LECTURE NOTE ON CONVECTION AND DIFFUSION

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ABSTRACT. Similarity structure.

1. SIMILARITY & SCALING INVARIANCE

The similarity structure of a PDE is from the crudest feature of the equation such as the number of derivatives and exponents. Even if it is based on such a crude feature, one can find a special solution using the similarity structure which is called a similarity solution. This solution keeps the most fundamental feature of the problem.

The technique to find the similarity structure can be applied to quite general problems. The preliminary requirement is that the problem under consideration should have the uniqueness at least. Let $u(x,t)$ be a solution to such a problem and consider a rescaled function v given by

$$u(x,t) = av(bx, ct), \quad a, b, c > 0. \tag{1.1}$$

The main step is to find the relation among a, b, c that guarantees for v to be the solution of the same problem including the initial condition and/or the boundary condition. Then the uniqueness of the problem implies $u = v$ and hence

$$u(x,t) = au(bx, ct), \tag{1.2}$$

which indicates that the solution should have certain structure provided by the equation. Survey of such behavior is the purpose of this section.

1.1. **Riemann Problem.** The first example is the Riemann problem of a conservation law,

$$\begin{aligned} &u_t + f(u)_x = 0, \\ &\lim_{t\downarrow 0} u(x,t) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \end{aligned} \tag{1.3}$$

One can easily see that the rescaled function v satisfies the initial value

$$\lim_{t\downarrow 0} v(x,t) = \begin{cases} au_l, & x < 0, \\ au_r, & x > 0. \end{cases}$$

Therefore, if v is expected to satisfy the same Riemann initial value in (1.3), one should pick $a = 1$. Now let's find the relation between b and c. One can easily check that

$$u_t = cv_t, \ f'(u)u_x = bf'(v)v_x \ \text{ and } \ u_t + f(u)_x = cv_t + bf(v)_x = 0.$$

Therefore, if $b = c$, then v satisfies the conservation law $v_t + f(v)_x = 0$.

The solution of the conservation law that satisfies the entropy condition is unique. For example for the convex scalar case the entropy condition is written by

$$\lim_{x\uparrow x_0} u(x,t) \geq \lim_{x\downarrow x_0} u(x,t).$$

For $b > 0$, such relations are not changed and hence v also satisfies the entropy condition. Since the solution is unique we may conclude $v(x,t) = u(x,t)$ and hence

$$u(x,t) = v(bx, bt) = u(bx, bt). \tag{1.4}$$

This relation indicates that the solution u is constant along any line that emanates from the origin. (See Figure 1.) In other words u is a function of x/t. Therefore, it is natural to introduce a new variable

$$\xi = \frac{x}{t} \tag{1.5}$$

One may say that the problem (1.3) is invariant under the scaling $u(x,t) \to u(bx,bt)$ and the variable ξ also has such a scaling invariance, i.e., even if we substitute bx in the place of x and bt in the place of t, ξ is not changed:

$$\frac{bx}{bt} = \frac{x}{t} = \xi.$$

Therefore, this variable ξ is called a similarity variable of the Riemann problem (1.3). One can rewrite the equation in terms of this similarity variable. Finding the similarity equation is good exercise for the chain rule. First, find

$$\xi_x = t^{-1},\ \xi_t = -\xi t^{-1},\ \frac{\partial}{\partial t}u(\xi) = u'\xi_t = -u'\xi t^{-1},\ \frac{\partial}{\partial x}f(u) = f(u)_\xi \xi_x = f'(u)u't^{-1}.$$

Next, substitute these to the problem (1.3) and obtain,

$$\begin{gathered} -\xi u' + f'(u)u' = 0, \\ \lim_{t\to-\infty} u(\xi) = u_l,\ \lim_{\xi\to+\infty} u(\xi) = u_r. \end{gathered} \tag{1.6}$$

Notice that the similarity structure of the Riemann problem is independent of the flux function $f(u)$. It can be a system of any kind as long as it provides proper uniqueness property. The nonlinearity or the power of $f(u)$ is not an issue. This fact is related to the choice of $a = 1$.

HOME WORK. Find the similarity structure in the following problem.

$$\begin{aligned} &u_t = u_{xx}, \\ &\lim_{t\downarrow 0} u(x,t) = \begin{cases} u_l, & x<0, \\ u_r, & x>0. \end{cases} \end{aligned}$$

1.2. L^1 **solutions to convection equations.**

Let $u(x,t)$ be an L^1 solution to

$$\begin{aligned} &u_t + f(u)_x = 0, \\ &\lim_{t\downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbf{R}). \end{aligned}$$

Since the total mass is preserved, one should consider a scaling that preserve the total mass. If not, there is no chance that v and u are identical. Since we consider one space dimension, we set $a = b$, i.e., the scaling considered is

$$u(x,t) = av(ax,ct), \qquad a,c>0. \tag{1.7}$$

Unlike the Riemann problem case the relation of scaling of invariance depends on the flux function f since $a \neq 1$. Therefore, we can not consider the flux function with the generality of the Reimann problem. Here, we consider a scalar case under a power law:

$$\begin{aligned} &u_t + f(u)_x = 0, \quad f(u) = |u|^q/q,\ q>1 \\ &\lim_{t\downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbf{R}). \end{aligned} \tag{1.8}$$

One can easily check that

$$u_t = acv_t,\ u_x = a^2 v_x \ \text{ and } \ u_t + f'(u)u_x = acv_t + a^{q+1}f'(v)v_x = 0.$$

Therefore, if $c = a^q$, then v also satisfies the same convection equation. Now we consider similarity variables,

$$\xi = xt^{-1/q}\ , \quad u(x,t) = t^{-1/q}\tilde{u}(\xi,t). \tag{1.9}$$

Then the quantities ξ and $\tilde{u}$ are not changed under the rescaling $(x,t)\to(ax,a^qt)$ and $u\to u/a$. One may derive an equation satisfied by these similarity variables. Using the chain rule, one may obtain

$$\xi_x=t^{-1/q},\ \xi_t=-\xi t^{-1}/q,\ u_x=t^{-2/q}\tilde{u}_\xi,\ u_t=t^{-1/q}t^{-1}\tilde{u}/q-t^{-1/q}\tilde{u}_\xi\xi t^{-1}/q+t^{-1/q}\tilde{u}_t.$$

Substitute these quantities to the equation and obtain

$$u_t+\operatorname{sign}(u)|u|^{q-1}u_x=t^{-1/q}\big(\tilde{u}_t+t^{-1}\big(\operatorname{sign}(\tilde{u})|\tilde{u}|^{q-1}\tilde{u}_\xi-(\xi\tilde{u}/q)_\xi\big)\big)=0.$$

Therefore, $\tilde{u}$ satisfies

$$\tilde{u}_t+t^{-1}(|\tilde{u}|^q/q-\xi\tilde{u}/q)_\xi=0.$$

Since the independent variable t appears explicitly in the equation, we consider a new time scale $s=\ln t$. The full similarity variables are

$$\xi=\frac{x}{\sqrt[q]{t}},\quad s=\ln t,\quad \tilde{u}(\xi,s)=\sqrt[q]{t}\,u(x,t). \tag{1.10}$$

Under the is exponentially fast time scale, $t=e^s$, the only difference is $\tilde{u}_t=\tilde{u}_st^{-1}$ and the equation is turned into

$$\tilde{u}_s+(|\tilde{u}|^q/q-\xi\tilde{u}/q)_\xi=0,\quad \tilde{u}(\xi,0)=u(\xi,1). \tag{1.11}$$

Notice that $s=0$ corresponds to $t=1$ and, hence we have $\tilde{u}(\xi,0)=u(\xi,1)$. It is also common use the variables $s=\ln(t+1)$, $\xi=x/\sqrt[q]{t+1}$, $\tilde{u}=\sqrt[q]{t+1}\,u$. In the case one have $\tilde{u}(\xi,0)=u(\xi,0)$ and the computation is slightly more complicate.

