Solutions for Homework 4

MAS501 Analysis for Engineers, Spring 2011

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1. Let $\{a_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} a_n = L,$$

where L is a real number. Show that the sequence of their arithmetic means also converges to L, that is,

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L.$$

Hints:

(a) Let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Then it suffices to show that $\limsup b_n = \liminf b_n = L$.

(b) $\limsup b_n = L$ is equivalent to $L - \epsilon \le \limsup b_n \le L + \epsilon$ for every $\epsilon > 0$.

(c) For any $\epsilon > 0$, *eventually* it holds that $L - \epsilon < a_n < L + \epsilon$ (by the hypothesis).

Proof. (R. Johnsonbaugh and W. E. Pfaffenberger) Let $\epsilon > 0$. There exists a positive integer N such that if $n \ge N$, then

$$L - \epsilon < a_n < L + \epsilon.$$

Let

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{for } n \ge N.$$

Now

$$b_n = \frac{a_1 + a_2 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n}$$

and since

$$\frac{(n-N)(L-\epsilon)}{n} < \frac{a_{N+1} + \cdots + a_n}{n} < \frac{(n-N)(L+\epsilon)}{n},$$

we have

$$\frac{C}{n} + \frac{(n-N)(L-\epsilon)}{n} < b_n < \frac{C}{n} + \frac{(n-N)(L+\epsilon)}{n}$$

where

$$C = a_1 + a_2 + \cdots + a_N.$$

Hence we conclude that

$$L - \epsilon \le \limsup_{n\to\infty} b_n \le L + \epsilon \quad \text{for every } \epsilon > 0$$

and so

$$\limsup_{n\to\infty} b_n = L.$$

Similarly,

$$\liminf_{n\to\infty} b_n = L.$$

Therfore the proof is complete. ∎

2. Prove that if a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Hint: Note that $a_n^2 \le |a_n|$ eventually. (Why?)

Proof. Because the series $\sum_{n=1}^{\infty} |a_n|$ converges, we have $|a_n| \to 0$ as $n \to \infty$. Thus $|a_n| \le 1$ ev. and $a_n^2 \le |a_n|$ ev., which means that there exists a positive integer $N > 1$ such that

$$a_n^2 \le |a_n| \quad \text{for every } n \ge N.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^2 &= \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2 \\ &\le \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} |a_n| < \infty \end{aligned}$$

and the proof is complete. □

3. Suppose that f is continuous at every point of $[a, b]$ and $f(x) = 0$ if x is rational. Prove that $f(x) = 0$ for every x in $[a, b]$.

Hint: You may use the fact that the set of rational numbers $\mathbf{Q}$ is dense in the Euclidean space $\mathbf{R}$.

Proof. (노재형 씨 답안 기반) Let x be a real number in $[a, b]$. Because the set of rational numbers $\mathbf{Q}$ is dense in the Euclidean space $\mathbf{R}$, there is a sequence $\{x_n\} \subset [a, b] \cap \mathbf{Q}$ such that $x_n \to x$. Then by continuity of f, we have

$$f(x_n) \to f(x) \quad \text{as } n \to \infty.$$

But $f(x_n) = 0$ for every n and so $f(x) = 0$. □