Homework 6

MAS501 Analysis for Engineers, Spring 2011

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1. Suppose $f'(x)$ exists and is bounded on all of $\mathbf{R}$. Prove that f is uniformly continuous on $\mathbf{R}$.

   Proof. Because $f'(x)$ is bounded, there is a $M > 0$ such that

   $$|f'(x)| \leq M \quad \text{for all } x \in \mathbf{R}.$$

   Given $\epsilon > 0$, take $\delta = \epsilon/M$. Then whenever

   $$|a - b| < \delta = \frac{\epsilon}{M}, \quad a, b \in \mathbf{R},$$

   we have

   $$\begin{aligned} |f(a) - f(b)| &= |f'(x)|\,|a - b| \quad \text{for some } x \text{ between } a \text{ and } b \\ &\leq M|a - b| < \epsilon. \end{aligned}$$

   by the mean value theorem. Thus f is uniformly continuous on $\mathbf{R}$. □

2. Suppose f is defined in an open interval containing x, and suppose $f''(x)$ exists. Show that

   $$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

   Show by an example that the limit may exist even if $f''(x)$ does not.
   Hint: Use L'Hospital's rule.

   Proof. Let $f_1(h) := f(x+h) + f(x-h) - 2f(x)$ and $f_2(h) := h^2$. Then

   $$\lim_{h\to 0} f_1(h) = \lim_{h\to 0} f_2(h) = 0$$

   and

   $$\lim_{h\to 0} \frac{f_1'(h)}{f_2'(h)} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

   Therefore we can use the L'Hospital's theorem to conclude that

   $$\lim_{h\to 0} \frac{f_1(h)}{f_2(h)} = \lim_{h\to 0} \frac{f_1'(h)}{f_2'(h)} = f''(x).$$

   □

   Example. There are many examples including

   $$f(x) := \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   We know $f''(0)$ does not exist. But the limit exists;

   $$\lim_{h\to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0.$$

   □