Homework 4

MAS501 Analysis for Engineers, Spring 2011

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1. Let $\{a_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} a_n = L,$$

where L is a real number. Show that the sequence of their arithmetic means also converges to L, that is,

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L.$$

Hints:

(a) Let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Then it suffices to show that $\limsup b_n = \liminf b_n = L$.

(b) $\limsup b_n = L$ is equivalent to $L - \epsilon \le \limsup b_n \le L + \epsilon$ for every $\epsilon > 0$.

(c) For any $\epsilon > 0$, *eventually* it holds that $L - \epsilon < a_n < L + \epsilon$ (by the hypothesis).

2. Prove that if a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Hint: Note that $a_n^2 \le |a_n|$ eventually. (Why?)

3. Suppose that f is continuous at every point of $[a, b]$ and $f(x) = 0$ if x is rational. Prove that $f(x) = 0$ for every x in $[a, b]$.

Hint : You may use the fact that the set of rational numbers $\mathbf{Q}$ is dense in the Euclidean space $\mathbf{R}$.