Homework 5

MAS501 Analysis for Engineers, Spring 2011

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1. A *contraction mapping* on Ω is a function f from the metric space (Ω, d) into itself satisfying

$$d(f(x), f(y)) \leq c\, d(x, y)$$

for some c, $0 \leq c < 1$ and all x and y in Ω.

   (a) Prove that a contraction mapping on Ω is uniformly continuous on Ω.

   (b) Prove that if Ω is *complete*, then the equation $f(x) = x$ is solvable in Ω and the solution is unique.

   Hints:

   i. Take any element in Ω, say x_1. And let $x_{n+1} := f(x_n)$ for $n = 1, 2, \cdots$. Prove that the sequence $\{x_n\}$ is Cauchy in Ω and so it converges to some element $x \in \Omega$ (by completeness of Ω).

   ii. Show this element x is a solution to $f(x) = x$, that is, $d(f(x), x) = 0$.

   iii. Finally show that the solution to $f(x) = x$ is unique in Ω. What happens when $x \in \Omega$ and $y \in \Omega$ are solutions to the equation $f(x) = x$?

   Note: If $f(x) = x$, we call the point x a *fixed point.* Also this type of result is called a *fixed point theorem* and has many applications.

2. Let f be a *uniformly continuous* mapping from Ω to Ω', where Ω and Ω' are metric spaces. Show that if $\{x_n\}$ is a Cauchy sequence in Ω, then $\{f(x_n)\}$ is a Cauchy sequence in Ω'. When f is just continuous, does the result still hold?