We have checked that the function $v(x,t)$ given by

$$u(x,t)=av(ax,a^qt)$$

satisfied the differential equation (1.8). Therefore, one may ask if $u=v$. To see that one should check the initial value. Under the above scaling one can easily check that

$$v(x,0)=u(x/a,0)/a.$$

Therefore, the only L^1 function that provides the identical initial value after the rescaling is the dirac-measure type $M\delta(x)$. Such a solution is usually called a source-type solution and we denote this source-type solution in this note by $\rho(x,t)$, i.e., $\rho(x,t)$ is a solution having initial value

$$\lim_{t\downarrow 0}\rho(x,t)=M\delta(x). \tag{1.12}$$

Similarly set

$$\tilde{\rho}(\xi,s)=\sqrt[q]{t}\rho(x,t).$$

If positive solutions are considered, then the source-type solution is unique. However, there is no unique solution if sign-changing solutions are allowed. In that case two invariant variables are introduced, which are

$$P=-\inf_x U(x,t),\quad Q=P+\lim_{x\to\infty}U(x,t),\quad \text{where }\ U(x,t)=\int_{-\infty}^x u(y,t)dy. \tag{1.13}$$

It is well known that the positive quantities P and Q are constant in time if the flux is convex (i.e., $f''(u)\ge 0$) and $Q-P=M$.If these invariant constants are prescribed, then the source-type solution is unique and hence we may conclude that,

$$\text{if } P \text{ and } Q \text{ are prescribed, then }\ \rho(x,t)=a\rho(ax,a^qt).$$

Therefore, the value of $\rho(x,t)$ along the curve $x/\sqrt[q]{t}=\mathit{constant}$ is decided completely by the value at a single point, and the resacled value $\tilde{\rho}$ is constant along it. (See Figure 2.) Therefore, $\tilde{\rho}$ is a function of ξ only if the initial value is given by the dirac-measure. This similarity solution is computed explicitly in the followings. Since $\tilde{\rho}_s=0$ and $\tilde{\rho}$ is compactly supported,

$$(|\tilde{\rho}|^q-\xi\tilde{\rho})_\xi=0\Rightarrow|\tilde{\rho}|^q-\xi\tilde{\rho}=0\Rightarrow\tilde{\rho}=0\text{ or }\operatorname{sign}(\tilde{\rho})|\tilde{\rho}|^{q-1}=\xi. \tag{1.14}$$

The solution that has the invariants P, Q and that satisfies the entropy condition should be given by

$$\tilde\rho(\xi) = \begin{cases} (f')^{-1}(\xi), & -a_P < \xi < a_Q, \\ 0, & \text{otherwise}, \end{cases} \tag{1.15}$$

where the end points of the support is given by the relation

$$-\int_{-a_P}^{0} \tilde\rho(\xi)d\xi = P, \quad \int_0^{a_Q} \tilde\rho(\xi)d\xi = Q.$$

From these relations, a_P and a_Q are explicitly computed:

$$a_P = \Big(\frac{qP}{q-1}\Big)^{\frac{q-1}{q}}, \qquad a_Q = \Big(\frac{qQ}{q-1}\Big)^{\frac{q-1}{q}}. \tag{1.16}$$

This self-similar solution is called an N-wave due to its shape. (See Figure 3.) If the positive solutions are considered, then $p = 0$ and $q = M$ and the N-wave has simply a triangle like shape.

By changing the similarity variables back to the original ones, one can easily transform the N-wave in terms of the original variables:

$$\rho(x,t) = \begin{cases} (f')^{-1}(x/t), & -a_P \sqrt[q]{t} < \xi < a_Q \sqrt[q]{t}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.17}$$

Suppose that the flux is given by $f(u) = |u|^2 + |u|^3$. In this case the problem has no scaling invariance. However, $\rho(x,t)$ in (1.17) is well defined for all convex flux, $f''(u) \ge 0$, and is the source-type solution. Since $\max_x \tilde\rho$ is of order $O(1)$, the source-type solution in the original variable has the order

$$\max_x \rho(x,t) = O(t^{-1/q}) \quad \text{as} \quad t \to \infty. \tag{1.18}$$

1.3. L^1 **solutions to diffusion equations.** Let $u(x,t)$ be a positive L^1 solution to a nonlinear diffusion equation

$$u_t = \phi(u)_{xx},$$
$$\lim_{t\downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbf{R}),$$

where the diffusion function ϕ is given by

$$\phi(u) = u^m, \quad m > 0. \tag{1.19}$$

Consider a scaling of

$$u(x,t) = av(ax, ct), \qquad a, c > 0. \tag{1.20}$$

One can easily check that

$$u_t = acv_t,\ \phi(u)_{xx} = a^{m+2}\phi(v)_{xx} \ \text{ and } \ u_t - \phi(u)_{xx} = acv_t - a^{m+2}\phi(v)_{xx} = 0.$$

Therefore, if $c = a^{m+1}$, then v also satisfies the same diffusion equation. Now we consider similarity variables. Consider

$$\alpha = m+1, \quad s = \ln t\ , \quad \xi = xt^{-1/\alpha}\ , \quad u(x,t) = t^{-1/\alpha}\tilde u(\xi, s). \tag{1.21}$$

Then the quantities ξ and $\tilde u$ are not changed under the rescaling $(x,t) \to (ax, a^\alpha t)$ and $u \to u/a$. One may derive an equation satisfied by these similarity variables. Using the chain rule, one obtain

$$\xi_x = t^{\frac{-1}{\alpha}},\ \xi_t = \frac{-\xi t^{-1}}{\alpha},\ \phi(u)_{xx} = t^{\frac{-1}{\alpha}}t^{-1}\phi(\tilde u)_{\xi\xi},\ u_t = -\frac{t^{\frac{-1}{\alpha}}t^{-1}}{\alpha}\tilde u - \frac{t^{\frac{-1}{\alpha}}t^{-1}}{\alpha}\xi\tilde u_\xi + t^{\frac{-1}{\alpha}}t^{-1}\tilde u_s.$$

Substituting these quantities to the equation we obtain

$$u_t - \phi(u)_{xx} = t^{-1-1/\alpha}(\tilde u_s - (\phi(u)_\xi + \xi\tilde u/\alpha)_\xi) = 0.$$

Therefore, $\tilde u(\xi, s)$ satisfies

$$\tilde u_s - (\phi(u)_\xi + \xi\tilde u/\alpha)_\xi = 0, \quad \tilde u(\xi, 0) = u(\xi, 1). \tag{1.22}$$

Let $\rho(x,t)$ be the positive source-type solution with initial value $\lim_{t\downarrow 0} u(x,t) = M\delta(x)$. Then, due to the similarity property its similarity quantity $\tilde\rho(\xi, s)$ is a function of ξ only and satisfies

$$(\phi(\tilde\rho)_\xi + \xi\tilde\rho/\alpha)_\xi = 0.$$

We may do the similar computation as the convection case:

$$(\phi(\tilde\rho)_\xi + \xi\tilde\rho/\alpha)_\xi = 0 \Rightarrow \phi(\tilde\rho)_\xi + \xi\tilde\rho/\alpha = 0 \Rightarrow \rho = 0 \text{ or } \frac{\phi'(\tilde\rho)}{\tilde\rho}\tilde\rho_\xi + \frac{\xi}{\alpha} = 0 \Rightarrow \tilde\rho = 0 \text{ or } g(\tilde\rho)_\xi = -\frac{\xi}{\alpha},$$

where

$$g'(x) = \phi'(x)/x, \quad (\text{i.e.,} \quad g(x) = \frac{m}{m-1}x^{m-1} \text{ for } \phi(x) = x^m).$$

Therefore, we may conclude that

$$\tilde\rho(\xi) = \max\big(0, g^{-1}(C - \xi^2/(2\alpha))\big), \tag{1.23}$$

where C is a constant that should be decided by the total mass relation $\int \tilde\rho(\xi)d\xi = M$.

Under the change of variable, one can easily transform the N-wave in terms of the original variables:

$$\rho(x,t) = \max\big(0, g^{-1}(Ct^{\frac{1-m}{m+1}} - x^2/(2\alpha t))\big) = \max\big(0, g^{-1}([Ct^{\frac{2}{m+1}} - x^2/(2\alpha)]/t)\big). \tag{1.24}$$

Notice that, unlike the convection case, the expression of $\rho(x,t)$ depends on the power m. It is not obvious to find the explicit formula for the source-type solution for a general case when the diffusion is not given by the power law.

