Homework 9

MAS501 Analysis for Engineers, Spring 2011

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1. Prove that $m^*(\{a\}) = 0$ when a is a real number.

   Hint: Recall definition of the outer measure.

2. Let E and F be two measurable sets. Prove that if $E \supset F$ and $m(F) < \infty$, then

$$m(E - F) = m(E) - m(F).$$

   Hint: It suffices to show that $m(E) = m(E - F) + m(F)$.

3. Let $\{E_n\}$ be a decreasing sequence of measurable sets, that is, $E_n \supset E_{n+1}$ for each n.

   (a) Prove that if $m(E_1) < \infty$, we have

$$m\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} m(E_n).$$

   Hint: Let $F_n := E_n - E_{n+1}$. Then

$$E_1 - \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$$

   and the set F_n are *pairwise disjoint*. Use countable additivity of the measure

$$m\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \sum_{n=1}^{\infty} m(F_n)$$

   and apply the result of Problem 2 in this homework.

   (b) Show that the condition $m(E_1) < \infty$ is necessary.

   Hint: Find a decreasing sequence of measurable sets $\{E_n\}$ with

$$\bigcap_{n=1}^{\infty} E_n = \emptyset \quad \text{and} \quad m(E_n) = \infty \text{ for each } n.$$

4. If A is any set, we define the *characteristic function* χ_A of the set A to be the function given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

   Prove that the function χ_A is measurable if and only if A is measurable. (We assume the domain of χ_A is measurable.)

   Hint: Recall definition of the measurable function.