Solutions for Homework 2

MAS501 Analysis for Engineers, Spring 2011

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1. (Discrete metric) Let Ω be a set.

   (a) Prove that a function $d : \Omega \times \Omega \to \mathbf{R}$ defined by

   $$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

   is a metric on Ω. The metric d is called the *discrete metric* for Ω.

   Proof. (주재율 씨 답안 기반) Positivity and symmetry is obvious. Only nontrivial thing is to check triangle inequality. Assume the triangle inequality does *not* hold; then there are some points $x, y, z \in \Omega$ such that

   $$d(x,z) > d(x,y) + d(y,z).$$

   If $d(x,z) = 0$, then the strict inequality does not make sense so we have $d(x,z) = 1$. Again by the strict inequality we have the right hand side $d(x,y) + d(y,z)$ should be zero. So $d(x,y) = d(y,z) = 0$, i.e., $x = y = z$. But then $d(x,z) = 0$ and this is a contradiction. □

   (b) Let (Ω, d) be a metric space with discrete metric. Find all the isolated points of Ω. Now can you see why the metric is called *discrete*?

   Answer. (김동재 씨 답안 기반) By definition of the discrete metric, we have for any $x \in \Omega$,

   $$B_{1/2}(x) = \{y \in \Omega : d(x,y) < 1/2\} = \{x\}.$$

   And this is an open set which is not containing any other element in Ω except x. Therefore, *all points in* Ω *are isolated points.*

   (윤성환 씨의 코멘트) It is reasonable that the metric d is called *discrete* because the corresponding metric space (Ω, d) makes all the points in Ω *isolated.* □

2. (Set operations and compactness)

   (a) Let $K_1, \cdots, K_n$ be a finite collection of compact subsets of a metric space Ω. Prove that $\bigcup_{i=1}^n K_i$ is compact. Show (by example) that this result does not generalize to infinite unions.

   Proof. (장혜령 씨 답안 기반) Let J be an index set and $\{G_j\}_{j \in J}$ be an open covering of $\bigcup_{i=1}^n K_i$. Then the open covering $\{G_j\}_{j \in J}$ also covers all of $K_1, \cdots, K_n$. Hence for each K_i, we can find a finite subcovering $\left\{H_k^{(i)}\right\}_{k=1}^{n_i} \subset \{G_j\}$ of K_i. Then obviously $\bigcup_{i=1}^n \left(\left\{H_k^{(i)}\right\}_{k=1}^{n_i}\right) \subset \{G_j\}$ is finite and

   $$\bigcup_{i=1}^n K_i \subset \bigcup_{i=1}^n \bigcup_{k=1}^{n_i} H_k^{(i)}.$$

   This implies that $\bigcup_{i=1}^n \left(\left\{H_k^{(i)}\right\}_{k=1}^{n_i}\right)$ is a finite subcovering. Therefore $\bigcup_{i=1}^n K_i$ is compact. □

   Examples. (이효정 씨의 반례) Consider sets of single element $\{i\}$ for each $i \in \mathbf{N}$. Then these sets are compact in the Euclidean space $\mathbf{R}$. But we have

   $$\bigcup_{i=1}^{\infty} \{i\} = \mathbf{N}.$$

   Because $\mathbf{N}$ is *not* bounded, it is *not* compact by the Heine-Borel theorem.

(김도훈 씨의 반례) Let $K_i := \left[0, 1 - \frac{1}{i}\right]$. Then all K_i's are compact in the Euclidean space $\mathbf{R}$. But we have

$$\bigcup_{i=1}^{\infty} K_i = [0, 1).$$

Because this set is *not* closed, it is *not* compact by the Heine-Borel theorem. □

(b) Let I be an index set and $\{K_i : i \in I\}$ be a collection of compact subsets of a metric space Ω. Prove that $\bigcap_{i \in I} K_i$ is compact.

Proof. (장혜령 씨 답안 기반) Define $K := \bigcap_{i \in I} K_i$. Since each K_i is compact, it is closed and bounded. Then the set K, an arbitrary intersections of closed sets, is also closed so K^c is open. Now let $\{G_j\}_{j \in J}$ be an open covering of K, i.e., $K \subset \bigcup_{j \in J} G_j$. Hence it holds that

$$\Omega = K \cup K^c \subset \bigcup_{j \in J} G_j \cup K^c.$$

So $\bigcup_{j \in J} G_j \cup K^c = \Omega$. Now fix any index $i_0 \in I$. Then obviously

$$K_{i_0} \subset \Omega = \bigcup_{j \in J} G_j \cup K^c,$$

that is, $\{G_j\}_{j \in J} \cup \{K^c\}$ is an open covering of K_{i_0}. Because K_{i_0} is compact, we can find a finite collection of sets $\{H_k\}_{k=1}^n \subset \{G_j\}_{j \in J}$ such that

$$K_{i_0} \subset \bigcup_{k=1}^{n} H_k \cup K^c.$$

Hence it holds that

$$K = K \cap K \subset K_{i_0} \cap K \subset \left(\bigcup_{k=1}^{n} H_k \cup K^c \right) \cap K = \bigcup_{k=1}^{n} H_k.$$

This implies that $\{H_k\}_{k=1}^n$ is a finite subcovering of K. Therefore the set K is compact. □

(c) Let K be a compact subset of Euclidean space $\mathbf{R}$ and let $y \in \mathbf{R}$. Prove that the set $\{x + y : x \in K\}$ is compact.

Proof. (이효정 씨 답안 기반) Define $T := \{x + y : x \in K\}$. (Note that intuitively this set is a translation of the set K.) We will show that the set T is closed and bounded.

(T is closed) Let $\{t_n\} \subset T$ be a sequence of real numbers such that $t_n \to t$ as $n \to \infty$. Since each t_n is an element of T, it can be written as $t_n = x_n + y$ for some $x_n \in K$. Thus it holds that

$$K \ni x_n = t_n - y \to t - y \quad \text{as } n \to \infty.$$

Because the compact set K is closed, we can conclude that $t - y \in K$. Therefore

$$t = (t - y) + y \in T.$$

We just proved that the limit is also in T, so that the set T is closed.

(T is bounded) Since the compact set K is bounded, it holds that $K \subset B_r(x)$ for some $r > 0$ and $x \in K$. Then we can easily see that $T \subset B_r(x + y)$. Thus T is bounded.

By the Heine-Borel theorem, the set T is compact. □