Solutions for Homework 9

MAS501 Analysis for Engineers, Spring 2011

---

1. Prove that $m^*(\{a\}) = 0$ when a is a real number.

   Proof. For every $\epsilon > 0$, we have $\{a\} \subset (a-\epsilon, a+\epsilon)$. So by definition of the outer measure,

   $$0 \leq m^*(\{a\}) \leq l\big((a-\epsilon, a+\epsilon)\big) = 2\epsilon.$$

   Since $\epsilon > 0$ is arbitrary, the proof is complete. □

2. Let E and F be two measurable sets. Prove that if $E \supset F$ and $m(F) < \infty$, then

   $$m(E - F) = m(E) - m(F).$$

   Proof. Because the set F is measurable, we have

   $$m^*(E) = m^*(E \cap F) + m^*(E - F)$$

   by definition. Since the set E is measurable and $E \supset F$, it holds that

   $$m(E) = m(F) + m(E - F).$$

   Finally subtracting $m(F) < \infty$ from both sides completes the proof. □

3. Let $\{E_n\}$ be a decreasing sequence of measurable sets, that is, $E_n \supset E_{n+1}$ for each n.

   (a) Prove that if $m(E_1) < \infty$, we have

   $$m\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} m(E_n).$$

   Proof. Let $F_n := E_n - E_{n+1}$. Then

   $$E_1 - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$$

   and $\{F_n\}$ is a collection of pairwise disjoint sets. Also by the result of Problem 2, we have

   $$\begin{aligned} m(E_1) - m\Big(\bigcap_{n=1}^{\infty} E_n\Big) &= m\Big(E_1 - \bigcap_{n=1}^{\infty} E_n\Big) = m\Big(\bigcup_{n=1}^{\infty} F_n\Big) \\ &= \sum_{n=1}^{\infty} m(F_n) \quad \text{by countable additivity of the measure} \\ &= \sum_{n=1}^{\infty} m(E_n - E_{n+1}) = \sum_{n=1}^{\infty} \big(m(E_n) - m(E_{n+1})\big) \\ &= m(E_1) - \lim_{n\to\infty} m(E_n). \end{aligned}$$

   Since $m(E_1) < \infty$, the result follows. □

(b) Show that the condition $m(E_1) < \infty$ is necessary.

Proof. Let $E_n := (n, \infty)$. Then $\{E_n\}$ is a decreasing sequence of measurable sets but

$$\bigcap_{n=1}^{\infty} E_n = \emptyset \quad \text{and} \quad m(E_n) = \infty \text{ for each } n.$$

Hence we have

$$0 = m\Big(\bigcap_{n=1}^{\infty} E_n\Big) \neq \lim_{n\to\infty} m(E_n) = \infty.$$

So the condition $m(E_1) < \infty$ is necessary in the previous problem. □

4. If A is any set, we define the *characteristic function* χ_A of the set A to be the function given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that the function χ_A is measurable if and only if A is measurable. (We assume the domain of χ_A is measurable.)

Proof. Let D be the domain of χ_A. First assume the set A is measurable. Then we have

$$\{x \in D : \chi_A(x) > \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ D & \text{if } \alpha < 0 \end{cases}$$

is measurable in any case. Hence the function χ_A is measurable. On the other hand, if the function χ_A is measurable, then by definition $\{x \in D : \chi_A(x) > 0\} = A$ is measurable. □