,  $c_1 = c_3 = 1, c_2 = 0$ . I am not sure if there is a paper for (3.39) for the case when all of these three terms are together.)

If  $\alpha = 1/q$ , then we may say that the convection dominates the evolution. Similarly, if  $\alpha = 1/(m+1)$  or  $\alpha = 1/(2p-2)$ , then the nonlinear diffusion operator  $\sigma_d$  or the p-Laplacian operator  $\sigma_p$  dominates the evolution, respectively. Combining (3.10) and (3.39) we obtain the convergence order

$$\|u(t) - \rho(t)\|_1 = O(t^{-\alpha}), \quad \alpha = \max\left(\frac{c_1}{q}, \frac{c_2}{m+1}, \frac{c_3}{2p-2}\right), \quad \text{as } t \rightarrow \infty \quad (3.40)$$

for the general case in (3.8).

For the case  $\sigma = \phi(u)_x$  the order of  $\rho$  is  $t^{-1/(m+1)}$  and hence the convergence order in (3.12) is

$$\|\rho(t) - u_c(t)\|_1 = O(t^{\frac{-1}{m+1}} t^{\frac{-m}{m+1}}) = O(t^{-1})$$

as  $t \rightarrow \infty$ .

**Remark 12** *If  $1 < p < 3/2$ , the decay order in (3.40) is higher than  $1/t$ . It seems that this case the diffusion speed is so fast that the time translation has bigger impact than space translation. The similar phenomenon is observed for the nonlinear diffusion when the diffusion is so fast that the total mass is not preserved.*

**Remark 13** *For the nonlinear diffusion in multi-dimensions,  $\sigma = \Delta(u^m)$ , the convergence order of magnitude of the source solution has been known by Dolbeault & del Pino [2] and Otto [15]. For  $p$ -Laplacian  $\sigma = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  with  $p$  in a certain range  $L^1$  convergence order has been obtained by Dolbeault & del Pino [3] for dimension  $n \geq 2$ , which is lower than the magnitude of solutions. However, under the presence of several phenomena any convergence order is not known until a recent work by Carrillo & Fellner [1]. In [1] the same convergence order has been shown for the case  $\sigma = \sigma_c + \sigma_d$  with  $m + 1 < q$  and  $1 \leq m < 2$ .*

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