Solutions for Homework 8

MAS501 Analysis for Engineers, Spring 2011

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1. Let f be continuous on $[a, b]$ and α be a jump function having jumps at the points $x_1, x_2, \cdots, x_N$.

   (a) Prove that $\int_a^b f\, d\alpha$ exists.
   Hint: Use Theorem 6.1.2 in the textbook.

   Proof. Let $x_0 := a$ and $x_{N+1} := b$. Then α is a constant, say a_n, in (x_n, x_{n+1}) for $n = 0, 1, \cdots, N$ and you can easily verify that

   $$V(\alpha, [x_n, x_{n+1}]) \le |f(x_n) - a_n| + |f(x_{n+1}) - a_n| < \infty, \quad n = 0, 1, \cdots, N.$$

   Hence by Theorem 6.3.2 (e), we have

   $$V(\alpha, [a, b]) = \sum_{n=0}^{N} V(\alpha, [x_n, x_{n+1}]) < \infty.$$

   So α is of bounded variation on $[a, b]$ and by Theorem 6.3.3, α is expressible as the difference of two increasing functions. Therefore the result follows from Theorem 6.1.2 and Theorem 6.2.1 (c). □

   (b) Show that

   $$\int_a^b f\, d\alpha = \sum_{n=1}^{N} f(x_n)\, c_n,$$

   where $c_n := \alpha(x_n^+) - \alpha(x_n^-)$, $1 \le n \le N$.

   Proof. Let $y_n := (x_{n-1} + x_n)/2$ for $n = 1, \cdots, N+1$. Then α has only one jump at x_n in $[y_n, y_{n+1}]$. Hence by (a), $\int_{y_n}^{y_{n+1}} f\, d\alpha$ exists. First assume that $y_n < x_n < y_{n+1}$ and choose a partition P of $[y_n, y_{n+1}]$ so that $[x_n - h, x_n + h]$, $h > 0$, is one of the subintervals of P. Then the Riemann-Stieltjes sum is

   $$\begin{aligned} S(P, f, \alpha) &= f(t)\big[\alpha(x_n + h) - \alpha(x_n - h)\big] \quad \text{for some } t \in [x_n - h, x_n + h] \\ &= f(t)\, c_n. \end{aligned}$$

   Because $\int_{y_n}^{y_{n+1}} f\, d\alpha$ exists, the left hand side converges to $\int_{y_n}^{y_{n+1}} f\, d\alpha$ as $|P| \to 0$. Also if $|P| \to 0$, then $h \to 0$ and by continuity of f, the right hand side converges to $f(x_n)\, c_n$. We've just proved that

   $$\int_{y_n}^{y_{n+1}} f\, d\alpha = f(x_n)\, c_n.$$

   When $y_n = x_n$ or $y_{n+1} = x_n$, we can prove the same result holds by similar fashion. Also we have

   $$\int_a^{y_1} f\, d\alpha = \int_{y_{N+1}}^{b} f\, d\alpha = 0$$

   because α is a constant on $[a, y_1]$ and $[y_{N+1}, b]$. Finally

   $$\int_a^b f\, d\alpha = \int_a^{y_1} f\, d\alpha + \sum_{n=1}^{N} \int_{y_n}^{y_{n+1}} f\, d\alpha + \int_{y_{N+1}}^{b} f\, d\alpha = \sum_{n=1}^{N} f(x_n) c_n.$$

   □

2. Give an example of a function α which is continuous on $[0,1]$ and differentiable on $(0,1)$ such that $\alpha \in BV[0,1]$, but α' is unbounded on $(0,1)$.

   Answer. Define $\alpha(x) := \sqrt{x}$ on $[0,1]$. Then obviously α is continuous on $[0,1]$ and differentiable on $(0,1)$. Also $\alpha'(x) = 1/(2\sqrt{x}) \to \infty$ as $x \to 0^+$; α' is unbounded on $(0,1)$. But α is increasing on $[0,1]$ hence of bounded variation. $\square$