Solutions for Homework 5

MAS501 Analysis for Engineers, Spring 2011

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1. A *contraction mapping* on Ω is a function f from the metric space (Ω, d) into itself satisfying

$$d(f(x), f(y)) \leq c\, d(x, y)$$

for some c, $0 \leq c < 1$ and all x and y in Ω.

(a) Prove that a contraction mapping on Ω is uniformly continuous on Ω.

Proof. Take arbitrary $\epsilon > 0$. Then $d(x, y) < \epsilon$ implies that

$$d(f(x), f(y)) \leq c\, d(x, y) < \epsilon.$$

Therefore a contraction mapping is uniformly continuous. □

(b) Prove that if Ω is *complete*, then the equation $f(x) = x$ is solvable in Ω and the solution is unique.

Hints:

i. Take any element in Ω, say x_1. And let $x_{n+1} := f(x_n)$ for $n = 1, 2, \cdots$. Prove that the sequence $\{x_n\}$ is Cauchy in Ω and so it converges to some element $x \in \Omega$ (by completeness of Ω).

ii. Show this element x is a solution to $f(x) = x$, that is, $d(f(x), x) = 0$.

iii. Finally show that the solution to $f(x) = x$ is unique in Ω. What happens when $x \in \Omega$ and $y \in \Omega$ are solutions to the equation $f(x) = x$?

Note: If $f(x) = x$, we call the point x a *fixed point*. Also this type of result is called a *fixed point theorem* and has many applications.

Proof. Take an element in Ω, say x_1. And let $x_{n+1} := f(x_n)$ for $n = 1, 2, \cdots$. We claim that the sequence $\{x_n\}$ is Cauchy in Ω. From the observation

$$d(x_3, x_2) = d(f(x_2), f(x_1)) \leq c\, d(x_2, x_1)$$

we can easily deduce that for every $m \geq 2$,

$$d(x_{m+1}, x_m) \leq c^{m-1}\, d(x_2, x_1).$$

If $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &\leq c^{n-2}\, d(x_2, x_1) + \cdots + c^{m-1}\, d(x_2, x_1) \\ &= c^{m-1}(c^{n-m-1} + \cdots + 1)\, d(x_2, x_1) \\ &\leq c^{m-1} \frac{1}{1-c} d(x_2, x_1) \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Therefore $d(x_n, x_m) \to \infty$ as $n, m \to \infty$ and the sequence $\{x_n\}$ is Cauchy in Ω.

Because Ω is complete, the sequence $\{x_n\}$ converges to an element in Ω, say x_0. Then we have

$$x_0 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x_0)$$

by continuity of the contraction mapping. So x_0 is a solution to the equation $f(x) = x$.

Finally we show the uniqueness of the solution. Assume y is also a solution to the equation $f(x) = x$. Then it holds that

$$d(x_0, y) = d(f(x_0), f(y)) \leq c\, d(x_0, y).$$

If $d(x_0, y) \neq 0$, we have $1 \leq c < 1$, which is a contradiction. Hence $d(x_0, y) = 0$, i.e., $y = x_0$. □

2. Let f be a *uniformly continuous* mapping from Ω to Ω', where Ω and Ω' are metric spaces. Show that if $\{x_n\}$ is a Cauchy sequence in Ω, then $\{f(x_n)\}$ is a Cauchy sequence in Ω'. When f is just continuous, does the result still hold?

   Proof. Let $\epsilon > 0$ be an arbitrary number. Since f is uniformly continuous, there is $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f(x), f(y)) < \epsilon$. Because $\{x_n\}$ is Cauchy, we can take a natural number N such that

   $$d(x_n, x_m) < \delta \quad \text{for all } n, m \geq N.$$

   Therefore we have

   $$d(f(x_n), f(x_m)) < \epsilon \quad \text{for all } n, m \geq N,$$

   which means that the sequence $\{f(x_n)\}$ is Cauchy in Ω'.

   If f is just continuous, the result does *not* hold. Let $\Omega = \mathbf{R} \setminus \{0\}$ and $\Omega' = \mathbf{R}$. Consider a sequence $\{x_n = \frac{1}{n}\}$ in Ω and a function $f : \Omega \to \Omega'$ given by

   $$f(x) = \frac{1}{x}.$$

   Then $x_n \to 0$ as $n \to \infty$ so the sequence $\{x_n\}$ is Cauchy in Ω (because it is convergent in $\mathbf{R}$). But the sequence $f(x_n) = n \to \infty$ as $n \to \infty$ so this sequence is *not* Cauchy (because it is divergent and $\Omega' = \mathbf{R}$ is *complete*). □