Solutions for Homework 7

MAS501 Analysis for Engineers, Spring 2011

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1. Suppose that f'' exists on $[0,1]$ and that $f(0) = f(1) = 0$. Suppose also that

$$|f''(x)| \le K \quad \text{for } x \in (0,1).$$

Prove that

$$\left|f'\left(\frac{1}{2}\right)\right| \le \frac{K}{4}$$

and

$$|f'(x)| \le \frac{K}{2} \quad \text{for } x \in [0,1].$$

Proof. Let $x \in [0,1]$ be a fixed number. Then by Taylor's formula with remainder,

$$\begin{aligned} f(0) &= f(x) + f'(x)(0-x) + \frac{f''(\xi_0)}{2}(0-x)^2 \quad \text{for some } \xi_0 \in (0,1), \\ f(1) &= f(x) + f'(x)(1-x) + \frac{f''(\xi_1)}{2}(1-x)^2 \quad \text{for some } \xi_1 \in (0,1). \end{aligned}$$

Subtracting the second equality from the first equality we have

$$0 = -f'(x) + \frac{f''(\xi_0)}{2}x^2 - \frac{f''(\xi_1)}{2}(1-x)^2.$$

Thus, it holds that

$$\begin{aligned} |f'(x)| &\le \frac{K}{2}\big(x^2 + (1-x)^2\big) \\ &= \frac{K}{2}\big(1 - 2x(1-x)\big) \le \frac{K}{2}. \end{aligned}$$

In particular, if $x = 1/2$, we have

$$\left|f'\left(\frac{1}{2}\right)\right| \le \frac{K}{2}\big((1/2)^2 + (1-1/2)^2\big) = \frac{K}{4}.$$

□

Note: The values of $f(0)$ and $f(1)$ do not matter; only the equality matters.

2. Prove that if f is a continuous nonnegative function on $[a,b]$ and

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ for all x in $[a,b]$.

Proof. Assume the contrary; then there is a $x_0 \in [a,b]$ such that $f(x_0) > 0$. Because f is continuous, for sufficiently small $\epsilon > 0$, there is a $\delta > 0$ such that $f(x) \ge \epsilon$ whenever $|x - x_0| < \delta$. Therefore

$$\begin{aligned} \int_a^b f(x)\,dx &\ge \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \\ &\ge \int_{x_0-\delta}^{x_0+\delta} \epsilon\,dx \\ &= 2\epsilon\delta > 0, \end{aligned}$$

which is a contradiction. □

Another proof. (전현성 씨 답안 기반) For $x \in [a, b]$, define

$$F(x) := \int_a^x f(t)\,dt.$$

(Note that because f is continuous on $[a, b]$, the Riemann integral in the right hand side exists.) Since f is nonnegative,

$$0 \le \int_a^x f(x)\,dx \le \int_a^b f(x)\,dx = 0.$$

Hence for every $x \in [a, b]$, we have

$$F(x) = \int_a^x f(x)\,dx = 0.$$

Finally by the Fundamental theorem of calculus it holds that

$$f(x) = F'(x) = 0$$

for all x in $[a, b]$. □