Since $\max_x \tilde\rho$ is of order $O(1)$, the source-type solution in the original variable has the order

$$\max_x \rho(x,t) = O(t^{-1/(m+1)}) \quad \text{as} \quad t \to \infty. \tag{1.25}$$

HOME WORK.

(1) Show that the self-similar solution $\rho(x,t)$ is the Gaussian for the linear heat equation case $m = 1$.
(2) Show that the support of $\rho(x,t)$ is compact for any given time $t > 0$ if $m > 1$.
(3) Show that the support of $\rho(x,t)$ is strictly positive compact on the whole real line for any given time $t > 0$ if $m > 1$.
(4) Discuss the similarity structure for the multidimensional case

$$u_t = \Delta u^m, \quad m > 0.$$

1.4. **p-Laplacian.** Let $u(x,t)$ be a positive L^1 solution to a different kind of nonlinear diffusion equation, so called the p-Laplacian,

$$\begin{aligned} &u_t = (|u_x|^{p-2}u_x)_x, \quad p > 1, \\ &\lim_{t\downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbf{R}). \end{aligned} \tag{1.26}$$

One may similarly study the similarity structure of the p-Laplacian equation. From the similarity variables one can show the asymptotic magnitude of the source-type solution:

$$\max_x \rho(x,t) = O(t^{-1/(2p-2)}) \quad \text{as} \quad t \to \infty. \tag{1.27}$$

HOME WORK.

(1) Find the similarity variables of the solutions to (1.26).
(2) Find the equation that the similarity variables satisfy.
(3) Find the explicit formula that the self-similar solution satisfies.
(4) Verify that the self-similar solution $\rho(x,t)$ is the Gaussian for the linear heat equation case $p = 2$.
(5) Find the range of the power $p > 1$ that the self-similar solution $\rho(x,t)$ is compactly supported.
(6) Discuss the similarity structure for the multidimensional case

$$u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u), \quad p > 1.$$

One can easily see that

$$u(x,t) = av(ax, a^{2p-2}t)$$

is the invariance scaling of the p-Laplacian. Therefore, the similarity variables are

$$\alpha = 2p-2, \quad \xi = xt^{1/\alpha}, \quad s = \ln t, \quad \tilde{u}(\xi, s) = t^{1/\alpha} u(x,t) \tag{1.28}$$

and they satisfy

$$\tilde{u}_s - (|\tilde{u}_\xi|^{p-2}\tilde{u}_\xi + \xi\tilde{u}/\alpha)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1).$$

Let $\rho(x,t)$ be the source-type solution and $\tilde{\rho}$ be the self-similar correspondence. Then, $\tilde{\rho}$ is the solution of the ordinary differential equation

$$|\tilde{\rho}_\xi|^{p-2}\tilde{\rho}_\xi + \xi\tilde{\rho}/\alpha = 0.$$

How to solve it?

1.5. **Co-existence of several different phenomena.** Let $u(x,t)$ be a positive L^1 solution to a mixed problem,

$$\begin{aligned} &u_t = (u^m)_{xx} + (|u_x|^{p-2}u_x)_x - (u^q/q)_x, \quad m > 0, q > 1, p > 1, \\ &\lim_{t \downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbf{R}). \end{aligned} \tag{1.29}$$

If

$$q = m + 1 = 2p - 2,$$

then the problem has a invariance scaling and hence we may find the similarity structure of the problem. If not, there is no exact similarity property. However, one may find the dominant factor of the evolution. We may conclude that

$$\max_x |u(x,t)| = O(t^{-1/\alpha}) \quad \text{as} \quad t \to \infty, \qquad \alpha = \min(q, m+1, 2p-2). \tag{1.30}$$

If $\alpha = q$, then we may say that the convection dominates the evolution. Similarly the PME type diffusion dominates the evolution if $\alpha = m+1$ and the p-Laplacian type diffusion decide the asymptotics if $\alpha = 2p-2$.

HOME WORK.

(1) Suppose that $\alpha = q$ in (1.30). Transform the equation (1.29) using the similarity variables for the convection case (1.10) and observe that the coefficients for other terms decays to zero.
(2) Discuss what decide the similarity structure of the problems. What is the basic difference between the convection and diffusion when one decides the scaling invariance.
(3) When the magnitude of the solution decays faster than the order $1/t$?

1.6. L^p **norm preserving similarity.** So far we have mostly considered L^1 norm preserving similarity structure. If one considers a problem that preserves L^p-norm, then one may consider L^p-norm preserving similarity structure. Hence, one may develop similarity structure depending on the solution space under consideration.

2. Characteristic curves for convection equations

S Let $u(x,t)$ satisfy

$$u_t + a(x,t)u_x = b(x,t), \quad \lim_{t \downarrow 0} u(x,t) = u_0(x) \in L^1(\mathbb{R}). \tag{2.1}$$

A curve in the xt-plane, $x = x(t)$, is called a characteristic curve of (2.1) if it satisfies

$$x'(t) = a(x(t), t).$$

One can easily check that

$$\frac{d}{dt}u(x(t),t) = u_t + x'(t)u_x = u_t + a(x(t),t)u_x = b(x(t),t).$$

In other words, the solution satisfies an ODE on the characteristic curve. In particular, if $b = 0$, u is constant along the characteristic curve $x = x(t)$. Furthermore, if the characteristic speed is decided by the solution u only, the characteristic curve is a line.

2.1. **Scalar conservation laws.** Consider

$$u_t + f(u)_x = u_t + f'(u)u_x = 0 \tag{2.2}$$

under the assumption that f is smooth and satisfies

$$f(0) = f'(0) = 0, \quad f''(u) \geq 0.$$

In this case characteristic curves satisfy

$$x'(t) = f'(u(x(t), t))$$

and

$$\frac{d}{dt}u(x(t), t) = u_t + x'(t)u_x = u_t + f'(u(x(t), t))u_x = 0.$$

This shows that u is constant along the characteristic curve and therefore the slope of the curve is constant, i.e, characteristic curves are straight lines. (If the waves speed $a(x, t)$ is discontinuous, then one may consider minimal or maximal characteristics.)

If characteristics collide to each other, then a discontinuity may appear. In that case, the speed of the propagation of the discontinuity, or the shock speed, is given by the Rankini-Hugoniot jump condition,

$$\text{shock speed } s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{and } s = \frac{f(u)}{u} \text{ if } u_- = 0.$$

We take a primitive of the solution as the potential of the problem,

$$U(x, t) = \int_{-\infty}^{x} u(y, t)dy. \tag{2.3}$$

Integrating (2.3) on $(-\infty, x)$ gives

$$U_t + f(u) = U_t + \frac{f(u)}{u}U_x = 0.$$

Here, potential U is constant along the characteristic $x'(t) = f(u)/u$. However, this characteristic curve is not the same the characteristic of u and is not a straight line any more. This curve is related to the shock characteristics when a non-zero value is connected to the zero state.

2.2. **Isentropic Gas Dynamics.** In this section $\rho(x, t)$ represents the density of a given gas, $u(x, t)$ is the velocity, $m = \rho u$ is the momentum, and p is the pressure. In the isentropic gas, the pressure is assumed to be a function of density only $p = p(\rho)$ and they satisfies the equation

$$\begin{aligned} &\rho_t + m_x = 0, \\ &m_t + (m^2/\rho + p(\rho)\,)_x = 0. \end{aligned} \tag{2.4}$$

We may consider characteristic curves of this system of two equations. There are two sets of characteristic curves from the two equations.

