Solutions for Homework 3

MAS501 Analysis for Engineers, Spring 2011

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1. Let A and B be nonempty sets of real numbers and assume $\sup A$ and $\sup B$ are finite. Prove that

$$\sup(A+B) = \sup A + \sup B,$$

where

$$A + B = \{a + b : a \in A, b \in B\}.$$

What can we say when either of $\sup A$ or $\sup B$ is *not* finite?

Proof. (이호 씨 답안 기반) For every $a \in A$, we have $a \leq \sup A$. Also for every $b \in B$, it holds that $b \leq \sup B$. Hence $a + b \leq \sup A + \sup B$ for all $a \in A$ and $b \in B$ so that $\sup A + \sup B$ is an upper bound of $A + B$. Therefore we have

$$\sup(A+B) \leq \sup A + \sup B.$$

On the other hand, by theorem 2.4.3, there are two sequences $\{a_n\} \subset A$ and $\{b_n\} \subset B$ such that

$$\sup A - \frac{1}{n} < a_n \leq \sup A,$$
$$\sup B - \frac{1}{n} < b_n \leq \sup B.$$

Because $a_n + b_n \in A + B$ we have

$$\sup A + \sup B - \frac{2}{n} < a_n + b_n \leq \sup(A+B).$$

Now taking the limit $n \to \infty$ yields

$$\sup A + \sup B \leq \sup(A+B)$$

and the proof is complete.

Even if either of $\sup A$ or $\sup B$ is not finite, the equality still *holds*. Note that the supremum of a nonempty set cannot be $-\infty$. Hence without loss of generality, we may assume that $\sup A = \infty$. Then there is a sequence $\{a_n\} \subset A$ such that $a_n \to \infty$. Now let b be a real number in the set B. Then $a_n + b \to \infty$ and so $\sup(A+B) = \infty$. Therefore the equliaty holds;

$$\sup(A+B) = \infty = \sup A + \sup B.$$

□

2. (Product metric space and Cauchy sequence) Let Ω_1 and Ω_2 be metric spaces and $\Omega_1 \times \Omega_2$ be the product metric space. (For the definition of product metric space, refer to Homework #1.)

   (a) Prove that if a sequence $\{(x_n, y_n)\}$ is Cauchy in $\Omega_1 \times \Omega_2$, then the sequence $\{x_n\}$ is Cauchy in Ω_1 and the sequence $\{y_n\}$ is Cauchy in Ω_2. Is the converse true?

   Proof. (홍비 씨 답안 기반) Because the sequence $\{(x_n, y_n)\}$ is Cauchy in $\Omega_1 \times \Omega_2$, by definition, it holds that

$$d_{\Omega_1 \times \Omega_2}\big[(x_n, y_n), (x_m, y_m)\big] = d_{\Omega_1}(x_n, x_m) + d_{\Omega_2}(y_n, y_m) \to 0 \quad \text{as } n, m \to \infty.$$

   Hence by positivity of the metric, we have $d_{\Omega_1}(x_n, x_m) \to 0$ as $n, m \to \infty$ and $d_{\Omega_2}(y_n, y_m) \to 0$ as $n, m \to \infty$. And this implies that the sequence $\{x_n\}$ is Cauchy in Ω_1 and the sequence $\{y_n\}$ is Cauchy in Ω_2.

   The converse is also *true*. Assume $\{x_n\}$ is a Cauchy sequence in Ω_1 and $\{y_n\}$ is a Cauchy sequence in Ω_2. Then by definition, $d_{\Omega_1}(x_n, x_m) \to 0$ as $n, m \to \infty$ and $d_{\Omega_2}(y_n, y_m) \to 0$ as $n, m \to \infty$. Therefore it holds that

$$d_{\Omega_1 \times \Omega_2}\big[(x_n, y_n), (x_m, y_m)\big] = d_{\Omega_1}(x_n, x_m) + d_{\Omega_2}(y_n, y_m) \to 0 \quad \text{as } n, m \to \infty.$$

   This implies that the sequence $\{(x_n, y_n)\}$ is Cauchy in $\Omega_1 \times \Omega_2$. □

   (b) Prove that if Ω_1 and Ω_2 are complete, then the product metric space $\Omega_1 \times \Omega_2$ is also complete. Is the converse true?

   Proof. (옥정슬 씨 답안 기반) Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $\Omega_1 \times \Omega_2$. Then by (a), $\{x_n\}$ is a Cauchy sequence in Ω_1 and $\{y_n\}$ is a Cauchy sequence in Ω_2. Because Ω_1 and Ω_2 are complete, there are two elements $x \in \Omega_1$ and $y \in \Omega_2$ such that $x_n \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$. Then it holds that

$$d_{\Omega_1 \times \Omega_2}\big[(x_n, y_n), (x, y)\big] = d_{\Omega_1}(x_n, x) + d_{\Omega_2}(y_n, y) \to 0 \quad \text{as } n \to \infty,$$

   that is, $(x_n, y_n) \to (x, y)$. This means that every Cauchy sequence in $\Omega_1 \times \Omega_2$ is convergent so $\Omega_1 \times \Omega_2$ is complete.

   The converse is also *true*. Let $\{x_n\}$ be a Cauchy sequence in Ω_1 and $\{y_n\}$ be a Cauchy sequence in Ω_2. Then by (a), the sequence $\{(x_n, y_n)\}$ is also Cauchy in $\Omega_1 \times \Omega_2$. Because $\Omega_1 \times \Omega_2$ is complete, there is an element $(x, y) \in \Omega_1 \times \Omega_2$ such that $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Then we have

$$d_{\Omega_1}(x_n, x) \le d_{\Omega_1 \times \Omega_2}\big[(x_n, y_n), (x, y)\big] \to 0 \quad \text{as } n \to \infty,$$

   that is, $x_n \to x$ as $n \to \infty$. This means that every Cauchy sequence in Ω_1 is convergent so Ω_1 is complete. By the same reasoning, we have $y_n \to y$ as $n \to \infty$ and so we can conclude that Ω_2 is also complete. □