Solutions for Homework 1

MAS501 Analysis for Engineers, Spring 2011

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1. A real number r is said to be *algebraic* if r is a root of a polynomial with rational coefficients. If r is not algebraic, r is said to be *transcendental*. Show that the set of all transcendental numbers is uncountable. (It is known that e and π are transcendental.)

   Answer: First we claim that the set of all polynomials with rational coefficients is countable.

   Proof of Claim. (배영오 씨 답안 기반) Let P_n be the set of all polynomials of degree n with rational coefficients:

   $$P_n = \{c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n : c_n \in \mathbf{Q} \setminus \{0\} \text{ and } c_0, c_1, \cdots, c_{n-1} \in \mathbf{Q}\}.$$

   Then $|P_n| = |\mathbf{Q} \setminus \{0\} \times \mathbf{Q}^{n-1}|$. Because a Cartesian product of a finite number of countable sets $\mathbf{Q} \setminus \{0\} \times \mathbf{Q}^{n-1}$ is countable, the set P_n is also countable. Now note that the set of all polynomials with rational coefficients P can be represented as $P = \bigcup_{n=1}^{\infty} P_n$, which is a countable union of countable sets. Hence the set P is countable. □

   Proof. (여병철 씨 답안 기반) Let A be the set of all algebraic numbers. We know that the set of all polynomials with rational coefficients is countable. Also, since *each such polynomial has a finite number of roots*, the set A is countable. But the real line $\mathbf{R}$ is uncountable. Hence the set of all transcendental numbers, which is $\mathbf{R} \setminus A$ by definition, must be uncountable. □

2. Let (Ω_1, d_1) and (Ω_2, d_2) be metric spaces.

   (a) Prove that $(\Omega_1 \times \Omega_2, d)$ is a metric space, where d is defined by the formula

   $$d[(x_1, x_2), (y_1, y_2)] = d_1(x_1, y_1) + d_2(x_2, y_2).$$

   The space $(\Omega_1 \times \Omega_2, d)$ is called the *product metric space.*

   Answer: We check three condtions a metric should satisfy.

   (positivity) $d[(x_1, x_2), (y_1, y_2)] = d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$. And the equality holds only when $d_1(x_1, y_1) = d_2(x_2, y_2) = 0$, i.e., when $(x_1, x_2) = (y_1, y_2)$.

   (symmetry)

   $$\begin{aligned} d[(x_1, x_2), (y_1, y_2)] &= d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, x_1) + d_2(y_2, x_2) \\ &= d[(y_1, y_2), (x_1, x_2)]. \end{aligned}$$

   (triangle inequality) Let $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$. Then we have

   $$\begin{aligned} d[(x_1, x_2), (z_1, z_2)] &= d_1(x_1, z_1) + d_2(x_2, z_2) \\ &\leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) \\ &= d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d_2(y_2, z_2) \\ &= d[(x_1, x_2), (y_1, y_2)] + d[(y_1, y_2), (z_1, z_2)]. \end{aligned}$$

   (b) Let $\mathbf{R} \times \mathbf{R}$ be the product metric space constructed from the Euclidean space $\mathbf{R}$. Then obviously the metric of $\mathbf{R} \times \mathbf{R}$ is different from the metric of Euclidean space $\mathbf{R}^2$. Although, we can prove that any open set of one metric space is also an open set of the other metric space. (This means their topology are same; they have same "nearness".) Prove it.

Proof. (김준우 씨 답안 기반) Let $d_{\mathbf{R}\times\mathbf{R}}$ be the product metric of the product metric space $\mathbf{R}\times\mathbf{R}$. Also let $d_{\mathbf{R}^2}$ be the usual metric of the Euclidean space $\mathbf{R}^2$. Then by definition, we have

$$\begin{aligned} d_{\mathbf{R}\times\mathbf{R}}\big[(x_1,x_2),(y_1,y_2)\big] &= |x_1-y_1|+|x_2-y_2|, \\ d_{\mathbf{R}^2}\big[(x_1,x_2),(y_1,y_2)\big] &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \end{aligned}$$

for any $x_1,x_2,y_1,y_2\in\mathbf{R}$. Because

$$\begin{aligned} (|x_1-y_1|+|x_2-y_2|)^2 &= (x_1-y_1)^2+(x_2-y_2)^2+2|x_1-y_1|\,|x_2-y_2| \\ &\geq (x_1-y_1)^2+(x_2-y_2)^2, \end{aligned}$$

we have

$$d_{\mathbf{R}\times\mathbf{R}}[\mathbf{x},\mathbf{y}] \geq d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}]$$

where $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$. Now observe that for any $a,b\in\mathbf{R}$ it holds that

$$0 \leq (a-b)^2 = 2(a^2+b^2)-(a+b)^2$$

hence

$$|a+b| \leq \sqrt{2}\sqrt{a^2+b^2}.$$

Now put $a=|x_1-y_1|$ and $b=|x_2-y_2|$ then we have

$$d_{\mathbf{R}\times\mathbf{R}}[\mathbf{x},\mathbf{y}] \leq \sqrt{2}\,d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}].$$

In summary, we've just showed that for any $\mathbf{x},\mathbf{y}\in\mathbf{R}^2$,

$$d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}] \leq d_{\mathbf{R}\times\mathbf{R}}[\mathbf{x},\mathbf{y}] \leq \sqrt{2}\,d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}].$$

Using these inequalities, we can easily verify that for any positive real number r, it holds that

$$\begin{aligned} \{\mathbf{y}\in\mathbf{R}^2 : d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}] < r\} &\subset \{\mathbf{y}\in\mathbf{R}^2 : d_{\mathbf{R}\times\mathbf{R}}[\mathbf{x},\mathbf{y}] < \sqrt{2}\,r\} \\ &\subset \{\mathbf{y}\in\mathbf{R}^2 : d_{\mathbf{R}^2}[\mathbf{x},\mathbf{y}] < \sqrt{2}\,r\} \end{aligned}$$

This relation means that every open ball of the Euclidean space $\mathbf{R}^2$ centered at some point *contains* an open ball of the product metric space $\mathbf{R}\times\mathbf{R}$ centered at the point and vice versa. Now let $\mathcal{O}$ be an open set in the Euclidean space $\mathbf{R}^2$. Then by definition, for every $\mathbf{x}\in\mathcal{O}$, there is an open ball of $\mathbf{R}^2$ centered at $\mathbf{x}$ that is entirely contained in $\mathcal{O}$. But we just proved that every open ball of $\mathbf{R}^2$ centered at $\mathbf{x}$ contains an open ball of $\mathbf{R}\times\mathbf{R}$ centered at $\mathbf{x}$. So for every $\mathbf{x}\in\mathcal{O}$ we can find an open ball of $\mathbf{R}\times\mathbf{R}$ centerd at $\mathbf{x}$ that is entirely contained in $\mathcal{O}$. This means that the set $\mathcal{O}$ is an open set in $\mathbf{R}\times\mathbf{R}$. Hence we can conclude that any *open set* of the Euclidean space $\mathbf{R}^2$ is an open set of the product metric space $\mathbf{R}\times\mathbf{R}$ and vice versa. (The proof of converse is similar.) □