Set potentials as

$$R(x, t) = \int_{-\infty}^{x} \rho(y, t)dy, \quad M(x, t) = \int_{-\infty}^{x} m(y, t)dy.$$

Then, after the integration of the equation (2.4), one obtains

$$\begin{aligned} &R_t + uR_x = 0, \\ &M_t + (u + p(\rho)/m\,)M_x = 0. \end{aligned} \tag{2.5}$$

We may consider the characteristic curves of these equations for potentials which are different from the ones of (2.4). The first relation in (2.5) implies that R is constant along the characteristic curve

$$x_1'(t; x_0) = u(x_1, t), \quad x_1(0; x_0) = x_0.$$

This characteristic curves are potential curves of the velocity field u and hence gives the trace of a moving particle which is also called particle trajectories. These characteristic curves cannot cross each other and hence the total mass of the gas on the both sides of it should remain constant, i.e.,

$$\frac{d}{dt}\int_{-\infty}^{x_1(t;x_0)} \rho(x,t)dx = 0.$$

Under the power law $p(\rho)=\rho^\gamma,\ \gamma\ge 1$, the characteristic curve for the momentum potential is given by

$$x_2'(t;x_0) = (u+\rho^{\gamma-1}/u)(x_2,t), \quad x_2(0;x_0)=x_0.$$

This indicates that the characteristic curve on which the momentum potential is constant has higher speed than the particle trace has. In particular, if the velocity u approaches to the zero, the speed of the second characteristic curve increases to infinity (what it mean ?). If the density ρ approaches to zero and $\gamma>1$, the two characteristic curves, x_1 and x_2, becomes parallel to each other.

3. Similarity curves for advection-diffusion equations

The similarity variables in (1.5),(1.10), (1.21), and (1.28) are obtained after finding invariance scaling relations first. However, it is possible to find similarity variables without knowing the relation. For example, if one considers a conservation law, one may simply set

$$s=\ln(t),\ \xi = xt^\lambda,\ u(x,t)=t^{n\lambda}v(\xi,s),\ u^m(x,t)=t^{mn\lambda}v^m(\xi,s), \tag{3.1}$$

where n is the space dimension. Then,

$$\begin{aligned}
s_t &= t^{-1},\\
\nabla_x\cdot\xi &= nt^\lambda,\\
\xi_t &= \lambda\xi t^{-1},\\
\nabla_x u &= t^{(n+1)\lambda}\nabla_\xi v,\\
\Delta_x u &= t^{(n+2)\lambda}\Delta_\xi v,\\
\Delta_x u^m &= t^{(mn+2)\lambda}\Delta_\xi v^m,\\
u_t &= n\lambda t^{n\lambda-1}v + t^{n\lambda}\nabla_\xi v(\xi,s)\cdot\xi_t + t^{n\lambda}v_s(\xi,s)s_t\\
&= n\lambda t^{n\lambda-1}v + t^{n\lambda-1}\lambda\xi\cdot\nabla_\xi v(\xi,s) + t^{n\lambda-1}v_s(\xi,s).
\end{aligned}$$

Substitute some of these into a given equation and find parameter λ that makes the equation autonomous. For example, if your equation is

$$u_t = \Delta u^m,$$

then you will obtain

$$n\lambda t^{n\lambda-1}v + t^{n\lambda-1}\lambda\xi\cdot\nabla_\xi v(\xi,s) + t^{n\lambda-1}v_s(\xi,s) = t^{(mn+2)\lambda}\Delta_\xi v^m.$$

If we take λ by

$$n\lambda - 1 = (mn+2)\lambda \ \Rightarrow\ \lambda = \frac{1}{n-mn-2}, \tag{3.2}$$

then the equation becomes autonomous, which is

$$v_s + \frac{n}{n-mn-2}v + \frac{\xi}{n-mn-2}\cdot\nabla_\xi v(\xi,s) = \Delta_\xi v^m.$$

In summary, the similarity variables and the transformed equation are

$$\xi = xt^{\frac{1}{n-mn-2}},\ s=\ln(t),\ v(\xi,s) = t^{\frac{n}{2+mn-n}}u(x,t), \tag{3.3}$$

$$v_s + \nabla_\xi\cdot\Big(\frac{\xi v}{n-mn-2}\Big) = \Delta_\xi v^m. \tag{3.4}$$

These are the same similarity variables and equation given in (1.21) and (1.22) for the one space dimension $n=1$.

4. Long time asymptotics in diffusion and convection

Let $u(x,t)$ be a positive solution of

$$u_t = \frac{\partial}{\partial x}\sigma(u, u_x), \quad \lim_{t\downarrow 0} u(x,t) = u_0(x) \geq 0, \quad t > 0,\, x \in \mathbb{R}, \tag{4.1}$$

where the limit is understood in L^1 sense and the positive initial value u_0 is compactly supported and integrable. We may assume without loss of generality that

$$\mathrm{supp}(u_0) \subset [0, L] \text{ and } \int u_0(x)dx = 1 \tag{4.2}$$

after an appropriate normalization. We consider three classical examples.

The convection is the case that σ is a function of u only. For this case we consider a smooth flux function,

$$\sigma_c(u) := f(u), \qquad f(0) = f'(0) = 0. \tag{4.3}$$

The extra conditions $f(0) = f'(0) = 0$ can be assumed without loss of generality after appropriate translations. In many papers convex flux functions are considered under the power law $f(u) = -u^q/q$, $q > 1$, and the case with $q = 2$ is the well known Burgers equation. We will consider without the convexity assumption on the flux $f(u)$.

For the diffusion case the nonlinear diffusion,

$$\sigma_d(u, u_x) := \phi'(u)u_x = \phi(u)_x, \qquad m > 0, \tag{4.4}$$

is considered, where ϕ is assumed to satisfy

$$\phi \in C[0, +\infty) \cap C^1(0, +\infty), \quad \phi(0) = 0 \text{ and } \phi'(u) \geq 0 \text{ for all } u > 0. \tag{4.5}$$

In this case the diffusion speed, $\phi'(u)$, is a function of the solution u. In the study of porous medium equations the power law, $\phi(u) = u^m$, is mostly considered for $m > 0$ and the case $m = 1$ is the classical heat equation.

The last case considered is the one when the diffusion speed is a function of the gradient of the solution. Consider

$$\sigma_p(u_x) := \psi(u_x), \tag{4.6}$$

where ψ is an odd function, $\psi(-v) = -\psi(v)$, and satisfies the same condition as ϕ in (4.5) for $v := u_x > 0$, i.e.,

$$\psi \in C[0, +\infty) \cap C^1(0, +\infty), \quad \psi(0) = 0 \text{ and } \psi'(v) \geq 0 \text{ for all } v > 0. \tag{4.7}$$

The p-Laplacian is the case under power law $\psi(v) = \mathrm{sign}(v)|v|^{p-1}, p > 1$.

In the entropy diminishing technique one of the main steps is to transform the problem using time dependent similarity variables. Then nontrivial steady solutions appear and one may find explicit canonical solutions such as N-waves or Barenblatt type solutions. Intermediate asymptotics is then obtained via diminishing of entropy difference between the solution and a steady state.

In this section we consider the cases in

$$0 \neq \sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c, \qquad c_i = 0 \text{ or } 1. \tag{4.8}$$

In these cases there exist two or more phenomena of different similarity scales and there is no exact similarity variables that provide any nontrivial steady solution. Let $\rho(x,t) \geq 0$ be the source solution to (4.1), i.e.,

$$\rho_t = \frac{\partial}{\partial x}\sigma(\rho, \rho_x), \quad \lim_{t\downarrow 0} \rho(x,t) = \delta(x), \quad x \in \mathbb{R},\; t \in \mathbb{R}^+. \tag{4.9}$$

If the operator σ is given by (4.3), (4.4) or (4.6), under the power law, the canonical solution $\rho(x,t)$ is given explicitly due to the similarity structure of each phenomenon. However, such an explicit solution does not exist for the general case in (4.8). Therefore, we should continue the study of asymptotic decay without such an explicit solution. It is well known that the difference between a general solution $u(x,t)$ to (4.1),(4.2) and the source-type solution $\rho(x,t)$ decays to zero in L^1 sense,

$$\|u(t)-\rho(t)\|_1 \to 0 \quad \text{as} \quad t\to\infty.$$

Here and in the remainder of this note, we use the notation $\|\cdot\|_1$ to denote the L^1-norm and $u(t)$ to denote the function on the real line given by $u(t)|_x = u(x,t)$. Since the L^1-norm of a solution is preserved, the L^1-norm is a natural way to measure the distance between two solutions. Note that one may obtain asymptotic orders in L^p-norm, $p\ge 1$ via classical interpolation arguments.

