Solutions for Homework 10

MAS501 Analysis for Engineers, Spring 2011

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1. Let f and g be bounded measurable functions defined on a set E of finite measure. Prove the following statements:

   (a) If $f \le g$ a.e., then
   $$\int_E f \le \int_E g.$$

   Proof. Let ψ be a *simple* function satisfying $\psi \ge g - f$. Since $g - f \ge 0$ a.e., we have $\psi \ge 0$ a.e. From this it follows that
   $$\int_E \psi \ge 0,$$
   whence
   $$\int_E g - f \ge 0.$$
   The proof is complete because $\int_E g - f = \int_E g - \int_E f$. □

   (b) It holds that
   $$\left| \int_E f \right| \le \int_E |f|.$$

   Proof. Since $\pm f \le |f|$, by Problem 1a, we have
   $$\pm \int_E f = \int_E \pm f \le \int_E |f|,$$
   whence $\left| \int_E f \right| \le \int_E |f|$. □

   (c) If $A \le f(x) \le B$ a.e., then
   $$A\, m(E) \le \int_E f \le B\, m(E).$$

   Proof. The proof is immediate from Problem 1a and the fact that $\int_E 1 = m(E)$. □

2. Find examples showing the following statements are true:

   (a) We may have strict inequality in Fatou's Lemma.

   Example. Define $f_n := \chi_{[n,n+1)}$. Then $f_n \to f \equiv 0$ and
   $$0 = \int_{\mathbf{R}} f < \liminf_{n\to\infty} \int_{\mathbf{R}} f_n = 1.$$
   □

   (b) Monotone Convergence Theorem need not hold for decreasing sequence of functions.

Example. Define $f_n := \chi_{[n,\infty)}$. Then $f_n \searrow f \equiv 0$ and

$$0 = \int_{\mathbf{R}} f \neq \lim_{n\to\infty} \int_{\mathbf{R}} f_n = \infty.$$

□

3. Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

Proof. Let D be the domain of f (which is a measurable set). Define

$$E := \{x \in D : f(x) > 0\}$$
$$E_n := \{x \in D : f(x) \geq 1/n\}.$$

Then we have $E = \bigcup_{n=1}^{\infty} E_n$. Also by nonnegativity of f, it follows that

$$0 = \int_D f \geq \int_D \chi_{E_n} f = \int_{E_n} f \geq \int_{E_n} \frac{1}{n} = \frac{1}{n} m(E_n),$$

whence $m(E_n) = 0$ for every n. Therefore, by countable subadditivity, we have

$$m(E) = m\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} m(E_n) = 0.$$

This means $f = 0$ a.e. by definition. □