The main result is the following theorem:

Theorem 4.1. *Let $\sigma_c, \sigma_d, \sigma_p$ and σ be given by (4.3)–(4.8), $u(x,t)$ be the solution to (4.1)-(4.2) and $\rho(x,t)$ be the source-type solution (4.9). Then,*

$$\|u(t)-\rho(t)\|_1 \le 2L \max_x \rho(x,t). \tag{4.10}$$

For the case with convection only, $\sigma=\sigma_c$, its asymptotics has been studied in [7] and hence we consider the other 6 possible cases in this paper. To obtain a convergence of the next order one should place the solution $u(x,t)$ at the correct position. Then one may obtain an intermediate asymptotics in L^1-norm, which is

$$\|u_c(t)-\rho(t)\|_1 = O(1/t) \quad \text{as} \quad t\to\infty, \tag{4.11}$$

where $u_c(x,t) := u(x+c,t)$. In the literature the convergence order of $1/t$ has been also studied in various cases. We could not obtain this convergence order with the generality in Theorem 4.1. The convergence order corresponding to the order $1/t$ is obtained for the following two cases:

Theorem 4.2. *Let $u(x,t)$ be the solution of (4.1)-(4.2), σ_d and σ_p be given by (4.4) and (4.6), respectively, and $\rho(x,t)$ be a canonical solution (4.9). Then,*
(i) Let $\sigma=\sigma_d$. If $c=\int xu_0(x)dx\in\mathbb{R}$ is the center of mass, then there exists $T>0$ such that

$$\|\rho(t)-u_c(t)\|_1 \le 8T \max_x \big(\rho(x,t)\phi(\rho(x,t))\big). \tag{4.12}$$

(ii) Let $\sigma=\sigma_p$ and u_0 is symmetric and continuous, and satisfies

$$\mathrm{supp}(u_0)=[-L,L],\ u_0(x)>0 \textit{ for } x\in(-L,L) \textit{ and } \int u_0(x)dx=1. \tag{4.13}$$

Then, there exists $T>0$ such that

$$\|\rho(t)-u(t)\|_1 \le 2T \max_x \big(\psi(\rho_x(x,t))\big). \tag{4.14}$$

Similar convergence order has been shown in [7] for the case of $\sigma=\sigma_c$ where the flux is convex: If $c=\min(\mathrm{supp}(u_0))$ and there exist $\epsilon, t_0>0$ such that

$$\int_{-\infty}^x \rho(y,t_0)dy \le \int_{-\infty}^x u(y+c,0)dy, \quad 0<x<\epsilon, \tag{4.15}$$

then there exists $T>0$ such that

$$||\rho(t)-u_c(t)||_1 \le 2T\, f\big(\max_x \rho(x,t)\big). \tag{4.16}$$

One of the main goals of this article is to introduce a potential comparison technique in the simple setting of one space dimension. However, the technique has been originally developed for several space dimension for nonlinear diffusion equation [8]. This note consists as followings. In Section 2 several preliminary steps are constructed including the potential comparison principle and then Theorem 4.1 is proved using the potential comparison technique. Section 3 is for the porous medium type equations and Theorem 4.2 (i) is proved.

In Section 4 the derivative of the solution to the p-Laplacian type equation (7.1) is understood as the sign-changing solution to the porous medium equation. The technique for the porous medium equation is naturally applied to show Theorem 4.2 (ii). To obtain the convergence orders in Theorems explicitly we consider special cases under the power laws in Section 5. If f, ϕ and ψ are given by (8.1), the convergence orders in (4.12), (4.14) and (4.16) turn out to be the order $1/t$.

5. L^1 CONVERGENCE ORDER OF THE MAGNITUDE OF SOLUTIONS

In this section we show Theorem 4.1. For the convection equation case, $\sigma = \sigma_c$, the entropy solutions are considered and the convergence order has been shown in [6], Theorem 1 (13), by trapping a solution between two rarefaction waves. In this chapter we consider the integrals of the solution $u(x,t)$ and the canonical one $\rho(x,t)$ as their potentials,

$$U(x,t) = \int_{-\infty}^{x} u(y,t)dy, \qquad R(x,t) = \int_{-\infty}^{x} \rho(y,t)dy. \tag{5.1}$$

Integrating (4.1) over $(-\infty, x)$ yields

$$U_t = \sigma(u, u_x) = \sigma(U_x, U_{xx}). \tag{5.2}$$

To justify this Hamilton-Jacobi type equation we should have

$$\lim_{x\to-\infty} \sigma(u(x,t), u_x(x,t)) = 0 \text{ and } \frac{\partial}{\partial t}\int_{-\infty}^{x} u(y,t)dy = \int_{-\infty}^{x} \frac{\partial}{\partial t} u(y,t)dy,$$

which can be easily verified.

It is well known that, if $u(x,0) \geq \tilde{u}(x,0)$, then the solutions $u, \tilde{u}$ to (4.1) satisfies $u(x,t) \geq \tilde{u}(x,t)$ for all $t > 0$. However, one can not expect this relation between u and ρ even after a time and space translation since ρ and u share the same total mass. In the following proposition we show that a similar comparison principle holds between potentials.

Proposition 5.1 (Comparison Principle). *Let σ be one of the 7 cases in Theorem 4.1, u and $\tilde{u}$ be solutions of (4.1) and U and $\tilde{U}$ be their potentials given by their integrals as in (5.1), respectively. Then, if $U(x,0) \geq \tilde{U}(x,0)$, then $U(x,t) \geq \tilde{U}(x,t)$ for all $t \geq 0$.*

Proof. Let $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$ where $c_i = 0$ or 1. One can easily check that the potential difference $E = U - \tilde{U}$ satisfies

$$\text{Ł}[E] := (c_1 a(x,t) + c_2 b(x,t))E_{xx} + (c_3 c(x,t))E_x - E_t = 0, \;\; E(x,0) \geq 0,$$

where

$$a := \frac{\phi'(u)u_x - \phi'(\tilde{u})\tilde{u}_x}{u_x - \tilde{u}_x}, \quad b := \frac{\psi(u_x) - \psi(\tilde{u}_x)}{u_x - \tilde{u}_x}, \quad c := \frac{f(u) - f(\tilde{u})}{u - \tilde{u}}.$$

Notice that $b(x,t) \geq 0$ and $c(x,t) \geq 0$ since $f(u)$ and $\mathrm{sign}(u_x)\psi(|u_x|)$ are increasing functions with respect to $u \geq 0$ and u_x, respectively. However, $a(x,t)$ may change its sign. In the followings we check if the maximum principle type arguments hold. First observe the uniform decay of E, i.e.,

$$|E(x,t)| \leq \int_{-\infty}^{x} |u(y,t) - \tilde{u}(y,t)|dy \leq \|u(x,t) - \tilde{u}(x,t)\|_1 \to 0$$

as $t \to \infty$. Suppose that there exists $t \geq 0$ and $x \in \mathbb{R}$ such that $E(x,t) < 0$. Then the uniform convergence and the initial condition $E(x,0) \geq 0$ imply that there exists a global minimum point (x_0, t_0).

First consider the case $c_1 = 1$. Since the solutions $u, \tilde{u}$ are continuous, E is differentiable and hence

$$E_x(x_0, t_0) = 0.$$

Therefore, $u(x_0,t_0) = \tilde{u}(x_0,t_0)$ and $a(x_0,t_0) = \phi'(u(x_0,t_0))$. The point x_0 should be an interior point of the support of $u(\cdot,t_0)$ or $\tilde{u}(\cdot,t_0)$ (otherwise $E(x_0,t_0) = 0$) and hence $a(x_0,t_0) > 0$ and the equation is uniformly parabolic in a small neighborhood of (x_0,t_0). The maximum principle now easily derives a contradiction.

Now consider the case $c_1 = 0$ and $c_2 = 1$. (In this case the coefficient b may diverges and hence the maximum principle is not obvious.) Since $E(x,t_0)$ is not a constant function and x_0 is a minimum point, there exists $\delta > 0$ such that $E(x,t_0) > E(x_0,t_0)$ for $x \in (x_0 - \delta, x_0)$ after retaking x_0 if needed. By taking smaller $\delta > 0$ if needed, we may assume $E_x(\cdot,t_0) < 0$ and $E_{xx}(\cdot,t_0) > 0$ on

$(x_0-\delta, x_0)$. If $b(x_0,t_0)>0$, the problem is uniformly parabolic in neighborhood of (x_0,t_0) again. Therefore, consider the case when $b(x_0,t_0)=0$. Let $W=E+\varepsilon z$, where

$$z(x)=1-e^{x_0-x} \quad \text{and } 0<\varepsilon<\frac{E(x_0,t_0)-E(x_0-\delta,t_0)}{z(x_0-\delta)}.$$

Then, since $z(x_0)=0$ and $z_x(x_0)=1$, we get $W(x_0,t_0)=E(x_0,t_0)$, $W_x(x_0,t_0)=\varepsilon>0$ and $W(x_0-\delta,t_0)>E(x_0,t_0)$. Therefore, $W(\cdot,t_0)$ has an interior minimum point in $(x_0-\delta,x_0)$. Let (y_0,t_0) be the minimum point. Since $E(x_0,t_0)$ is the global minimum of E and the auxiliary function z depends on x variable only, this minimum point is also a minimum of W on $Q:=(x_0-\delta,x_0)\times(0,T)$ for any $T>0$. Notice that, since $E_x(\cdot,t_0)\neq 0\neq E_{xx}(\cdot,t_0)$ in the interval $(x_0-\delta,x_0)$, both of $b(x,t)$ and $c(x,t)$ are strictly positive. Therefore, Ł is uniformly parabolic in a neighborhood of (y_0,t_0) and

$$Ł[W]=Ł[E]+\varepsilon Ł[z]=\varepsilon(bz_{xx}-c_3cz_x)=-\varepsilon(b+c_3c)\le 0$$

on it. The assumption that (y_0,t_0) is a interior minimum point of W contradicts to the maximum principle. The case $c_1=c_2=0$ and $c_3=1$ is done in [7] and hence omitted. □

Notice that this comparison principle holds for potentials given by integrals of solutions (5.1) and that it seems to hold under a more general operator $\sigma=\sigma(u,u_x)$. The next step is to place the potential U between two potentials of canonical solutions.

Corollary 5.2 (Trapped!). *Let u be a solution of (4.1)-(4.2) and ρ be the canonical solution of (4.9). Then their potentials U,R given by (5.1) satisfies*

$$R(x-L,t)\le U(x,t)\le R(x,t) \text{ for all } t>0,\ x\in\mathbb{R}. \tag{5.3}$$

Proof. Since $\rho(x,0)=\delta(x)$, $R(x,0)$ is the heavy side function

$$R(x,0)=\begin{cases}0, & \text{if } x<0,\\ 1, & \text{if } x\ge 0.\end{cases}$$

Therefore the restriction (4.2) on the initial value u_0 implies that

$$R(x-L,0)\le U(x,0)\le R(x,0)$$

(**see Figure 1**). Proposition 5.1 implies that (5.3) holds for all $t>0$. □

The next step is to compute the convergence rate of potentials using (5.3) which comes easily:

Lemma 5.3 (Potential Convergence Rate). *Under the same conditions as in Corollary 5.2,*

$$\|U(x,t)-R(x,t)\|_\infty\le L\max_{x\in\mathbb{R}}\rho(x,t). \tag{5.4}$$

Proof. Using the comparison inequality (5.3) and the Mean Value Theorem, we obtain

$$\begin{aligned}|U(x,t)-R(x,t)| &\le |R(x-L,t)-R(x,t)|\\ &= LR_x(c,t)=L\rho(c,t)\le L\max_{x\in\mathbb{R}}\rho(x,t),\end{aligned}$$

where $c\in[x-L,x]$. Since the estimate is independent of the point x, the estimate is uniform as in (5.4). □

Now the last step is to transfer this potential convergence rate to the desired solution convergence rate. Following the approach in [8], one may employ the classical regularity theory in [10] to derive the estimate of $U_x-R_x(\equiv u-\tilde u)$ from $U-R$. It seems that there is a simpler approach under a conjecture that ρ is steeper than any other solution u.

Proposition 5.4. *Let $u(x,t)$ and $\rho(x,t)$ be solutions of (4.1)-(4.2) and (4.9), respectively. Then,*

$$\|\rho(t)-u(t)\|_1=2\|R(t)-U(t)\|_\infty\le 2L\max_x\rho(x,t). \tag{5.5}$$

Proof. Let $\sigma = c_1\sigma_d + c_2\sigma_p + c_3\sigma_c$ where $c_i = 0$ or 1. One can easily check that the potential difference $E = R - U$ satisfies

$$Ł[E] := (c_1 a(x,t) + c_2 b(x,t))E_{xx} + (c_3 c(x,t))E_x - E_t = 0,\ E(x,0) \geq 0,$$

where

$$\begin{aligned} a &:= \frac{\phi'(\rho)\rho_x - \phi'(u)u_x}{\rho_x - u_x}, \\ b &:= \frac{\mathrm{sign}(\rho_x)\psi(|\rho_x|) - \mathrm{sign}(u_x)\psi(|u_x|)}{\rho_x - u_x}, \\ c &:= \frac{f(\rho) - f(u)}{\rho - u}. \end{aligned}$$

Since $\mathrm{sign}(x)\psi(x)$ is an odd function with $\psi'(x) \geq 0$ for $x > 0$, the coefficient b is non-negative. Therefore, if $c_1 = 0$, we may apply the property of the nonincrease of the lap number [**?**]. If $c_1 = 1$, then the coefficient of the leading order term $a + c_2 b$ can be negative. However, the lap number changes at the points $\rho = u$ and $a = \phi'(u) > 0$ near such a point. Therefore, we may apply the argument similarly and show the nonincrease of the lap number. We may conclude that the number of sign changes is at most once since $\lim_{t\downarrow 0}\rho(x,t) = \delta(x)$, the Dirac-delta measure. Let $x(t)$ be the sign-changing point of $e = \rho - u$. Then, clearly,

$$\begin{aligned} \|u(t) - \rho(t)\|_1 &= \int |u(x,t) - \rho(x,t)|dx = 2\int_{-\infty}^{x} [\rho(x,t) - u(x,t)]dx \\ &= 2\|U(t) - R(t)\|_\infty. \end{aligned}$$

Furthermore, using the comparison inequality (5.3), we obtain

$$|U(x,t) - R(x,t)| \leq |R(x-L,t) - R(x,t)| = \int_{x-L}^{x} \rho(y,t)dy \leq L\max_x \rho(x,t).$$

Since the estimate is independent of the point x, the estimate is uniform and the inequality in (5.5) is obtained. □

6. Porous medium type nonlinear diffusion

In this section we show Theorem 4.2(i). For the nonlinear diffusion case, $\sigma = \sigma_d$, the equation (4.1) is written as

$$u_t = \phi(u)_{xx}, \quad \lim_{t\downarrow 0} u(x,t) = u_0(x) \geq 0, \quad x \in \mathbb{R},\ t > 0. \tag{6.1}$$

The source solution ρ satisfies

$$\rho_t = \phi(\rho)_{xx}, \quad \lim_{t\downarrow 0} \rho(x,t) = \delta(x), \quad x \in \mathbb{R},\ t > 0. \tag{6.2}$$

Under the power law, $\phi(u) = u^m$, $m > 0$, the canonical solution $\rho(x,t)$ is explicitly given by

$$\rho^{m-1}(\mathbf{x},t) = \max\Big\{0, At^{\frac{1-m}{m+1}} + \frac{1-m}{2m(m+1)}|\mathbf{x}|^2 t^{-1}\Big\}, \tag{6.3}$$

where the constant σ is decided by the relation $\int \rho(\mathbf{x},t)d\mathbf{x} = 1$. One can easily check that $\rho(x,t)$ has its maximum at $x = 0$ and

$$\max_{x\in\mathbb{R}} \rho(x,t) = A^{\frac{1}{m-1}} t^{\frac{-1}{m+1}} = O(t^{\frac{-1}{m+1}}). \tag{6.4}$$

For the fast diffusion range, $0 < m < 1$, the convergence order $1/t$ has been shown in [8].

To obtain a convergence order $1/t$ we have trapped the initial potential $U(x,0)$ between $R(x,t)$ and $R(x,t+T)$ in the convection case. However, the primitive of a solution does not decouple the equation enough. Therefore, we introduce Newtonian potentials:

$$\mathbf{u}(x,t) = \int_{-\infty}^{x} U(y,t)dy, \qquad \mathbf{r}(x,t) = \int_{-\infty}^{x} R(y,t)dy. \tag{6.5}$$

Let $\varphi(x) = \max\{0, x\}$. Then, one may check that $\Delta\varphi(x) = \delta(x)$ and $\mathbf{u} = \varphi * u$. Therefore, it is natural to call $\mathbf{u}$ a Newtonian potential. A typical way is to take $\varphi(x) = |x|/2$ as a kernel. Here, to show the connection to Section 5 more closely, the Newtonian potential is given by (6.5).

Proposition 6.1. *Let u be the solution of (6.1) and $\mathbf{u}$ be its Newtonian potential. Then*

$$\mathbf{u}_t(x,t) = \phi(u(x,t)) > 0 \qquad \text{for each } x \in \mathbb{R},\ t > 0. \tag{6.6}$$

Proof. Basically, we need to check if

$$\lim_{h\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{x}\frac{u(y,t+h)-u(y,t)}{h}dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{x}\lim_{h\to\infty}\frac{u(y,t+h)-u(y,t)}{h}dydx.$$

Under the power law $\phi(u) = u^m$, $m > 0$, it is clear for $m > 1$ since the integrands are compactly supported. For the fast diffusion case $0 < m \le 1$ we simply refer [8]. Therefore, we need to check for the case of non-power law. □

Proposition 6.2 (Potential comparison)**.** *Let $\mathbf{u}(x,t)$ and $\tilde{\mathbf{u}}(x,t)$ be the Newtonian potentials of two bounded solutions u and $\tilde{u}$ of (6.1), respectively. Then $\mathbf{u}, \tilde{\mathbf{u}}$ are continuous on the closure of $\mathbb{R}\times]0,\infty[$. If $\mathbf{u}(x,0) \le \tilde{\mathbf{u}}(x,0)$ for all $x \in \mathbb{R}$, then $\mathbf{u}(x,t) \le \tilde{\mathbf{u}}(x,t)$ for all $t > 0$.*

Proof. Let $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$. Then $\mathbf{v}$ satisfies

$$\mathbf{v}_t = \mathbf{u}_t - \tilde{\mathbf{u}}_t = \phi(u) - \phi(\tilde{u}) = \frac{\phi(u)-\phi(\tilde{u})}{u-\tilde{u}}\mathbf{v}_{xx}\big(\equiv a(x,t)\mathbf{v}_{xx}\big). \tag{6.7}$$

The coefficient $a(x,t)$ is non-negative since ϕ is an increasing function and hence the maximum principle implies that $\mathbf{v}(x,t)$ is nonnegative if the initial value $\mathbf{v}(x,0)$ is nonnegative. □

Lemma 6.3. *Let c be the center of mass $c = \int xu_0(x)dx$. Then, there exists $T > 0$ such that*

$$\mathbf{r}(x,t) \le \mathbf{u}(x+c,t) \le \mathbf{r}(x,t+T), \quad t > 0. \tag{6.8}$$

Proof. First we can easily check that $\partial_x\mathbf{u}(x,t) = U(x,t) \le 1$, $\partial_x\mathbf{u}(x,0) = 1$ for $x > L$ and

$$\mathbf{u}(L,0) = \int_0^L\int_0^x u_0(y)dydx = \Big[x\int_0^x u_0(y)dy\Big]_{-\infty}^{L} - \int_0^L xu_0(x)dx = L - c.$$

Therefore, $\mathbf{u}(x,0) = x - c$ for all $x \ge L$ and $\mathbf{u}(x,0) \ge x - c$. Clearly $\mathbf{u}(x,0) \ge 0$ and $\mathbf{u}(x,0) = 0$ for all $x < 0$. $\mathbf{r}(x,0)$ is easy to compute that we have

$$\mathbf{u}(x+c,0) \ge \mathbf{r}(x,0) = \begin{cases} 0, & x < 0, \\ x, & x > 0. \end{cases}$$

Considering the explicit formula (6.3) for the canonical solution $\rho(x,t)$ we can easily check that $\mathbf{r}(x,t) \to \infty$ as $t \to \infty$ uniformly on any compact interval. Therefore, there exists $T > 0$ such that

$$\mathbf{r}(x,0) \le \mathbf{u}(x+c,0) \le \mathbf{r}(x,T)$$

(**see Figure 3**). Now the comparison principle, Proposition 6.2, completes the proof. □

The convergence order of potentials is easily computed like followings:

Lemma 6.4 (Potential convergence order)**.** *Let $\mathbf{u}, \mathbf{r}$ be Newtonian potentials of u, ρ, respectively, and $c = \int xu_0(x)dx$. Then, there exists $T > 0$ depending on the initial value u_0 such that*

$$\|\mathbf{r}(t) - \mathbf{u}_c(t)\|_\infty = T\max_x\big(\phi(\rho(x,t))\big). \tag{6.9}$$

Proof. Let $T > 0$ be the constant that satisfies (6.8). Then,

$$\begin{aligned}|\mathbf{r}(x,t) - \mathbf{u}(x+c,t)| &\le |\mathbf{r}(x,t) - \mathbf{r}(x,t+T)| \\ &\le T\|\mathbf{r}_t\|_\infty = T\max_x\big(\phi(\rho(x,t))\big).\end{aligned}$$

Since the estimate is independent of x, it is a uniform estimate. □

Now we transfer this convergence order to solutions. Since the coefficient a in (6.7) is non-negative, the lap number decreases and hence there is at most two sign-changing points and hence $U(x,t) - R(x,t)$ may change its sign once. Therefore, by the same arguments in the proof of Proposition 5.4, we have

$$\|R(t) - U_c(t)\|_1 = 2\|\mathbf{r}(t) - \mathbf{u}_c(t)\|_\infty \le 2T \max_x \big(\phi(\rho(x,t))\big).$$

We need to convert this L^1 convergence rate to a uniform one to use the arguments again. This comes from the $L^1 - L^\infty$ interpolation and

$$\|R(t) - U_c(t)\|_\infty \le 2T \max_x \big(\phi(\rho(x,t))\big) \max_x \big(\rho(x,t)\big). \tag{6.10}$$

Since $\rho - u$ may changes its sign at most twice, the estimate corresponding to the first step is

$$\|\rho(t) - u_c(t)\|_1 \le 4\|R(t) - U_c(t)\|_\infty \le 8T \max_x \big(\phi(\rho(x,t))\big) \max_x \big(\rho(x,t)\big).$$

This completes the proof of Theorem 4.2(i).

7. p-Laplacian type nonlinear diffusion

In this section we show Theorem 4.2(ii). Consider the case $\sigma = \sigma_p$, i.e., u is the solution of

$$u_t = \psi(u_x)_x, \quad \lim_{t\downarrow 0} u(x,t) = u_0(x), \quad x \in \mathbb{R},\ t > 0, \tag{7.1}$$

where ψ is an odd function satisfying (4.7) and the initial value u_0 is symmetric and continuous, and satisfies

$$\mathrm{supp}(u_0) = [-L, L], \quad u_0(x) > 0 \text{ for } x \in (-L, L) \text{ and } \int u_0(x)dx = 1. \tag{7.2}$$

The source solution ρ satisfies

$$\rho_t = \psi(\rho_x)_x, \quad \lim_{t\downarrow 0} \rho(x,t) = \delta(x), \quad x \in \mathbb{R},\ t > 0. \tag{7.3}$$

Let

$$v = u_x, \qquad \varrho = \rho_x.$$

Then after differentiating (7.1) one can easily check that v and ϱ are solutions to the porous medium type equations

$$v_t = (\psi(v))_{xx}, \qquad \varrho_t = (\psi(\varrho))_{xx}. \tag{7.4}$$

The difference from the previous case is that v, ϱ are sign-changing solution with zero total mass, i.e.,

$$\int_{-\infty}^{\infty} v(x,t)dx = \int_{-\infty}^{\infty} \varrho(x,t)dx = 0.$$

Let $u, \tilde{u}$ be solutions to (7.1) and $v, \tilde{v}$ be their derivatives, respectively. We take the integrals $U, \tilde{U}$ of $u, \tilde{u}$ as their potentials. Then, $U, \tilde{U}$ are Newtonian potentials of $v, \tilde{v}$. We already saw in Proposition 5.1 that $E := U - \tilde{U} \ge 0$ for all $t \ge 0$ if $E(x,0) \ge 0$. In the proof we basically used the maximum principle to the relation

$$E_t = b(x,t)E_{xx}, \quad b(x,t) := \frac{\psi(v) - \psi(\tilde{v})}{v - \tilde{v}} \ge 0.$$

Now we construct an estimate related to the time translation. Since $\rho(x,t) \ge 0$ decays to zero as $t \to \infty$, preserves its initial mass, and is symmetric with respect to $x = 0$, we have, for any $t \ge 0$,

$$\begin{gathered} R(x,t) \to 1 \text{ as } x \to \infty,\ R(x,t) \to 0 \text{ as } x \to -\infty, \\ R_x(x,t) \to 0 \text{ as } t \to \infty,\ R(0,t) = 1/2 \text{ for } t \ge 0. \end{gathered}$$

Similarly, since $u(x,t)$ is symmetric, the potential U of the solution u satisfies

$$\begin{gathered} U(x,t) \to 1 \text{ as } x \to \infty,\ U(x,t) \to 0 \text{ as } x \to -\infty, \\ U_x(x,t) \to 0 \text{ as } t \to \infty,\ U(0,t) = 1/2 \text{ for } t \ge 0. \end{gathered}$$

The initial potentials $U(x,0), R(x,0)$ satisfy

$$U(x,0)=1 \text{ for all } x>L, \quad U(x,0)=0 \text{ for all } x<-L,$$
$$R(x,0)=1 \text{ for all } x>0, \quad R(x,0)=0 \text{ for all } x<0.$$

Under the extra assumption that $U_x(x_0,0)\neq 0$ at the point such that $U(x_0,0)=1/2$, one can easily see that there exist $T>0, c\in\mathbb{R}$ such that

$$R(x-c,T)\le U(x,0)\le R(x-c,0), \quad x>1/2a.$$
$$R(x-c,0)\le U(x,0)\le R(x-c,T), \quad x<1/2a.$$

It is clear that such an estimates as (6.8) is not possible since $R(x,t)$ and $U(x,t)$ cross to each other at the point $x=0$, i..e, $U(0,t)=R(0,t)=1/2$ for all $t>0$. However we may apply the maximum principle arguments by restricting the domain to $x\ge 0$.

Lemma 7.1. *There exists $T>0$ such that*

$$R(x,t+T)\le U(x,t)\le R(x,t), \quad t>0,\, x\ge 0. \tag{7.5}$$

Proof. The second inequality in (7.5) is already obtained and we show the first one. Since the initial value is continuous and $u_0(0)>0$, there exists $\varepsilon>0, \delta>0$ such that

$$u_0(x)>\varepsilon \qquad \text{for} \qquad 0<x<\delta.$$

Since $\rho(x,t)\to 0$ as $t\to\infty$, there exists $T>0$ such that $\max_x \rho(x,T)<\varepsilon$. Therefore, $R(x,T)\le U(x,0)$ for $0<x<\delta$. Since $R(x,t)\to 0$ uniformly as $t\to\infty$ on the domain $[\delta,L]$ and $U(x,0)>\varepsilon\delta$ on $[\delta,L]$, we have $R(x,T)\le U(x,0)$, $0<x<L$, after taking larger $T>0$ if needed. Since $U(x,0)=1$ for $x\ge L$ and $R(x,t)\le 1$ for $x\ge L$, we have

$$R(x,T)\le U(x,0), \qquad t>0,\, x\ge 0.$$

Now we may apply the maximum principle on the domain $\Omega := \{(x,t): 0\le x<\infty,\, 0\le t<\infty\}$. Since (7.5) hold on the boundary $\partial\Omega$, we may conclude that (7.5) holds on Ω. □

Now the convergence order is similarly estimated as

$$\|u-\varrho\|_1 \le 2\|R(t)-R(T+t)\|_\infty \le 2T\|R_t\|_\infty \le 2T\max_x \psi(\varrho(x,t)))$$

and the proof for Theorem 4.2(ii) is complete.

In the diffusion case we obtained extra convergence order of similarity scale since the solution is the second order derivative of its potential. Here we could not do that. However, since ϱ has zero total mass, we may expect that it may decay faster than ρ.

8. Explicit computation of convergence orders

In this section we compute the convergence order in Theorems 4.1 and 4.2 explicitly under power laws:

$$f(u)=u^q/q,\ \phi(u)=u^m,\ \psi(v)=\mathrm{sign}(v)|v|^{p-1},\ m>0, q>1, p>1. \tag{8.1}$$

If $\sigma=f(u)$, $\phi(u)_x$ or $\psi(u_x)$, the the source-type solution ρ is given explicitly and it is well known that

$$\max_{x\in\mathbb{R}}\rho(x,t)=O(t^{-\alpha}), \quad \alpha=\begin{cases} 1/q & \text{if } \sigma=f(u),\\ 1/(m+1) & \text{if } \sigma=\phi(u)_x,\\ 1/(2p-2) & \text{if } \sigma=\psi(u_x),\end{cases} \tag{8.2}$$

as $t\to\infty$. If σ is a sum of two or all of these three terms, the decay power α is simply the largest one among these three. For the general case $0\neq\sigma=c_1\sigma_c+c_2\sigma_d+c_3\sigma_p$, the decay rate of the solution ρ is

$$\max_{x\in\mathbb{R}}\rho(x,t)=O(t^{-\alpha}), \quad \alpha=\max\left(\frac{c_1}{q},\frac{c_2}{m+1},\frac{c_3}{2p-2}\right) \tag{8.3}$$

as $t\to\infty$. (This decay order has been shown for the convection-diffusion case in [5], i.e., $c_1=c_3=1, c_2=0$. I am not sure if there is a paper for (8.3) for the case when all of these three terms are together.)

If $\alpha = 1/q$, then we may say that the convection dominates the evolution. Similarly, if $\alpha = 1/(m+1)$ or $\alpha = 1/(2p-2)$, then the nonlinear diffusion operator σ_d or the p-Laplacian operator σ_p dominates the evolution, respectively. Combining (4.10) and (8.3) we obtain the convergence order

$$\|u(t)-\rho(t)\|_1 = O(t^{-\alpha}),\ \alpha = \max\Big(\frac{c_1}{q}, \frac{c_2}{m+1}, \frac{c_3}{2p-2}\Big),\ \text{as } t\to\infty \tag{8.4}$$

for the general case in (4.8).

For the case $\sigma = \phi(u)_x$ the order of ρ is $t^{-1/(m+1)}$ and hence the convergence order in (4.12) is

$$\|\rho(t)-u_c(t)\|_1 = O(t^{\frac{-1}{m+1}}t^{\frac{-m}{m+1}}) = O(t^{-1})$$

as $t\to\infty$.

Remark 8.1. If $1<p<3/2$, the decay order in (8.4) is higher than $1/t$. It seems that this case the diffusion speed is so fast that the the time translation has bigger impact than space translation. The similar phenomenon is observed for the nonlinear diffusion when the diffusion is so fast that the total mass is not preserved.

Remark 8.2. For the nonlinear diffusion in multi-dimensions, $\sigma = \Delta(u^m)$, the convergence order of magnitude of the source solution has been known by Dolbeault & del Pino [2] and Otto [15]. For p-Laplacian $\sigma = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ with p in a certain range L^1 convergence order has been obtained by Dolbeault & del Pino [3] for dimension $n\geq 2$, which is lower than the magnitude of solutions. However, under the presence of several phenomena any convergence order is not known until a recent work by Carrillo & Fellner [1]. In [1] the same convergence order has been shown for the case $\sigma=\sigma_c+\sigma_d$ with $m+1<q$ and $1\leq m<2$.

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